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UNIVERSAL MAPS AND ABSORBING SETS IN THE CLASSES OF COUNTABLE-DIMENSIONAL SPACES

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It is proved that there exists a map $f : X \to Q$ of countable-dimensional (c.d.) space $X \in \mathcal{A}_2$ which is soft in the class of c.d. spaces (i.e. each partial section of f parametrized by a closed subset of c.d. space can be extended over the whole space). The result is applied to constructing of absorbing sets in the classes of absolute Borelian and projective c.d. spaces. Some results are extended onto non-separable case and onto the case of pairs of spaces.

Throughout the paper, all the spaces are metrizable and all the maps are continuous. A space is said to be countable-dimensional (briefly, c.d.) if it can be represented as a countable union of its finite-dimensional subspaces. J. Nagata [13] has constructed a universal space $N_\omega = \{(x_i)_{i=1}^\infty \in Q \mid \text{only finite number of } x_i \text{ is rational}\}$ for separable c.d. spaces (Q denotes the Hilbert cube $[0,1]^\omega$). A brief survey of results concerning universal c.d. spaces can be found in [10]; see also [14].

It is proved in this paper that there exists a separable c.d. space X and a map $f : X \to Q$ which is universal for the class of maps with c.d. domain. Moreover, the map f is soft in the class of c.d. spaces, i.e., it satisfies the property of extension of sections parametrized by c.d. spaces (see the definition of c.d. soft map below).

The mentioned space X is of the absolute Borelian class $\mathcal{A}_2$ (i.e., it is an absolute $G_{\delta\sigma}$-space), as well as the Nagata space N_ω. Using the property of c.d. softness of f we can construct c.d. spaces of absolute Borelian classes $\mathcal{A}_\alpha$, $\alpha \geq 2$, $\mathcal{M}_\alpha$, $\alpha \geq 3$ and of absolute projective classes $\mathcal{P}_i$, $i \geq 1$ that are absolute retracts (ARs) and are universal for the respective classes. Further, these universal spaces can be transformed, via the standard construction, into absorbing sets in a copy of the Hilbert space l_2. Applying the powerful characterization results of the theory of absorbing sets [2] we can obtain topological characterizations of some universal c.d. spaces.

Note that our results may be considered as a (necessarily, partial) answer to the following question 3.1 (or 549) of [8]: *find more absorbing sets.*

In the last section, formally independent from the rest of the paper, we deal with strongly universal pairs and we use results of [5] and [14] in order to prove

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existence of pairs of normed spaces which are strongly universal in the class of pairs (K, L) where L is a countable-dimensional subspace of given absolute Borelian or projective class in a complete separable space K.

1. C.D.-INVERTIBLE AND C.D.-SOFT MAPS.

A map $f : X \to Y$ is said to be *c.d.-invertible* if for every c.d. space Z and every map $g : Z \to Y$ there exists a map $h : Z \to X$ such that $g = fh$.

A map $f : x \to Y$ is said to be *c.d.-soft* if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ \big\downarrow & & \big\downarrow f \\ Z & \xrightarrow[\psi]{} & Y \end{array}$$

where A is a closed subset of a c.d. space Z there exists a map $\Phi : Z \to X$ such that $\Phi \mid A = \varphi$ and $f\Phi = \psi$.

Note that the notion of c.d.-soft map is a natural counterpart of the notion of soft map introduced by E. V. Shchepin [15].

The proof of the following Lemma 1 is based on a factorization theorem due to B. A. Pasynkov. We do not formulate this theorem in the plain generality.

Theorem (see [1]). *Let $f : X \to Z$ be a map of a c.d. space X onto a separable space Z. Then there exists a separable c.d. space Y and maps $g : X \to Y$, $h : Y \to Z$ such that $f = gh$.*□

Lemma 1. *There exists a c.d.-invertible map $f : X \to Q$ where X is a separable c.d. space.*

Proof. There exists a set $\{f_\alpha : X_\alpha \to Q \mid \alpha \in \Gamma\}$ of maps of separable c.d. spaces into Q with the following property: for each map $g : X \to Q$ of separable c.d. space Y there exists an $\alpha \in \Gamma$ and a homeomorphism $h : Y \to X_\alpha$ such that $g = f_\alpha h$. Let $X' = \coprod\{X_\alpha \mid \alpha \in \Gamma\}$. By the factorization theorem, the map $f' : X' \to Q$ coinciding with f_α on each X_α admits a factorization $f' = fg$ where $f : X \to Q$ is a map of a separable c.d. space X. In order to prove that f is a required map, consider a map $f'' : X'' \to Q$ of a c.d. space X''. Let $f'' = g'h'$ be a factorization of f'' where $g' : Y' \to Q$ is a map of a separable c.d. space Y'. By the construction of X', there exists an $\alpha \in \Gamma$ and a homeomorphism $h : Y' \to X_\alpha$ such that $f_\alpha h = g'$. Then letting $g'' = ghh' : X'' \to X$ we obtain $fg'' = f''$.□

