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PRESERVATION OF METRIZABLE ABSOLUTE RETRACTS AND SOFT MAPS BY COVARIANT TOPOLOGICAL FUNCTORS

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Assume that a covariant functor $F : \mathcal{S}Metr \to \mathcal{S}Metr$ on the category of separable metric spaces preserves embeddings, homotopies and compacta. It is proven that (i) if $F(K) \in$ AR for every cell $K = [0,1]^n$, $n \in \mathbb{N}$, then the functor F preserves the class of separable absolute retracts; (ii) if $F(K) \in$ ANR for every finite simplicial complex K then F preserves the class of separable absolute neighborhood retracts having the homotopy type of a compact ANR; (iii) if $F(K) \in$ ANR for every countable locally finite simplicial complex K then the functor F preserves the class of separable absolute neighborhood retracts.

If additionally, the functor F preserves preimages then it preserves the class of soft maps between separable metric spaces if and only if the map $F(\mathrm{pr}_1) : F(Q \times Q) \to F(Q)$ is soft (here $\mathrm{pr}_1 : Q \times Q \to Q$ is the projection and Q is the Hilbert cube).

Introduction.

In the last decade, the theory of functors in the category of compacta has been formed as an independent branch of topology. In particular, geometric aspect of this theory, determined by relationship between the theory of retracts and the functor theory, have reached a considerable progress. This progress is stipulated by that long noticed fact that some functors (e.g., the hyperspace functor, the functor of probability measures, superextension etc.) essentially improve or at least preserve extensor properties of spaces. Notice, however, that there exist "nice" in many respects functors which fail to preserve the class of absolute (neighborhood) retracts. The simplest example is the countable power functor $(\cdot)^\omega$ which preserves the class of absolute retracts, but not the class of absolute neighborhood retracts. A more complicated example is given by the subfunctor P_2^Π of the probability measure functor P. Here $\Pi \subset [0,1]$ is the standard Cantor discontinuum and $P_2^\Pi(X) = \{\mu \in P(X) \mid \mu = t\delta_x + (1-t)\delta_y,\ x,y \in X \text{ and } t \in \Pi\}$, where δ_x is the Dirac measure supported by the point $x \in X$. It is easily seen that for every compactum X cotnaining more than one point, the space $P_2^\Pi(X)$ is not an absolute neighborhood retract. There is a large number of literature dedicated to

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the problem of preservation of various classes of absolute extensors by topological functors [2–9].

The theory of topological functors acting out of the category of compacta is not so developed. There is however, the construction $F_\beta : \mathcal{T}ych \to \mathcal{T}ych$ of extension of a normal functor $F : \mathcal{C}omp \to \mathcal{C}omp$ from the category of compacta onto the category of Tychonoff spaces, offered by A. Chigogidze [14]: for a normal functor $F : \mathcal{C}omp \to \mathcal{C}omp$ and a Tychonoff space X we let $F_\beta(X) = \{a \in F(\beta X) \mid \operatorname{supp}(a) \subset X\}$, where βX is the Stone-Čech compactification of X. Analogously to the compact case, the question was posed: when does the functor F_β preserve the class of absolute (neighborhood) retracts? In [3,18,24], the answer is given for some concrete functors.

In this note we shall prove that if a normal functor $F : \mathcal{C}omp \to \mathcal{C}omp$ transforms the finite-dimensional cells (countable locally finite simplicial complexes) into absolute (neighborhood) retracts then (i) the functor F_β preserves the class of separable absolute (neighborhood) retracts; (ii) if additionally F is a functor with finite support then for every integer $n \geq 0$ the functor F_β preserves the class of absolute (neighborhood) extensors in dimension n.

Analogical results are obtained for maps: the functor F_β preserves (locally) soft maps between separable metric spaces if and only if it preserves softness of the projection $\mathrm{pr}_1 : Q \times Q \to Q$ onto the Hilbert cube Q (it preserves local softness of the map $\mathrm{pr}_1 |U : U \to Q$ for every open set $U \subset Q \times Q$).

The results of this paper were announced in [17].

On homotopy preserving functors.

We denote by $\mathcal{T}op$ the category of topological spaces and their continuous maps and by ($\mathcal{S}\mathcal{M}etr$) $\mathcal{M}etr$ its full subcategory consisting of (separable) metric spaces (recall that a subcategory $\mathcal{C}$ of the category $\mathcal{T}op$ is full, provided for any objects X, Y of the category $\mathcal{C}$ the set of morphisms from X to Y consists of all continuous maps from X to Y).

