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ABSORBING SETS FOR COUNTABLE-DIMENSIONAL SPACES

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Absorbing sets for absolute Borel classes of countable-dimensional spaces are constructed.

0. All spaces under the discussion are assumed to be metrizable and separable. The notion of $\mathcal{C}$-absorbing set, $\mathcal{C}$ being a class of spaces, is introduced in [1]. In [1] a characterization theorem for the $\mathcal{C}$-absorbing sets is also proved.

Dobrowolski and Mogilski in [2] have asked to find more absorbing sets. In this paper we construct the absorbing sets for all absolute additive Borel classes for $\xi \geq 2$ and for multiplicative classes for $\xi \geq 3$ for countable-dimensional spaces.

Nagata has constructed the absolute $G_{\delta\sigma}$ space N universal for the class of countable-dimensional (briefly c.d.) spaces [5]. Some constructions of universal spaces for c.d. spaces can be found in [6].

In the sequel the absolute Borel classes will be denoted by $\mathcal{M}_\xi$ for the multiplicative classes and by $\mathcal{A}_\xi$ for the additive classes. (Thus G_δ is $\mathcal{M}_1$, F_σ is $\mathcal{A}_1$,...). The Borel classes of subsets of a space X are denoted by $\mathcal{A}_\xi(X)$ and $\mathcal{M}_\xi(X)$.

A space X is defined to be $\mathcal{C}$-universal, provided X contains a closed copy of each space from $\mathcal{C}$.

Denote by $\mathcal{M}_\xi^{cd}(\mathcal{A}_\xi^{cd})$ the intersection of the class $\mathcal{M}_\xi(\mathcal{A}_\xi)$ with the class of countable-dimensional spaces. M.M.Zarichnyi has proved that there exist $\mathcal{A}_\xi^{cd}$-universal spaces for $\xi \geq 2$ and $\mathcal{M}_\xi^{cd}$-universal spaces for $\xi \geq 3$ [9]. This result implies existence of $\mathcal{M}_\xi^{cd}$- and $\mathcal{A}_\xi^{cd}$-absorbing sets. Since the proof of Zarichnyi is based on the factorization theorem for countable-dimensional spaces and it contains no information about obtained universal spaces, it is reasonable to find natural realizations for these spaces.

The paper is organized as follows. In Section 1 we recall some known definitions and results. In Section 2 we construct an $\mathcal{A}_2^{cd}$-universal space N_2. In Section 3 we generalize our method to obtain $\mathcal{A}_\xi^{cd}$-universal spaces N_ξ for $\xi \geq 2$ and $\mathcal{M}_\xi^{cd}$-universal spaces G_ξ for $\xi \geq 3$. In Section 4 we obtain $\mathcal{A}_\xi^{cd}$- and $\mathcal{M}_\xi^{cd}$-absorbing sets.

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1. We don't give there any definitions of the fundamental notions of infinite-dimensional topology such as Z-set, homotopic negligibility etc. One can find them in [1].

Let $\mathcal{C}$ be a class of (separable metric) spaces. We say that $\mathcal{C}$ is topological if for every $C \in \mathcal{C}$ and every homeomorphism $h : C \to D$ it follows that $D \in \mathcal{C}$. A topological class $\mathcal{C}$ is hereditary with respect to closed subsets if every closed subset of any $C \in \mathcal{C}$ belongs to $\mathcal{C}$.

From now on, all the classes of spaces are assumed to be topological, additive and hereditary with respect to closed sets.

For a topological class $\mathcal{C}$ we can form the class $\mathcal{C}_\sigma$ that consists of all spaces C that can be written as $C = \cup_{n=1}^{\infty} C_n$, where C_n is a closed subset of C with $C_n \in \mathcal{C}$, $n = 1, 2, \ldots$.

A space X is strongly $\mathcal{C}$-universal if for every map $f : C \to X$ from a space $C \in \mathcal{C}$, for every closed subset $D \subset C$ such that $f|D : D \to X$ is a Z-embedding and for every $\mathcal{U} \in \mathrm{cov}(X)$, there exists a Z-embedding $h : C \to X$ such that $h|D = f|D$ and $(f,h) \prec \mathcal{U}$ (the symbol $(f,h) \prec \mathcal{U}$ means that for each $x \in X$ there is $U \in \mathcal{U}$ such that $\{f(x), h(x)\} \subset U$).