Recall the definition of the absolute Borelian additive (multiplicative) classes $\mathcal{A}_\alpha$ $(\mathcal{M}_\alpha)$, $0 \le \alpha < \omega_1$ and absolute projective classes $\mathcal{P}_i$, $i < \omega$ (see, e.g. [12] for more details). Let X be a space. Denote by $\mathcal{A}_0(X)$ the family of all open subsets of X and by $\mathcal{M}_0(X)$ the family of all closed subsets of X. Suppose that for each countable ordinal $\beta < \alpha$ the families $\mathcal{A}_\beta(X)$ and $\mathcal{M}_\beta(X)$ have been defined. Then let $\mathcal{A}_\alpha(X) = \cup_{i=1}^{\infty} X_i$ where $X_i \in \cup_{\beta<\alpha}\mathcal{M}_\beta(X)$ and let $\mathcal{M}_\alpha(X) = \cap_{i=1}^{\infty} X_i$ where $X_i \in \cup_{\beta<\alpha}\mathcal{A}_\beta(X)$.

For a complete separable X let $\mathcal{P}_0(X) = \cup_{\alpha<\omega_1}\mathcal{A}_\alpha(X)$. Let $\mathcal{P}_{2i+1}(X)$ be the family of subsets of X which are the images of members of $\mathcal{P}_{2i}(X)$ and let $\mathcal{P}_{2i}(X) = \{A \subset X \mid X \backslash A \in \mathcal{P}_{2i-1}(X)\}$.

Finally, let $\mathcal{C}$ be one of the symbols $\mathcal{A}_\alpha$, $\mathcal{M}_\alpha$ or $\mathcal{P}_i$. Then the absolute class $\mathcal{C}$ consists of all spaces E such that for every homeomorphic embedding $h : E \to X$ into a complete space X we have $h(E) \in \mathcal{C}(X)$.

Lemma 2. *There exists a c.d. invertible map $f : X \to Q$ where $X \in \mathcal{A}_2$ is a separable c.d. space.*

Proof. By Lemma 1, there exists a c.d.-invertible map $\widetilde{f} : \widetilde{X} \to Q$ of a separable c.d. space $\widetilde{X}$. Embed $\widetilde{X}$ into Nagata universal space N_ω. By the classical Lavrentiev theorem, $\widetilde{f}$ can be extended to a map $f : X \to Q$ where X is a G_δ-subset of N_ω that includes $\widetilde{X}$. Obviously, f is c.d.-invertible and $X \in \mathcal{A}_2$. □

Lemma 3. *For each separable c.d. space X there exists a separable c.d. AR-space Y containing X as a closed subset. Moreover, we can take $Y \in \mathcal{A}_2$ whenever $X \in \mathcal{A}_2$.*

Proof. M. Bestvina and J. Mogilski [3] have described an embedding of a space X as a linearly independent subset of a normed space such that the span $L(X)$ of X is c.d. whenever X is c.d. It is also remarked in [3] that $L(X) \in \mathcal{A}_2$ whenever $X \in \mathcal{A}_2$. Therefore, $L(X)$ is a required AR-space. □

Lemma 4. *Let B be a closed subset of a separable space $Y \in \mathcal{A}_2$. There exists a c.d.-invertible map $g : A \in Q$ such that A is a separable c.d. space, $A \in \mathcal{A}_2$ and the map $g|g^{-1}(B) : g^{-1}(B) \to B$ is a homeomorphism.*

Proof. Embed Y into Q and let $A' = f^{-1}(Y)$, $g' = f|A' : A' \to Y$ where $f : X \to Q$ is a c.d.-invertible map of a separable c.d. space $X \in \mathcal{A}_2$ (see Lemma 2).