Let $F : \mathcal{C} \to \mathcal{T}op$ be a functor from a full subcategory $\mathcal{C} \subset \mathcal{T}op$. We shall say that the functor F preserves

(i) *embeddings*, provided for every embedding $e : X \to Y$ of spaces $X, Y \in \mathcal{O}b\,\mathcal{C}$ the map $F(e) : FX \to FY$ is also an embedding;

(ii) *homotopies*, provided for every homotopy $\{h_t : X \to Y\}_{t\in[0,1]}$, where $X, Y \in \mathcal{O}b\,\mathcal{C}$, the homotopy $\{F(h_t) : FX \to FY\}_{t\in[0,1]}$ is continuous as a map from $FX \times [0,1]$ into FY.

Further we assume that for every $X \in \mathcal{O}b\,\mathcal{C}$ $X \times [0,1] \in \mathcal{O}b\,\mathcal{C}$. For every $X \in \mathcal{O}b\,\mathcal{C}$ define the map $j_X : F(X) \times [0,1] \to F(X \times [0,1])$ by $j_X(a,t) = F(i_t)(a)$, $(a,t) \in F(X) \times [0,1]$, where the map $i_t : X \to X \times [0,1]$ is determined by $i_t(x) = (x,t)$, $x \in X$.

Proposition 1. *If for every $X \in \mathcal{O}b\,\mathcal{C}$ the map $j_X : F(X) \times [0,1] \to F(X \times [0,1])$ is continuous then the functor F preserves homotopies.*

Proposition 2. *If the functor F preserves embeddings and for some $X \in \mathcal{O}b\,\mathcal{C}$ the map $j_X : F(X) \times [0,1] \to F(X \times [0,1])$ is continuous then for every $Y \in \mathcal{O}b\,\mathcal{C}$, $Y \subset X$, the map $j_Y : F(Y) \times [0,1] \to F(Y \times [0,1])$ is continuous as well.*

By $Q = [-1,1]^\omega$ we denote the Hilbert cube. It is well known that every separable metric space can be embedded into the Hilbert cube. This and Proposition 2 imply

Corollary 1. *An embedding preserving functor $F : \mathcal{SM}etr \to \mathcal{T}op$ preserves homotopies if and only if the map $j_Q : F(Q) \times [0,1] \to F(Q \times [0,1])$ is continuous.*

Finally we consider the example of an embedding preserving functor which does not preserve homotopies. For every topological space X let $\beta_d(X)$ be the Stone-Čech compactification of the discrete copy of X and for a map $f : X \to Y$ of topological spaces let $\beta_d(f) : \beta_d(X) \to \beta_d(Y)$ be the only extension of the map $f : X \to Y$ between X and Y endowed with the discrete topoogies. Obviously, the construction β_d determines the functor $\beta_d : \mathcal{T}op \to \mathcal{C}omp$. Moreover, by [14], the functor β_d is embedding preserving. However, since the map $j_X : \beta_d(X) \times [0,1] \to \beta_d(X \times [0,1])$ is discontinuous even for the singleton $X = \{*\}$, the functor β_d does not preserve homotopies.

Main Results.

By A(N)E we denote the collection of absolute (neighborhood) extensors for the class of metrizable spaces and by A(N)R the collection of metrizable A(N)R spaces.

Theorem 1. *Let $F : \mathcal{SM}etr \to \mathcal{SM}etr$ be a functor that preserves embeddings, homotopies and compacta.*

1) *If $F(K) \in AR$ for every cell $K = [0,1]^n$, $n \in \mathbb{N}$, then the functor F preserves the class of separable absolute retracts.*
2) *If $F(K) \in ANR$ for every finite simplicial complex K then F preserves the class of separable absolute neighborhood retracts having the homotopy type of a compact ANR;*
3) *If $F(K) \in ANR$ for every countable locally finite simplicial complex K then the functor F preserves the class of separable absolute neighborhood retracts.*

Let $F : \mathcal{M}etr \to \mathcal{T}op$ be an embedding preserving functor. For a couple $A \subset X$ of metric spaces we identify the space $F\,A$ with the subset $F(e)(F\,A) \subset F\,X$, where $e : A \to X$ is the identity embedding. We shall say that the embedding preserving functor $F : \mathcal{M}etr \to \mathcal{T}op$ preserves

(i) *closed intersections*, provided for every collection $\{X_\alpha\}_{\alpha \in A}$ consisting of closed subsets of a metric space X we have $F(\bigcap_{\alpha\in A} X_\alpha) = \bigcap_{\alpha \in A} F(X_\alpha)$;
(ii) *preimages*, if for every map $f : X \to Y$ of metric spaces and a subset $A \subset Y$ $F(f)^{-1}(F\,A) = F(f^{-1}(A))$.