A space X is countable-dimensional if X can be represented as the union of a sequence $X_1, X_2, \ldots$ of its subspaces such that $\dim X_i < \infty$ for $i = 1, 2, \ldots$ [4].

It is well-known that a space X is a c.d. space if and only if X can be represented as the union of a sequence $X_1, X_2, \ldots$ of subspaces such that $\dim X_i \leq 0$ for $i = 1, 2, \ldots$ [5].

By $\mathbb{R}^\omega$ we denote the countable infinite product of real lines. We say that a subset X of $\mathbb{R}^\omega$ is $\mathcal{C}$-absorbing if $\mathbb{R}^\omega \setminus X$ is locally homotopy negligible in $\mathbb{R}^\omega$, $X = \cup_{n=1}^{\infty} X_n$ where each X_n is a strong Z-set in X and $X_n \in \mathcal{C}$, and X is strongly $\mathcal{C}$-universal.

In [1] it is proved that if there is a $\mathcal{C}$-absorbing set Ω in $\mathbb{R}^\omega$ then $X \in AR$ is homeomorphic to Ω if and only if $X \in \mathcal{C}_\sigma$, X is strongly $\mathcal{C}$-universal, and $X = \cup_{i=1}^{\infty} X_i$ where each X_i is a strong Z-set in X.

Let A be a subset of X. By $W(X,A)$ we denote a subset in X^ω defined by $W(X,A) = \{(x_1, x_2, \ldots) \in X^\omega | x_i \in A \text{ for all but finitely many } i\}$.

By $\mathbb{P}$ we denote the subset in $\mathbb{R}$ consisting of all irrationals. Then the space $N = W(\mathbb{R}, \mathbb{P}) \subset \mathbb{R}^\omega$ is the universal Nagata's space for c.d. spaces, i.e., each c.d. space can be embedded (not necessarily as a closed subset) in N [5]. It is easy to see that N is both c.d. space and an absolute $\mathcal{A}_2$-space.

2. In this section we shall construct an $\mathcal{A}_2^{cd}$-universal space N_2.

Recall that $\mathbb{P}^\omega \cong \mathbb{P}$ (the symbol "$\cong$" means "homeomorphic to"). By M_1 we denote the set $(\mathbb{P} \setminus \{\sqrt{2}\})^\omega \subset \mathbb{P}^\omega \subset N$. Then the space N_2 is defined as follows: $N_2 = W(N, M_1)$. Since $W(N, \mathbb{P}^\omega) \cong N$, we have $N_2 \subset N$. It is easy to see that $N_2 \in \mathcal{A}_\xi^{cd}$.

We shall need the following Lemma from [5]:

Lemma A [5,Lemma 4.1]. *Let $R = \cup_{n=1}^{\infty} B_n$, $\dim B_n = 0$, U_m is a countable family of open sets, F_m is a countable family of closed sets and $F_m \subset U_m$.*

Then we can choose open sets U_{mr}, satisfying the next conditions:

1. $F_m \subset U_{mr} \subset \mathrm{cl}\, U_{mr} \subset U_{mr'} \subset \mathrm{cl}\, U_{mr'} \subset U_m$ *for every* $r < r'$,
2. $\mathrm{cl}\, U_{mr} = \cap\{U_{mr'} \mid r' > r\}$ $U_{mr} = \cup\{\mathrm{cl}\, U_{mr'} \mid r' < r\}$,
3. *Each point $p \in B_n$ belongs to at most $n-1$ boundaries of U_{mr}, where r runs over the rationals from the segment $[-\sqrt{2}; \sqrt{2}]$.*

Remark. It follows from the proof of the Lemma A that we can assume $F_m = \cap\{U_{mr} \mid r > -\sqrt{2}\}$ and $U_m = \cup\{U_{mr} \mid r < \sqrt{2}\}$.

It is easy to check that N can be represented as the union of an increasing sequence $A_1, A_2, \ldots$ of zero-dimensional absolute G_δ-subsets. From now on, we fix this represention $N = \cup_{n=1}^{\infty} A_n$.

Lemma 1. *Let $S \subset A_i$ for some $i \in \mathbb{N}$ and S is an absolute G_δ. Then there exists a map $\varphi : N \to N$ such that $\varphi^{-1}(M_1) = S$ and $\varphi(A_i) \subset \mathbb{P}^\omega$.*

Proof. Put $B_1 = A_i$, $B_2 = A_{i+1}$, etc. We can represent S as the intersection of a sequence $U_1, U_2, \ldots$ of open subsets of N. We can assume that the sequence $U_1, U_2, \ldots$ is decreasing. Choose any point $s \in S$ and put $F_i = \{s\}$ for each $i \in \mathbb{N}$. Then we can choose open subsets U_{ir} satisfying the conditions from Lemma A.