Now embed the map g' into a map $u : K \to L$ of compact spaces. Let $\mathcal{G}$ be the decomposition of K whose nontrivial elements are of the form $u^{-1}(b)$, $b \in \overline{B}$. The map u admits the factorization $u = u'q$ where $q : K \to K/\mathcal{G}$ is the quotient map. It is easy to see that we can take $A = q(A')$, $g = u'|A : A \to B$. □

Now we are going to prove existence of c.d.-soft map onto Q with a separable c.d. domain.

Theorem 1. *There exists a c.d.-soft map $f : X \to Q$ where X is a separable c.d. space and $X \in \mathcal{A}_2$.*

Proof. Let $q_1 : \widetilde{X}_1 \to Q = Z_0$ be a c.d.-invertible map of a separable c.d. space $\widetilde{X}_1 \in \mathcal{A}_2$. By Lemma 3, there exists a separable c.d. AR-space $Y_1 \in \mathcal{A}_2$ and a closed embedding $\widetilde{X}_1 \hookrightarrow Q \times Y_1$ such that $q_1 = p_0^1|\widetilde{X}_1$ (here p_0^1 denotes the projection of the product $Q \times Y_1$ onto the first factor).

By Lemma 4, there exists a c.d.-invertible map $q_2 : \widetilde{X}_2 \to Q \times Y_1 = Z_1$ such that $\widetilde{X}_2$ is a separable c.d. space, $\widetilde{X}_2 \in \mathcal{A}_2$, and the restriction $q_2|q_2^{-1}(\widetilde{X}_1) \to \widetilde{X}_1$ is a homeomorphism. Let $s_1 : \widetilde{X}_1 \to \widetilde{X}_2$ be a map such that $q_2 s_1 = 1_{\widetilde{X}_1}$.

Proceeding similarly we obtain a commutative diagram

$$\begin{array}{ccccccc}
Z_0 & \xleftarrow{p_0^1} & Z_1 & \xleftarrow{p_1^2} & Z_2 & \xleftarrow{p_2^3} & \dots \\
{\scriptstyle q_1}\uparrow & & {\scriptstyle q_2}\uparrow & & {\scriptstyle q_3}\uparrow & & \\
\widetilde{X}_1 & \xrightarrow[s_1]{} & \widetilde{X}_2 & \xrightarrow[s_2]{} & \widetilde{X}_3 & \xrightarrow[s_3]{} & \dots
\end{array}$$

where $Z_i = Q \times Y_1 \times \dots \times Y_i$, Y_i are separable c.d. AR-spaces, $Y_i \in \mathcal{A}_2$, $p_i^{i+1} : Z_{i+1} \to Z_i$ are the projections, $\widetilde{X}_i$ are closed c.d. subspaces of Z_i, q_i are c.d.-invertible maps such that the restrictions $q_i|q_i^{-1}(\widetilde{X}_{i-1}) : q_i^{-1}(\widetilde{X}_{i-1}) \to \widetilde{X}_{i-1}$ are maps such that $q_{i+1}s_i = 1_{\widetilde{X}_i}$.

Consider the inverse system $\mathcal{S} = \{Z_i, p_i^{i+1}\}$ and let $p_0 : \varprojlim \mathcal{S} \to Z_0 = Q$ be the limit projection. Let $X_i = \{(x_j)_{j=1}^{\infty} \mid x_j \in \widetilde{X}_j$ and $x_{j+1} = s_j(x_j)$ for every $j \geq i\} \subset \varprojlim \mathcal{S}$, $X = \cup_{i=0}^{\infty} X_i$, $f = p_0|X : X \to Z_0 = Q$. Obviously, X is a separable c.d. space, $X \in \mathcal{A}_2$, and X_i are closed subspaces of X.

We are going to show that the map f is c.d. soft. Given a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow f \\ D & \xrightarrow[\gamma]{} & Z_0 \end{array}$$

where C is a closed subset of a c.d. space D, construct a map $\beta : D \to X$ such that $\beta|C = \alpha$ and $f\beta = \gamma$. Let $\alpha = (\alpha_i)_{i=0}^{\infty}$ where $\alpha_i : C \to Z_i$ are the coordinate maps. Let $C_i = \alpha^{-1}(X_i)$ and let $D_1 \subset D_2 \subset \dots$ be a sequence of closed subsets of D such that $D_i \cap C = C_i$ and $D = \cup_{i=1}^{\infty} D_i$.