By $\exp : \mathcal{T}op \to \mathcal{T}op$ we denote the hyperspace functor. Recall [11] that $\exp X$ is the space of compact subsets of the space X, equipped with the Vietoris topology which base consists of the sets $\langle U_1, \dots, U_n \rangle = \{A \in \exp X \mid A \subset \bigcup_{i=1}^n U_i,\ A \cap U_i \neq \emptyset,\ 1 \le i \le n\}$, where $U_1, \dots, U_n$ run over the topology of X.

Let $F : \mathcal{M}etr \to \mathcal{T}op$ be a functor that preserves embeddings and closed intersections. For a point $a \in F\,X$ by $\mathrm{supp}(a) = \cap\{A \mid A$ is a closed subset of X such that $a \in F\,A\}$ the support of the point a is denoted. The functor F is defined to be a *functor with compact support*, provided the $\mathrm{supp}(a) \subset X$ is compact for every $a \in F\,X$, where X is a metric space; F is a *functor with compact continuous support*, provided the map $\mathrm{supp} : F\,X \to \exp X$ is continuous for every X.

For functors with compact continuous support the statement 3 of Theorem 1 admits the following improvement.

Proposition 3. *Let $F : \mathcal{SM}etr \to \mathcal{SM}etr$ be a functor with compact continuous support, preserving embeddings, homotopies and compacta. If $F(K) \in ANR$ for* *every finite simplicial complex then the functor F preserves the class of separable absolute neighborhood retracts.*

Proof. By Theorem 1, it is sufficient to prove that $F(K) \in \mathrm{ANR}$ for every countable locally finite simplicial complex. Fix $a \in F(K)$. We will prove that a has an open neighborhood that is an ANR (this will imply $F(K) \in \mathrm{ANR}$ [Hu]). Let L be a finite subcomplex such that L is a closed neighborhood of the compact set $\mathrm{supp}(a)$ in K. By the assumtion, $F(L) \in \mathrm{ANR}$. Moreover, since the hyperspace $\exp(L)$ is a closed neighborhood of the point $\mathrm{supp}(a)$ in $\exp(K)$ and the map $\mathrm{supp} : F(K) \to \exp(K)$ is continuous, $F(L) = \mathrm{supp}^{-1}(\exp(L))$ is a closed ANR-neighborhood of the point a in $F(K)$. Proposition is proven.

There is also the map analog of Theorem 1. We start from the map version of the concept of an A(N)E-space. A map $p : X \to Y$ of topological spaces is defined to be *(locally) soft*, provided for every metric space Z, its closed subspace $Z_0 \subset Z$ and maps $f : Z_0 \to X$, $g : Z \to Y$ with $p \circ f = g|Z_0$ there exists an extension $\bar{f} : U \to X$ of f such that $p \circ \bar{f} = g|U$, where $U = Z$ ($U \supset Z_0$ is an open set in Z) [12].

Theorem 2. *Let $F : \mathcal{SM}etr \to \mathcal{T}op$ be a functor preserving embeddings, homotopies and preimages. By $\mathrm{pr}_1 : Q \times Q \to Q$ we denote the projection onto the first factor ($Q = [-1,1]^\omega$ is the Hilbert cube).*
1) *The functor F preserves softness of maps of separable metric spaces if and only if the map $F(\mathrm{pr}_1)$ is soft.*
2) *The functor F preserves local softness of maps of separable metric spaces if and only if for every open set $U \subset Q \times Q$ the map $F(\mathrm{pr}_1 |U) : F(U) \to F(Q)$ is locally soft.*

Now we will use results from the theory of functors on the category of compacta. For a monomorphic intersection preserving functor $F : \mathcal{C}omp \to \mathcal{C}omp$ by $P_\beta : \mathcal{T}ych \to \mathcal{T}ych$ we denote the extension of F onto the category of Tychonoff spaces, offered by A. Chigogidze [14].

Recall that for a Tychonoff space X $F_\beta(X) = \{a \in F(\beta X) \mid \mathrm{supp}(a) \subset X\}$, where βX is the Stone-Čech compactification of X. If the functor F is monomorphic (preserves preimages) then the functor F_β preserves embedding (preimages) [14].

A functor $F : \mathcal{C}omp \to \mathcal{C}omp$ is defined to be continuous, provided it preserves the limits of inverse spectra [11]. By [15], if $F : \mathcal{C}omp \to \mathcal{C}omp$ is a monomorphic continuous intersection preserving functor then for every compactum X the map $j_X : F(X) \times [0,1] \to F(X \times [0,1])$ is continuous. The same is true for any monomorphic functor $F : \mathcal{C}omp \to \mathcal{C}omp$ preserving intersections, preimages, the singleton and the empty set [1]. This and Propositions 1,2 imply the following propositions stating that for some functors F_β Therems 1, 2 are applicable.