For each $i \in \mathbb{N}$ define the map $\varphi_i : N \to \mathbb{R}$ by the formula

$$\varphi_i(x) = \begin{cases} -\sqrt{2}, & x = s, \\ \inf\{r \mid x \in U_{ir}\}, & x \in U_i \setminus \{s\}, \\ \sqrt{2}, & x \notin U_i. \end{cases}$$

The maps φ_i are continuous and φ_i take on rational values if and only if $x \in B(U_{ir})$ for some r ($B(U_{ir})$ is the boundary of U_{ir}).

Now define the map $\varphi : N \to \mathbb{R}^\omega$ as the diagonal of the maps φ_i: $\varphi = (\varphi_i)_{i=1}^{\infty}$. Since for each $x \in N$ there exists $j \in \mathbb{N}$ such that $x \in B_j$, x lies in at most $j-1$ boundaries of U_{kr}, i.e., all but finitely many numbers $\varphi_k(x)$ are irrational. Hence $\varphi(N) \subset N \subset \mathbb{R}^\omega$.

Additionally, since $A_i = B_1$, we have $\varphi(A_i) \subset \mathbb{P}^\omega$. Since $x \in \cap_{i=1}^{\infty} U_i$ for each $x \in S$, we have $\varphi_i(x) \in \mathbb{P} \setminus \{\sqrt{2}\}$ for each $i \in \mathbb{N}$. On the other hand, if $x \notin S$ then there exists $i \in \mathbb{N}$ such that $x \notin U_i$, hence $\varphi_i(x) = \sqrt{2}$. Thus we have $\varphi^{-1}(M_1) = S$. Lemma is proved.

Theorem 1. *Let $F \subset N$, $F \in \mathcal{A}_2(N)$ (i.e., F is $G_{\delta\sigma}$-set in N). Then there exists a closed embedding $h : N \to W(N, \mathbb{P}^\omega)$ such that $h^{-1}(N_2) = F$ and the following two conditions are satisfied:*
1. *for each $i \in \mathbb{N}$ we have $\mathrm{pr}_j \circ h(A_i) \subset \mathbb{P}^\omega$ for each $j \geq i+1$,*
2. *$\mathrm{pr}_i \circ h(N \setminus F) \cap M_1 = \varnothing$ for each $i > 1$, where $\mathrm{pr}_i : W(N, \mathbb{P}) \to N$ is the natural projection.*

Proof. We can assume that $F = \cup_{i=1}^{\infty} F_i$ where each F_i is a G_δ-subset in A_i and the sequence $F_1, F_2, \ldots$ is increasing. For each $i \in \mathbb{N}$ let $h_i : N \to N$ be a map satisfying $h_i^{-1}(M_1) = F_i$ and $h_i(A_i) \subset \mathbb{P}^\omega$ (see Lemma 1).

The identity map of N is denoted by h_0. Define the map $h : N \to N^\omega$ as the diagonal $h = (h_i)_{i=0}^{\infty}$. Since each point $x \in N$ lies in some A_j, $h_i(x) \in \mathbb{P}^\omega$ for each $i \geq j$. Thus $h(N) \subset W(N, \mathbb{P}^\omega)$ and the condition 1 holds. The set $h(N)$ considered as the subset in the space $N \times W(N, \mathbb{P}^\omega) \cong W(N, \mathbb{P}^\omega)$, is the graph of the map $(h_i)_{i=1}^{\infty}$, hence h is a closed embedding.

Let $a \in F$. Then there exists $j_0 \in \mathbb{N}$ such that $a \in F_j$ for all $j \geq j_0$. It follows from Lemma 1 that $h_j(a) \in M_1$ for each $j \geq j_0$, hence $h(F) \subset N_2$. On the other hand if $a \notin F$ then for each $j \in \mathbb{N}$ we have $a \notin F_j$, hence $h_j(a) \notin M_1$. Thus the condition 2 is valid and Theorem is proved.

Corollary. *The space N_2 is $\mathcal{A}_2^{cd}$-universal.*

Since $W(N,\mathbb{P}^\omega)\cong N$ and $M_1\cong\mathbb{P}^\omega$, the following question arises naturally:

Are the spaces N and N_2 homeomorphic?