We shall construct the map $\beta : D \to X$ in the form $\beta = (\beta_j)_{j=1}^{\infty}$ where $\beta_j : D \to Z_j$ are the coordinate maps. Let $\beta_0 = \gamma$ and let $\beta_1 : D \to Z_1$ be a map such that $\beta_1|C = \alpha_1$ and $p_0^1\beta_1 = \beta_0 = \gamma$ (existence of such a map β_1 can be easily deduced from the fact that $Y_1 \in AR$). Since the map q_2 is c.d.-invertible, there exists a map $g_1 : D_1 \to \widetilde{X}_2$ such that $q_2 g_1 = \beta_1|D_1$. Note that $g_1|C_1 = s_1|C_1 = \alpha_2|C_1$ and, therefore, the map $\alpha_2 \cup g_1 : C \cup D_1 \to \widetilde{X}_2$ is well-defined. Since $Y_2 \in AR$, there exists a map $\beta_2 : D \to Z_2$ such that $p_1^2\beta_2 = \beta_1$ and $\beta_2|(C \cup D_1) = \alpha_2 \cup g_1$. Assume that for every $j \leq k$ a map $\beta_j : D \to Z_j$ such that $\beta_j|C = \alpha_j$ and $p_{j-1}^j \beta_j = \beta_{j-1}$ is defined. Since the map q_{k+1} is c.d.-invertible, there exists a map $g_k : D_k \to \widetilde{X}_{k+1}$ such that $q_{k+1}g_k = \beta_k|D_k$. Since $g_k|C_k = s_k|C_k = \alpha_{k+1}|C_1$, the map $\alpha_{k+1} \cup g_k : C \cup D_k \to \widetilde{X}_{k+1}$ is well-defined. Since $Y_{k+1} \in AR$, there exists a map $\beta_{k+1} : D \to Z_{k+1}$ such that $p_k^{k+1}\beta_{k+1} = \beta_k$ and $\beta_{k+1}|(C \cup D_k) = \alpha_{k+1} \cup g_k$. It is easy to see that the map $\beta = (\beta_j)_{j=0}^{\infty} : D \to X$ is well-defined and $\beta|C = \alpha$, $f\beta = \gamma$. □

Let $\mathcal{C}$ be a class of spaces. A map $f : X \to Y$ is said to be *fibrewise* $\mathcal{C}$*−universal* if for every maps $g : C \to D$, $j : D \to Y$, where $C \in \mathcal{C}$, there exists a closed embedding $i : C \to X$ such that $fi = jg$.

By $\mathcal{C}(c.d.)$ we denote the subclass of $\mathcal{C}$ consisting of all the separable c.d. elements of $\mathcal{C}$.

Theorem 2. *Let $\mathcal{C}$ be one of the classes $\mathcal{A}_\alpha$, $2 \leq \alpha < \omega_1$, $\mathcal{M}_\alpha$, $3 \leq \alpha < \omega_1$ or $\mathcal{P}_i$, $i < \omega$. There exists a fibrewise $\mathcal{C}(c.d.)$-universal map $f : X \to Q$ where $X \in \mathcal{C}(c.d.)$.*

Proof. Let $\widetilde{f} : \widetilde{X} \to Q$ be a c.d.-soft map where $\widetilde{X}$ is a separable c.d. space, $\widetilde{X} \in \mathcal{A}_2$. There exists a separable space $Y \in \mathcal{C}$ containing a closed copy of every separable $Z \in \mathcal{C}$ (see, e.g., [2 ,4]). Embed $\widetilde{X} \times Y$ into Q, take $X = \widetilde{f}^{-1}(\widetilde{X} \times Y)$, and denote by f the composition

$$X \xrightarrow{\widetilde{f}|X} \widetilde{X} \times Y \xrightarrow{pr_1} \widetilde{X} \xrightarrow{\widetilde{f}} Q.$$

Let $g : C \to D$ and $j : D \to Q$ be maps where $C \in \mathcal{C}(c.d.)$. Take a closed embedding $k : C \to Y$. By c.d.-invertibility of $\widetilde{f}$, there exists a map $i' : C \to \widetilde{X}$ such that $\widetilde{f}i' = jg$ and there exists a map $i : C \to X$ such that $(\widetilde{f}|X)i = (i', k)$. Obviously, i is a closed embedding. □

It would be interesting to compare Theorems 1 and 2 with the corresponding finite-dimensional results of A. Dranishnikov [9] and A. Chigogidze [6].

2. Absorbing sets for classes of c.d. spaces.

We recall briefly some necessary notions of the theory of absorbing sets; see [2] for details. Let $\mathcal{C}$ be a class of spaces satisfying the conditions: (a) $\mathcal{C}$ is topological, i.e., it contains all the homeomorphic images of its elements; (b) $\mathcal{C}$ is additive, i.e., $C \cup D \in \mathcal{C}$ whenever $C, D \in \mathcal{C}$; (c) $\mathcal{C}$ is hereditary with respect to closed subsets.