Proposition 4. *If $F : \mathcal{C}omp \to \mathcal{C}omp$ is a monomorphic continuous intersection preserv ing functor then the functor $F_\beta : \mathcal{T}ych \to \mathcal{T}ych$ preserves embedding, homotopies and compacta.*

Proposition 5. *If $F : \mathcal{C}omp \to \mathcal{C}omp$ is a monomorphic functor preserving intersections, preimages, singleton and emtpy set then the functor $F_\beta : \mathcal{T}ych \to \mathcal{T}ych$ preserves embeddings, homotopies and preimages.*

We say that $F : \mathcal{C}omp \to \mathcal{C}omp$ is a functor with finite support, provided the support $\mathrm{supp}(a)$ is finite for any compactum X and $a \in FX$. According to [5, Theorem 3], if for a normal functor $F : \mathcal{C}omp \to \mathcal{C}omp$ with finite support the functor F_β preserves the class of separable A(N)R-spaces then F_β preserves the class of separable metrizable A(N)R(n)-spaces for any $n \geq 0$.

Theorem 1, Propositions 3,4, and the known results on preserving of compact A(N)R's by concrete functors [6–9] imply

Corollary 1. *The functors* $\exp_\beta$, $\exp^c_\beta$, Γ_β, Γ^c_β, λ_β, G_β, $(N_k)_\beta$, Gr_β, $(\exp^c_n)_\beta$, $(SP^n_G)_\beta$, $(P_n)_\beta$, $(\lambda_n)_\beta$ *preserve the class of separable AR-spaces and the class of separable ANR-spaces having the homotopy type of compact ANR. The functor* $(\cdot)^\omega_\beta$ *preserves the class of separable AR-spaces and the functors* $\exp_\beta$, $\exp^c_\beta$, $(\exp_n)_\beta$ *preserve the class of separable ANRs. The functors* $(\exp_n)_\beta$, $(P_n)_\beta$, $(\lambda_n)_\beta$ *preserve the class of separable metrizable* $AE(n)$*-spaces and the functor* $(\exp_n)_\beta$ *preserves the class of separable merizable* $ANE(n)$*-spaces for every* $n \geq 0$*.*

Remark that preservation of the class of absolute (neighborhood) retracts by the functors exp and $\exp_n$ has been proven by N.T. Nhu [24].

A monomorphic intersection preserving functor $F : \mathcal{C}omp \to \mathcal{C}omp$ is called a functor of finite degree n, provided for every compactum X and $a \in FX$ $|\operatorname{supp}(a)| \leq n$. Basmanov's result [2], Theorem 1, [5, Theorem 3], and Proposition 3,4 imply

Corollary 2. *Let* $F : \mathcal{C}omp \to \mathcal{C}omp$ *be a normal functor of finite degree* n*. If* $F(\emptyset)$ *and* Fn *are finite-dimensional ANR-compacta then the functor* F_β *preserves the class of separable absolute retracts, the class of separable absolute neighborhood retracts having the homotopy type of a compact ANR, and the class of separable metrizable* $AE(n)$*-spaces for any* $n \geq 0$*. If additionally, the functor* F *has continuous support then the functor* F_β *preserves the class of separable ANR-spaces and the class of separable metrizable* $ANE(n)$*-spaces for every* $n \geq 0$*.*

Recall that $P : \mathcal{C}omp \to \mathcal{C}omp$ is the probability measure functor and $(\cdot)^\omega : \mathcal{C}omp \to \mathcal{C}omp$ is the countable power functor. It is well known [10] that for every soft map $f : X \to Y$ between metric compacta the maps $P(f) : P(X) \to P(Y)$ and $f^\omega : X^\omega \to Y^\omega$ are soft. This, Theorem 2, and Proposition 5 imply

Corollary 3. *The functors* P_β *and* $(\cdot)^\omega_\beta$ *preserve the class of soft maps between separable metric spaces.*

Proofs.

We start from some definitions.

A subset A of a topological space X is defined to be *homotopy negligible* (sf. [26, Th.2.4]) iff there exists a homotopy $H : X \times [0,1] \to X$ such that $H(X \times (0,1]) \cap A = \emptyset$ and $H(x,0) = x$ for every $x \in X$.

Let $p : X \to Y$ be a map. A subset $A \subset X$ is defined to be *fiber homotopy negligible* iff there exists a homotopy $H : X \times [0,1] \to X$ such that $H(X \times (0,1]) \cap A = \emptyset$ and $H(x,0) = x$, $p \circ H(x,t) = p(x)$ for every $x \in X$, $t \in [0,1]$.