In particular, can the space N_2 be closely embedded in N?

3. Now we construct $\mathcal{A}_\xi^{cd}$-universal spaces for $\xi\geq 2$ and $\mathcal{M}_\xi^{cd}$-universal spaces for $\xi\geq 3$. We shall need the $\mathcal{M}_\xi^0$-universal spaces where $\mathcal{M}_\xi^0$ is the subclass of $\mathcal{M}_\xi$ consisting of 0-dimensional spaces.

We use the standard procedure of Sikorski [7]. Let $H_1=\mathbb{P}^\omega\setminus M_1$. Assume that sets H_ξ and M_ξ are construct for some ordinal ξ. Then put $M_{\xi+1}=H_\xi^\omega\subset\mathbb{P}^\omega$ and $H_{\xi+1}=\mathbb{P}^\omega\setminus M_{\xi+1}$.

For the limit ordinal α we fix an increasing sequence of ordinal numbers $\xi_1,\xi_2,...$ such that ξ_i converges to α. Now assume that sets H_ξ and M_ξ are already construct for each $\xi<\alpha$. Then put $M_\alpha=\prod_{i=1}^\infty H_{\xi_i}$ and $H_\alpha=\mathbb{P}^\omega\setminus M_\alpha$. Note that $H_\alpha(M_\alpha)$ is the $\mathcal{A}_\alpha^0(\mathcal{M}_\alpha^0)$-universal space and $M_\alpha\in\mathcal{M}_\alpha^0$, $H_\alpha\in\mathcal{A}_\alpha^0$ for $\alpha>1$.

In the sequel, we shall identify the space N with $W(N,\mathbb{P})$ and the space $\mathbb{P}$ with $\mathbb{P}^\omega$.

Define the spaces N_ξ for $\xi>2$. For a non-limit ordinal $\xi=\alpha+1$ put $N_\xi=W(N,M_\alpha)\subset W(N,\mathbb{P})\cong N$. For a limit ordinal ξ fix a sequence ξ_i defined in the construction of the spaces M_ξ and define the space N_ξ as follows: $N_\xi=\{(q_i)\in N^\omega\mid$ there exists $n\in\mathbb{N}$ such that $q_i\in M_{\xi_n}$ for each $i\geq n\}$.

Theorem 2. *For each countable ordinal $\xi\geq 2$ the space N_ξ is $\mathcal{A}_\xi^{cd}$-universal.*

Proof. We shall prove the theorem by transfinite induction. For $\xi=2$ our theorem coincides with Theorem 1.

We shall prove that for each countable ordinal ξ the following statement is valid:

$(*)_\xi$ for each $F\in\mathcal{A}_\xi(N)$ there exists a closed embedding $\varphi:N\to W(N,\mathbb{P}^\omega)$ such that $\varphi^{-1}(N_\xi)=F$ and following two conditions are satisfied

1) for each $i\in\mathbb{N}$ $pr_j\circ\varphi(A_i)\subset\mathbb{P}^\omega$ for each $j\geq i+1$,

2) $pr_i\circ\varphi(N\setminus F)\cap M_{\xi-1}=\varnothing$ for each $i>1$.

If ξ is the limit ordinal the condition 2) should be modified:

2)' $pr_i\circ\varphi(N\setminus F)\cap M_{\xi_{i-1}}=\varnothing$ for each $i>1$.

We shall give the complete proof of $(*)_\xi$ only for $\xi=\beta+2$. The proof of $(*)_\xi$ for $\xi-1$ or a limit ordinal ξ is analogous.

Let $G\in\mathcal{A}_\xi(N)$. Then $G=\cup_{i=1}^\infty G_i$ where $G_i\in\mathcal{M}_{\beta+1}(N)$ and $G_i\subset A_i$. Then $N\setminus G_i\in\mathcal{A}_{\beta+1}(N)$ and by $(*)_{\beta+1}$ there exists a closed embedding $\varphi_i:N\to W(N,\mathbb{P}^\omega)$, satisfying all the conditions from $(*)_{\beta+1}$. For each $i\in\mathbb{N}$ define the map $h_i:N\to W(N,\mathbb{P}^\omega)$ by the formula $h_i=(pr_{i+j}\circ\varphi_i)_{j=1}^\infty$. It follows from the properties of the map φ that $pr_l\circ h_i(G_i)\subset\mathbb{P}^\omega\setminus M_\beta$ for each $l\in\mathbb{N}$, and for each point $x\in N\setminus G_i$ there exists $k\in\mathbb{N}$ such that $pr_s\circ h_i(x)\in M_\beta$ for each $s\geq k$.