As usual, by ANR we denote the class of absolute neighbourhood retracts. Two maps $f, g : X \to Y$ are said to be $\mathcal{U}$-close, where $\mathcal{U}$ is a cover of Y, if for each $x \in X$ the set $\{f(x), g(x)\}$ is contained in an element of $\mathcal{U}$. A homotopy $h : X \times I \to Y$ is called $\mathcal{U}$-homotopy provided that for each $x \in X$ the set $h(\{x\} \times I)$ is contained in some $U \in \mathcal{U}$.

A closed subset X of $Y \in$ANR is called (strong) Z-set if for every open cover $\mathcal{U}$ of Y there exists a map $f : Y \to Y$ which is $\mathcal{U}$-close to 1_Y and $f(Y) \cap X = \emptyset$ (respectively, $\overline{f(Y)} \cap X = \emptyset$). An embedding into Y is called Z-embedding if its image is a Z-set in Y.

A space $X \in$ANR is called strongly $\mathcal{C}$-universal if for every map $f : C \to X$ of a space $C \in \mathcal{C}$ and for every closed subset $D \subset C$ such that $f|d : D \to X$ is a Z-embedding and for every open cover $\mathcal{U}$ of X there exists a Z-embedding $g : C \to X$ which is $\mathcal{U}$-close to f and such that $g|D = f|D$.

A subset A of X is called locally homotopy negligible (in X) [16] if for every open subset U of X the inclusion $U \backslash A \hookrightarrow U$ is a weak homotopy equivalence, i.e. it induces an isomorphism of all homotopy groups.

Finally, a subset X of a copy E of the Hilbert space l_2 is called a $\mathcal{C}$-absorbing set (in E), if $E \backslash X$ is locally homotopy negligible in E, $X = \cup_{i=1}^{\infty} X_i$ where each X_i is a Z-set in X, $X_i \in \mathcal{C}$, and X is strongly $\mathcal{C}$-universal.

Lemma 5. *Let $f : X \to Q$ be a c.d.-soft map where X is a c.d. space. Then for every $Y \in$A(N)R we have $f^{-1}(Y) \in$A(N)R.*

Proof. We consider only the case $Y \in$AR. There exists a c.d. space $Z \in$AR containing $f^{-1}(Y)$ as a closed subset (see Lemma 3). Let $g : Z \to Y$ be an extension of the map $f|f^{-1}(Y) : f^{-1}(Y) \to Y$. By c.d.-softness of f, there exists a map $r : Z \to X$ such that $r|f^{-1}(Y) = 1_{f^{-1}(Y)}$ and $fr = g$. In particular, r is a retraction of Z onto $f^{-1}(Y)$ and therefore $f^{-1}(Y) \in$AR. □

Theorem 3. *Let $\mathcal{C}$ be one of the classes $\mathcal{A}_\alpha$, $2 \le \alpha < \omega_1$, $\mathcal{M}_\alpha$, $3 \le \alpha < \omega_1$ or $\mathcal{P}_i$, $i < \omega$. There exists an AR-space $X \in \mathcal{C}(c.d.)$ containing a closed copy of each space $C \in \mathcal{C}(c.d.)$.*

Proof. By Theorem 2, there exists a fibrewise $\mathcal{C}(c.d.)$-universal map $f : X \to Q$ where $X \in \mathcal{C}(c.d.)$. Note that, by Lemma 5, $X \in$AR. In order to show that X is a required space consider, for each $C \in \mathcal{C}$, the constant map $C \to \{*\}$ where $* \in Q$. By Theorem 2 there exists a closed embedding $i : C \to X$. □

Let $\mathcal{C}$ and X be those from the statement of Theorem 3. We will use the following denotations for the weak product $W(X,*) = \{(x_i)_{i=1}^{\infty} | x_i = * \text{ for all but finitely many } i\} \subset X^{\omega}$ ($*$ is an arbitrary point of X):

$\widehat{\Lambda}_{\alpha}$, whenever $\mathcal{C} = \mathcal{A}_{\alpha}$,

$\widehat{\Omega}_{\alpha}$, whenever $\mathcal{C} = \mathcal{M}_{\alpha}$,

$\widehat{\Pi}_i$, whenever $\mathcal{C} = \mathcal{P}_i$.