The proofs of the main results are based on the following simple remarks:

Lemma 1. *(sf. [26, Th.3.1]).*

(i) If $X \in$ *A(N)E and* $A \subset X$ *is a set with homotopy negligible complement then* $A \in$ *A(N)E;*

(ii) if $f : X \to Y$ *is a (locally) soft map and* $A \subset X$ *is a set with fiber homotopy negligible complement then the map* $f|A : A \to Y$ *is (locally) soft.*

Lemma 2. *Let* $F : \mathcal{M}etr \to \mathcal{T}op$ *be a functor that preserves embeddings and homotopies.*

(i) If $A \subset X \in \mathcal{M}etr$ *is a subset with the homotopy negligible complement in* X *then* $F(A)$ *is a set with homotopy negligible complement in* $F(X)$*;*

(ii) if $f : X \to Y$ is a map of metric spaces and $A \subset X$ is a set with fiber homotopy negligible complement in X then the complement $F(X)\backslash F(A)$ is fiber homotopy negligible in $F(X)$ (with respect to the map $F(f)$).

For a cardinal A by $l^2_A = \{(x_a)_{a\in A} \mid x_a \in \mathbb{R},\ a \in A$ and $\sum_{a\in A} x_a^2 < \infty\}$ we denote the standard Hilbert space of density A.

Proposition 6. *Let A be a cardinal and $F : \mathcal{M}etr \to \mathcal{T}op$ be a functor that preserves embeddings, preimages and homotopies. By $\mathrm{pr}_1 : l^2_A \times l^2_A \to l^2_A$ the projection onto the first factor is denoted.*

(i) If $F(\mathrm{pr}_1)$ is a soft map then the map $F(f)$ is soft for every soft map $f : X \to Y$ of metric spaces of density $\le A$;

(ii) if $F(\mathrm{pr}_1 |U) : F(U) \to F(l^2_A)$ is a locally soft map for any open set $U \subset l^2_A \times l^2_A$ then the map $F(f)$ is locally soft for every locally soft map $f : X \to Y$ of metric spaces of density $\le A$.

Proof. We shall prove only the statement (ii). Assume that the map $F(\mathrm{pr}_1 |U) : F(U) \to F(l^2_A)$ is locally soft for every open set $U \subset l^2_A \times l^2_A$. Fix a locally soft map $f : X \to Y$ of metric spaces of density $\le A$.

Claim 1. *For every open set $U \subset Y \times l^2_A$ the map $F(\mathrm{pr}_Y |U) : F(U) \to F(Y)$ is locally soft. Here $\mathrm{pr}_Y : Y \times l^2_A \to Y$ is the projection.*

Proof. Indeed, since $\mathrm{dens}(Y) \le A$, without loss of generality, Y can be assumed to be a subset in l^2_A. Let $U \subset Y \times l^2_A$ be an open set. Pick up an open set $V \subset l^2_A \times l^2_A$ such that $V \cap (Y \times l^2_A) = U$. Note that $U = (\mathrm{pr}_1 |V)^{-1}(Y)$.

By the assumption, $F(\mathrm{pr}_1 |V) : F(V) \to F(l^2_A)$ is a locally soft map. Since the functor F preserves preimages, $F(U) = F(\mathrm{pr}_1 |V)^{-1}(F(Y))$. Since the map $F(\mathrm{pr}_1 |V)$ is locally soft, so is the map $F(\mathrm{pr}_1 |V)|F(U) = F(\mathrm{pr}_1 |U) : F(U) \to F(Y)$. The observation that the maps $\mathrm{pr}_Y |U$ and $\mathrm{pr}_1 |U$ are homeomorphic completes the proof.

Claim 2. *For every normed space L of density A and every open set $U \subset Y \times L$ the map $F(\mathrm{pr}_Y |U) : F(U) \to F(Y)$ is locally soft.*

Proof. Let $\widehat{L}$ be the completion of L. By [27], $\widehat{L}$ being a Banach space of density A is homeomorphic to l^2_A. It is well known that the complement $\widehat{L}\backslash L$ is homotopy negligible in $\widehat{L}$, i.e. there exists a homotopy $H : \widehat{L} \times [0,1] \to \widehat{L}$ such that $H(\widehat{L} \times (0,1]) \subset L$ and $H(l,0) = l$ for $l \in \widehat{L}$. Define the homotopy $H_Y : Y \times \widehat{L} \times [0,1] \to Y \times \widehat{L}$ by $H_Y(y,l,t) = (y,H(l,t))$.

Let $U \subset Y\times L$ be an open set and let $\widehat{U} = (Y\times\widehat{L})\backslash \mathrm{cl}_{Y\times\widehat{L}}((Y\times L)\backslash U)$. Obviously, $U = \widehat{U} \cap (Y \times L)$.