Since $W(N,\mathbb{P}^\omega)\cong N$ and $(\mathbb{P}^\omega\setminus M_\beta)^\omega=H_\beta^\omega\cong M_{\beta+1}\subset\mathbb{P}^\omega\subset N$, we have $h_i^{-1}(M_{\beta+1})=G_i$ and $h_i(A_i)\subset\mathbb{P}^\omega$.

Denote by h_0 the identity map of N and define the map $\varphi:N\to W(N,\mathbb{P}^\omega)$ as the diagonal $\varphi=(h_i)_{i=0}^\infty$. It is easy to check that φ is a closed embedding and satisfies all the conditions from $(*)_\xi$. Theorem follows immediately from $(*)_\xi$.

Now we construct the spaces G_ξ for $\xi\geq 3$ putting $G_\xi=N\setminus N_\xi$. It is easy to check that $G_\xi\in\mathcal{M}_\xi^{cd}$ for each $\xi\geq 3$.

Theorem 3. *The space G_ξ is $\mathcal{M}_\xi^{cd}$-universal for $\xi \geq 3$.*

Proof. Let $L \in \mathcal{M}_\xi^{cd}$. We can consider L as a subset in N. Then $N \setminus L \in \mathcal{A}_\xi(N)$. It follows from the proof of Theorem 2 that there exists a closed embedding $\varphi : N \to W(N, \mathbb{P}^\omega)$ such that $\varphi^{-1}(N_\xi) = N \setminus L$. Then $\varphi^{-1}(N \setminus N_\xi) = L$ and $\varphi|L$ is a closed embedding in $G_\xi = N \setminus N_\xi$.

It is remarked in [9] that $\mathcal{A}_1^{cd}$- and $\mathcal{M}_1^{cd}$-universal spaces fail to exist, and the question is formulated whether there exists $\mathcal{M}_2^{cd}$-universal space. Unfortunately, our methods do not work in this case.

4. Now we can construct $\mathcal{A}_\xi^{cd}(\mathcal{M}_\xi^{cd})$-absorbing sets. This construction is based on the spaces $N_\xi(G_\xi)$.

Note that the subset N_ξ has locally homotopy negligible complement in $\mathbb{R}^\omega$. (It follows from the inclusions $W(\mathbb{R}, *) \subset N_\xi \subset \mathbb{R}^\omega$, where $*$ is any point in $\mathbb{R}$). Since $\mathbb{R}^\omega$ is AR, N_ξ is AR, too [8]. Consider the space $W(N_\xi, *)$, which is AR. This space is strongly $\mathcal{F}_0$-universal where $\mathcal{F}_0$ is the class of all closed subsets of $W(N_\xi, *)$ [1]. But the class $\mathcal{F}_0$ coincides with the class $\mathcal{A}_\xi^{cd}$. It is easy to check that $W(N_\xi, *)$ can be represented as a countable union of strong Z-sets. Since our reasoning is also valid for G_ξ, $\xi \geq 3$, the following theorem holds:

Theorem 4. *$\Lambda_\xi^{cd} = W(N_\xi, *)$ is an $\mathcal{A}_\xi^{cd}$-absorbing set for each $\xi \geq 2$ and $\Omega_\xi^{cd} = W(G_\xi, *)$ is an $\mathcal{M}_\xi^{cd}$-absorbing set for $\xi \geq 3$.*

The following theorem follows immediately from Theorem 4 and from [1].

Theorem 5. *Let $X \in \mathcal{A}R$. Then $X \cong \Lambda_\xi^{cd}(\Omega_\xi^{cd})$ if and only if $X \in \mathcal{A}_\xi^{cd}(\mathcal{M}_\xi^{cd})$, X is strongly $\mathcal{A}_\xi^{cd}(\mathcal{M}_\xi^{cd})$-universal and $X = \cup_{i=1}^\infty X_i$ where each X_i is a strong Z-set in X.*

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Added in proof. When the paper was accepted for publication the author received the preprint of J. Mogilski [10] where it is shown that the product $\sigma \times N$ is an $\mathcal{A}_2^{cd}$-absorbing set.

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