The above denotations correspond to these of [2,4] for absorbing sets in absolute Borelian and projective classes.

Theorem 4. *The space $\widehat{\Lambda}_{\alpha}$, $2 \le \alpha < \omega_1$ (respectively, $\widehat{\Omega}_{\alpha}$, $3 \le \alpha < \omega_1$; $\widehat{\Pi}_i$, $i < \omega$) is $\mathcal{A}_{\alpha}$(c.d.)-(respectively, $\mathcal{M}_{\alpha}$(c.d.)-, $\mathcal{P}_i$(c.d.)$-$)absorbing set.*

Proof. This is a consequence of a general result of [2]. The necessary background is given by Theorem 3. □

Simple arguments show that there exists no $\mathcal{C}(c.d.)$-absorbing sets whenever $\mathcal{C} \in \{\mathcal{A}_1,$
$\mathcal{M}_0, \mathcal{M}_1\}$. By similarity, we consider only the case $\mathcal{C} = \mathcal{A}_1$.

Assume that there exists $\mathcal{A}_1(c.d.)$-universal space $Y = \cup_{i=1}^{\infty} Y_i$ where Y_i are compact c.d. spaces. It is well known [1], any compact space K is c.d. if and only if the small transfinite dimension $\operatorname{ind} K$ is defined. Let $\operatorname{ind} Y_i = \alpha_i$ and $\alpha = \sup\{\alpha_i \mid i \in \mathbb{N}\} + 1$. It is easy to construct a compact c.d. space M with the following property: each nonempty open subset of M contains a copy of a compact space Z with $\operatorname{ind} Z = \alpha$.

Assuming that M can be embedded in Y we conclude, by Baire category arguments, that some $M \cap Y_i$ contains a copy of Z. This gives a contradiction.

The following characterization theorem is a direct consequence of [2, Theorem 5.3].

Theorem 5. *A space $Y \in AR$ is homeomorphic to $\widehat{\Lambda}_{\alpha}$, $2 \le \alpha < \omega_1$ (respectively, $\widehat{\Omega}_{\alpha}$, $3 \le \alpha < \omega_1$, $\widehat{\Pi}_i$, $i < \omega$) if and only if*

(i) $Y = \cup_{i=1}^{\infty} Y_i$ where $Y_i \in \mathcal{A}_{\alpha}(c.d.)$ (respectively, $Y_i \in \mathcal{M}_{\alpha}(c.d.)$, $Y_i \in \mathcal{P}_i(c.d.)$) and Y_i are strong Z-sets in Y;

(ii) Y is strongly $\mathcal{A}_{\alpha}(c.d.)$-universal (respectively, strongly $\mathcal{M}_{\alpha}(c.d.)$-universal, strongly $\mathcal{P}_i(c.d.)$-universal). □

We can consider Ω-manifolds where Ω is one of the spaces $\widehat{\Lambda}_{\alpha}$, $\widehat{\Omega}_{\alpha}$, or $\widehat{\Pi}_i$. Similarly to [2], the Triangulation Theorem, the Open Embedding Theorem, the Characterization Theorem, and the Factor Theorem can be proved for Ω-manifolds.

Here we prove a version of Miller's theorem for Ω-manifolds. Recall that a map $g : Y \to Z$ is called *fine homotopy equivalence* provided for every open cover $\mathcal{U}$ of Z there exists a map $h : Z \to Y$ such that gh is $\mathcal{U}$-homotopic to 1_Z and hg is $f^{-1}(\mathcal{U})$-homotopic to 1_Y (here $f^{-1}(\mathcal{U}) = \{f^{-1}(U) \mid U \in \mathcal{U}\}$).

Theorem 6. *Let Ω be as above. For every separable $Z \in ANR$ there exists Ω-manifold Y and a fine homotopy equivalence $g : Y \to Z$.*

Proof. Let $f : X \to Q$ be a c.d.-soft map where X is a separable c.d. space, $X \in \mathcal{A}_2$ (see Theorem 1). We consider Z as a subset of Q, let $Y = \Omega \times f^{-1}(Z)$, and let $g = f|f^{-1}(Z) \circ pr_2$. The result immediately follows from the Factor Theorem: $\Omega \times K$ is Ω-manifold for each $K \in \mathrm{ANR} \cap \mathcal{A}_2(c.d.)$. □

3. Nonseparable case.

Some constructions of universal spaces for c.d. spaces of weight $\leq \tau > \omega$ can be found in [10]. In this section we apply Theorem 1 to obtain some new examples of nonseparable universal c.d. spaces.