Let $W = H_Y^{-1}(\widehat{U}) \subset Y\times\widehat{L}\times[0,1]$. Evidently that W is an open set in $Y\times\widehat{L}\times[0,1]$ and $\widehat{U}\times\{0\} \subset W$. It is not difficult to construct a continuous function $\mu : \widehat{U} \to (0,1]$ such that $\{(u,t) \in \widehat{U}\times[0,1] : t \le \mu(u)\} \subset W$. Define the homotopy $\widehat{H} : \widehat{U}\times[0,1] \to \widehat{U}$ by $\widehat{H}(u,t) = H_Y(u,\mu(u)t)$. Obviously $\widehat{H}(\widehat{U} \times (0,1]) \subset \widehat{U} \cap (Y \times L) = U$, $\widehat{H}(u,0) = u$ and $\mathrm{pr}_Y \circ\widehat{H}(u,t) = \mathrm{pr}_Y(u)$ for $u \in \widehat{U}$ and $t \in [0,1]$. Hence the complement $\widehat{U}\backslash U$ is fiber homotopy negligible in $\widehat{U}$. Note that $\widehat{U}$ is an open set in $Y \times \widehat{L}$. By Claim 1, the map $F(\mathrm{pr}_Y |\widehat{U})$ is locally soft (recall that $\mathrm{pr}_Y : Y \times \widehat{L} \to Y$ is the natural projection). Since the complement $\widehat{U}\backslash U$ is fiber homotopy negligible, Lemma 2 yields that the complement $F(\widehat{U})\backslash F(U)$ is fiber homotopy negligible with respect to the map $F(\mathrm{pr}_Y)$. By Lemma 1, the map $F(\mathrm{pr}_Y |U) : F(U) \to F(Y)$ is locally soft.

Now, we can complete the proof of Proposition 6.

By [19, Corollary II.1.1], there exists a normed space L and a closed embedding $e : X \to L$. Since $\mathrm{dens}(X) \leq A$, we may assume $\mathrm{dens}(L) = A$.

Define the closed embedding $i : X \to Y \times L$ by $i(x) = (f(x), e(x))$ (recall that $f : X \to Y$ is the given locally soft map). Further, we shall identify X with its image $i(X) \subset Y \times L$. Since the map $f : X \to Y$ is locally soft, there exists an open neighborhood $U \subset Y \times L$ of X and a retraction $r : U \to X$ with $\mathrm{pr}_Y \circ r = \mathrm{pr}_Y |U$. This retraction induces the retraction $F(r) : F(U) \to F(X)$ such that $F(\mathrm{pr}_Y) \circ F(r) = F(\mathrm{pr}_Y |U)$. By Claim 2, the map $F(\mathrm{pr}_Y |U) : F(U) \to F(X)$ is locally soft. Then the map $F(f) = F(\mathrm{pr}_Y |X)$, being a fiber retract of $F(\mathrm{pr}_Y |U)$, is locally soft. Proposition 6 is proved.

Proof of Theorem 2. Again, we shall prove only the statement 2. Recall that $\mathrm{pr}_1 : Q \times Q \to Q$ is the projection. Assume that for every open subset $U \subset Q \times Q$ the map $F(\mathrm{pr}_1 |U) : F(U) \to F(Q)$ is locally soft. Since the pseudointerior $s = (-1, 1)^\omega$ has the homotopy negligible complement in the Hilbert cube $Q = [-1, 1]^\omega$, by analogy with the above proof, one can prove that for every $Y \subset Q$ and every open set $U \subset Y \times s$ the map $F(\mathrm{pr}_Y |U) : F(U) \to F(Y)$ is locally soft (here $\mathrm{pr}_Y : Y \times s \to Y$ is the projection). Now, note that the pseudointerior s is homeomorphic to the separable Hilbert space l_2 [16]. Consequently, for every open subset $U \subset l_2 \times l_2$ the map $F(\mathrm{pr}_1 |U) : F(U) \to F(l_2)$ is locally soft. By Proposition 6, the map $F(f) : F(X) \to F(Y)$ is locally soft for every locally soft map $f : X \to Y$ between separable metric spaces.

By analogy with the above proofs, one can prove

Proposition 7. *Let A be a cardinal and $F : \mathcal{M}etr \to \mathcal{T}op$ be an embedding and homotopy preserving functor.*
(1) If $F(l^2_A) \in AE$ then $F(X) \in AE$ for every absolute retract X of density $\leq A$.
(2) If $F(U) \in ANE$ for every open set $U \subset l^2_A$ then $F(X) \in ANE$ for every absolute neighborhood retract X of density $\leq A$.