A map $f : X \to Y$ is said to be *completely zero-dimensional* with respect to some admissible metric on X [11] if for every $y \in Y$ and every $\varepsilon > 0$ there exists a neighbourhood U of Y such that the set $f^{-1}(U)$ can be represented as a discrete union of open sets of diameter $< \varepsilon$ (in this metric). As in [7], we apply zero-dimensional maps to obtain soft maps of nonseparable spaces. Recall that $l_2(\tau)$ denotes the Hilbert space of weight τ.

Theorem 7. *For each $\tau \geq \omega$ there exists a c.d.-soft map $f : X \to l_2(\tau)$ where X is a c.d. space of weight τ, $X \in \mathcal{A}_2$.*

Proof. Let $g : Y \to Q$ be a c.d.-soft map where $Y \in \mathcal{A}_2$ is a c.d. space. By [1 ,11], there exists a completely zero-dimensional map $h : l_2(\tau) \to Q$ (with respect to some admissible metric on $l_2(\tau)$). Let X be a fibered product of Y and $l_2(\tau)$ with respect to maps g and h. The arguments of [7] allow us to show that the natural projection $f : X \to l_2(\tau)$ is the required map. □

Corollary. *Let $\mathcal{C}$ be one of the classes $\mathcal{A}_\alpha$, $2 \leq \alpha < \omega_1$ or $\mathcal{M}_\alpha$, $3 \leq \alpha < \omega_1$. For every $\tau \geq \omega$ there exists a c.d. space $C \in \mathcal{C}$ of weight τ containing a closed copy of each c.d. space $D \in \mathcal{C}$ of weight $\leq \tau$.*

Proof. Let $f : X \to L_2(\tau)$ be a c.d.-soft map of a c.d. space $X \in \mathcal{A}_2$ of weight τ. There exists a subspace $Y \subset l_2(\tau)$, $Y \in \mathcal{C}$ which contains a closed copy of each space $Z \in \mathcal{C}$ of weight $\leq \tau$ (the construction of such a space Y is left to the reader). It is easy to see that we can take $C = f^{-1}(Y)$. □

4. Absorbing pairs and c.d. spaces

The following definition is due to R. Cauty (see [5]). Let (K, L) be a pair of spaces. A pair (X, Y) is called *strongly* (K, L)*-universal* if, for every closed subset D of K, every map $f : K \to X$ whose restriction to D is a Z-embedding and satisfies the condition $(f|D)^{-1}(Y) = D \cap L$ and every open cover $\mathcal{U}$ of X, there exists a Z-embedding $g : K \to X$ which is $\mathcal{U}$-close to f and satisfies the conditions $g|D = f|D$ and $g^{-1}(Y) = L$. Let $(\mathcal{K}, \mathcal{L})$ be a pair of classes of spaces. A pair (X, Y) is called *strongly* $(\mathcal{K}, \mathcal{L})$*-universal* if it is strongly (K, L)-universal for every pair (K, L) where $K \in \mathcal{K}$, $L \in \mathcal{L}$.

For each $t \in I$ let $Q_t = \{(x_i)_{i=1}^\infty \in Q \mid x_1 = t\}$. We need the following result of R. Pol.

Theorem [14 , Corollary 2.5]. *There exists a c.d. $G_{\sigma\delta}$-set $E_\infty \subset Q$ such that for every c.d. $G_{\sigma\delta}$-set G in Q there is an irrational $t \in I$ with $G \cap Q_t = E_\infty \cap Q_t$.* □

Let P be the set of irrationals in I. Note that, by the construction, the set E_∞ from the above Theorem is a subset of $P \times Q'$ where $Q' = \prod_{i=2}^\infty I_i$, $I_i = I$ for each i.

It is easy to choose a sequence of disjoint subsets $T_1, T_2, \ldots$ of I homeomorphic to P with the property that each nondegenerate interval in I contains some T_i. Let

$T = \cup_{i=1}^{\infty} T_i$. For each i let $E_\infty^{(i)}$ be the set obtained from $E_\infty \subset P \times Q'$ by means of homeomorphism of P onto T_i, and let $E_\infty^* = \cup_{i=1}^{\infty} E_\infty^{(i)}$.

For each subset $X \subset Q'$ let $E_\infty^*(X) = E_\infty^* \cap (T \times X)$.