Proposition 8. *Let $F : \mathcal{S}Metr \to \mathcal{T}op$ be an embedding and homotopy preserving functor.*
(1) If $F(Q) \in AE$ then $F(X) \in AE$ for every separable absolute retract X.
(2) If $F(U) \in ANE$ for every open set $U \subset Q$ then $F(X) \in ANE$ for every separable absolute neighborhood retract X.

Proof of Theorem 1. Assume that $F : \mathcal{S}Metr \to \mathcal{S}Metr$ is a functor preserving embeddings, homotopies and compacta and such that $F(K) \in \mathrm{ANR}$ for every (countable locally) finite simplicial complex K.

At first, we shall prove that $F(K \times Q) \in \mathrm{ANR}$ for every (countable locally) finite simplicial complex K. For, fix such a complex K and note that the space $K \times Q$ is the limit of the inverse sequence

$$\cdots \to K \times I^3 \xrightarrow{r_3} K \times I^2 \xrightarrow{r_2} K \times I \xrightarrow{r_1} K,$$

where $r_n(k, t_1, \dots, t_n) = (k, t_1, \dots, t_{n-1})$ for $(k, t_1, \dots, t_n) \in K \times I^n$, $n \in \mathbb{N}$ (recall that $I = [0, 1]$ is the segment). By $r^n : K \times Q \to K \times I^n$ we denote the limit projections. The maps $F(r^n) : F(K \times Q) \to F(K \times I^n)$ induce the map $R : F(K \times Q) \to \varprojlim F(K \times I^n)$. Here $\varprojlim F(K \times I^n)$ is the limit of the inverse sequence

$$\cdots \to F(K \times I^3) \xrightarrow{F(r_3)} F(K \times I^2) \xrightarrow{F(r_2)} F(K \times I) \xrightarrow{F(r_1)} F(K).$$

Claim 3. *The map $R : F(K \times Q) \to \varprojlim F(K \times I^n)$ is an embedding such that the image $R(F(K \times Q))$ has the homotopy negligible complement in $\varprojlim F(K \times I^n)$.*

Proof. Denote by αK the Aleksandrov compactification of K. The above reasonings allow us to construct the map $R_\alpha : F(\alpha K \times Q) \to \varprojlim F(\alpha K \times I^n)$. Moreover, since the functor F preserves embeddings, $R = R_\alpha | F(K \times Q)$. Therefore, to prove that R is an embedding, it is sufficiently to prove that the map R_α is an embedding. Since the functor F preserves compacta, $R_\alpha : F(\alpha K \times Q) \to \varprojlim F(\alpha K \times I^n)$ is a map between compacta. Hence, injectiveness of R_α will imply that R_α is an embedding.

Denote by $i^n : \alpha K \times I^n \to \alpha K \times Q$ the embedding defined by $i^n(k, t_1, \dots, t_n) = (k, t_1, \dots, t_n, 0, 0, \dots) \in K \times Q$, $n \in \mathbb{N}$. Obviously that $r^n \circ i^n = \mathrm{id} | \alpha K \times I^n$, $n \in \mathbb{N}$.

By $\omega_0 = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ we denote the convergent sequence. Define the map $\lambda : \alpha K \times Q \times \omega_0 \to \alpha K \times Q$ by the formula

$$\lambda(a, q, t) = \begin{cases} (a, q), & \text{if} \quad t = 0; \\ i^n \circ r^n(a, q), & \text{if} \quad t = \frac{1}{n},\ n \in \mathbb{N}. \end{cases}$$

It is easily verified that the map λ is continuous. Since the functor F preserves homotopies, the map $\Lambda = F(\lambda) \circ j_{\alpha K \times Q} | F(\alpha K \times Q) \times \omega_0 : F(\alpha K \times Q) \times \omega_0 \to F(\alpha K \times Q)$ is continuous.

Now fix $a, b \in F(\alpha K \times Q)$ be two distinct points. Since the map $\Lambda : F(\alpha K \times Q) \times \omega_0 \to F(\alpha K \times Q)$ is continuous and $\Lambda(a, 0) = a \neq b = \Lambda(b, o)$, there exists $n \in \mathbb{N}$ such that $\Lambda(a, \frac{1}{n}) \neq \Lambda(b, \frac{1}{n})$. But $\Lambda(a, \frac{1}{n}) = F(i^n) \circ F(r^n)(a)$ and $\Lambda(b, \frac{1}{n}) = F(i^n) \circ F(r^n)(b)$. Hence $F(i^n) \circ F(r^n)(a) \neq F(i^n) \circ F(r^n)(b)$. This implies $F(r^n)(a) \neq F(r^n)(b)$ and $R_\alpha(a) = \varprojlim F(r^n)(a) \neq R_\alpha(b)$, i.e., the map $R_\alpha : F(\alpha K \times Q) \to \varprojlim F(\alpha K \times I^n)$ is injective. Consequently, the map $R : F(K \times Q) \to \varprojlim F(K \times I^n) \subset \varprojlim F(\alpha K \times I^n)$ is an embedding.

Now let us show that the image $R(F(K \times Q))$ has the homotopy negligible complement in $\varprojlim F(K \times I^n)$. Denote by $R^n : \varprojlim F(K \times I^n) \to F(K \times I^n)$, $n \in \mathbb{N}$ the limit projections. For every $n \in \mathbb{N}$ consider the homotopy $h_n : K \times I^n \times [0,1] \to K \times I^n$ defined by $h_n(k, t_1, \dots, t_n, t) = (k, t_1, \dots, t_{n-1}, t \cdot t_n)$. Since the functor F preserves homotopies, the induced homotopy $H_n : F(K \times I^n) \times [0,1] \to F(K \times I^n)$ is continuous. Now consider the homotopy $H : \varprojlim F(K \times I^n) \times [0,1] \to \varprojlim F(K \times I^n)$ defined by the formula

$$H(a, t) = \begin{cases} a, & \text{if} \quad t = 0; \\ R \circ F(i^n) \circ H_n(R^n(a), 2^{n+1}t - 1), & \text{if} \quad 2^{-(n+1)} \le t \le 2^{-n}. \end{cases}$$

(Recall that $i^n : K \times I^n \to K \times Q$ is an embedding, right inverse to the retraction $r^n : K \times Q \to K \times I^n$). One can readily verify that the homotopy H is continuous. By the construction, $H(a, 0) = a$ for every $a \in \varprojlim F(K \times I^n)$ and $H(\varprojlim F(K \times I^n) \times (0, 1]) \subset R(F(K \times Q))$. Hence, the complement of the set $R(F(K \times Q))$ is homotopy negligible in $\varprojlim F(K \times I^n)$.

A map $r : X \to Y$ is defined to be a *strait deformation retraction*, provided there exist a map $i : Y \to X$ and a homotopy $H : X \times [0,1] \to X$ such that $r \circ i = \mathrm{id} | Y$ and $r \circ H(x, t) = r(x)$, $H(x, 0) = x$, $H(x, 1) = i \circ r(x)$ for every $(x, t) \in X \times [0,1]$. By [20, Corollary 3.4], the limit of an inverse sequence of ANR-spaces is an ANR, provided the bonding maps are strait deformation retractions. It is easily seen that homotopy preserving functors preserve strait homotopy retractions. Since the bonding maps $r_n : K \times I^n \to K \times I^{n-1}$ are strait deformation retractions, so are the maps $F(r_n) : F(F \times I^n) \to F(K \times I^{n-1})$. By the assumption, each space $F(K \times I^n)$

is an absolute neighborhood retract. Therefore, the inverse limit $\varprojlim F(K \times I^n)$ is an ANR. Since the set $R(F(K \times Q))$ has the homotopy negligible complement in $\varprojlim F(K \times I^n) \in$ ANR, by Lemma 1, $F(K \times Q) \in$ ANR. Since the set $K \times s$ has the homotopy negligible complement in $K \times Q$, by Lemmas 1, 2, $F(K \times s) \in$ ANR.

If $F(I^n) \in$ AR for every $n \in \mathbb{N}$ then, slightly modifying the above proof, we obtain $F(Q) \in$ AR. By Proposition 8, $F(X) \in$ AR for every separable absolute retract X.

To prove the statement 2 of Theorem 1, fix a separable ANR X having the homotopy type of compact ANR. By [13, 44.2], every compact ANR is homotopy equivalent to a finite simplicial complex. Therefore, there exists a finite simplicial complex K homotopy equivalent to X. By [26], there exists a complete-metrizable ANR $\widetilde{X} \supset X$ containing X such that the complement $\widetilde{X} \backslash X$ is homotopy negligible in $\widetilde{X}$. Then the complete-metrizable ANR $\widetilde{X}$ is homotopy equivalent to the polyhedron K. By [22,27], the s-manifolds $\widetilde{X} \times s$ and $K \times s$ are homeomorphic. Since $F(K \times s) \in$ ANR, this implies $F(\widetilde{X} \times s) \in$ ANR and, consequently, $F(\widetilde{X}) \in$ ANR (note that $\widetilde{X}$ is a retract of $\widetilde{X} \times s$). Since the set $X \subset \widetilde{X}$ has the homotopy negligible complement in $\widetilde{X}$, by Lemmas 1, 2, $F(X) \in$ ANR.

The proof of the statement 3 can be worked out by analogy with the above proof (use the fact that each separable ANR is homotopy equivalent to a countable locally finite simplicial complex [23]).

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