Lemma 6. *Let $\mathcal{C} \in \{\mathcal{M}_\alpha \mid \alpha \geq 3\} \cup \{\mathcal{A}_\alpha \mid \alpha \geq 2\} \cup \{\mathcal{P}_i \mid i \geq 1\}$. There exists a subset $X \subset Q'$, $X \in \mathcal{C}$ such that $E_\infty^*(X)$ is a c.d. dense subset in Q, $E_\infty^*(X) \in \mathcal{C}$, and the following condition holds: for each pair (C, D) where $D \in \mathcal{C}(c.d.)$ there exists an embedding $f : C \to E_\infty^*(X)$ such that $f^{-1}(E_\infty^*(X)) = D$.*

Proof. There exists a dense subset $X \subset Q'$, $X \in \mathcal{C}$ with the following property: for each pair (C, D) where $D \in \mathcal{C}$ there exists an embedding $g : C \to Q'$ such that $g^{-1}(X) = D$ (see [2] for the Borelian classes and [4] for the projective classes). There exists $t \in T$ such that

$$E_\infty^*(X) \cap Q_t = Q_t \cap (T \times D).$$

Now the required embedding can be defined by the formula: $f(c) = g(c, t)$, $c \in C$.

Note that for each finite subset $A \subset X$ each nondegenerate subinterval of I contains a point t' such that $\{t'\} \times A \subset E_\infty^*(X)$. Therefore, $E_\infty^*(X)$ is dense in Q. Obviously, $E_\infty^*(X) \in \mathcal{C}$. □

Fix $\mathcal{C}$ and X as in Lemma 6. There exists an embedding $h : Q \to H$ into a separable Hilbert space H such that $h(Q)$ is a linearly independent subset of H. Let L be the closure of the linear span of $h(Q)$ in H and let M be the linear span of $h(E_\infty^*(X))$.

As usual, $\prod_{l_2} L$ denotes the countable l_2-product of L, i.e.,

$$\prod_{l_2} L = \{(x_i)_{i=1}^\infty \mid x_i \in L, \sum_{i=1}^\infty ||x_i||^2 < \infty\}$$

equipped with the norm

$$|||(x_i)_{i=1}^\infty||| = (\sum_{i=1}^\infty ||x_i||^2)^{1/2},$$

and $\sum_{l_2} M$ denotes the subspace of $\prod_{l_2} M$ consisting of all eventually zero sequences. Using the results of [5] we can immediately prove the following

Theorem 8. *Let L and M be as above, then the pair $(\prod_{l_2} L, \sum_{l_2} M)$ is strongly $(\mathcal{M}_1, \mathcal{C}(c.d.))$-universal.* □

The following theorem is a direct consequence of Theorem 8 and of [5, Theorem 2.1].

Theorem 9. *Let X be a topological copy of l_2,$Y \subset X$, and let L and M be as above. The pair (X, Y) is homeomorphic to $(\prod_{l_2} L, \sum_{l_2} M)$ if and only if Y is $\mathcal{C}(c.d.)$-absorbing and (X, Y) is strongly $(\mathcal{M}_1, \mathcal{C}(c.d.))$-universal.* □

Note that, in particular, Theorem 9 gives realizations of the spaces $\widehat{\Lambda}_\alpha$, $\widehat{\Omega}_\alpha$, and $\widehat{\Pi}_i$ as pre-Hilbert spaces.

5. Remarks and open questions.

Let $f : X \to Q$ be the c.d.-soft map of the c.d. space X from Theorem 1.

Question 1. *Characterize the map f topologically.*

For this purpose it is necessary to develop a fibrewise version of the theory of absorbing sets [2].

Question 2. *Describe the topology of the pre-images $f^{-1}(\Lambda_\alpha), f^{-1}(\Omega_\alpha)$, and $f^{-1}(\Pi_i)$. Can these spaces be represented as countable unions of strong Z-sets?*

It can be shown that, in general, strong Z-sets fail to be preserved by pre-images of soft maps.

Question 3. *Find apparent constructions of the spaces $\widehat{\Lambda}_\alpha$, $\widehat{\Omega}_\alpha$, and $\widehat{\Pi}_i$.*

Recently T. Radul [17] modified Nagata's construction in order to obtain models of spaces $\widehat{\Lambda}_\alpha$ and $\widehat{\Omega}_\alpha$. J. Mogilski [18] proved that $\sigma \times N_\omega$ is an $\mathcal{A}_2$(c.d.)-absorbing set.

Question 4. *Find topological copies of the spaces $\widehat{\Lambda}_\alpha$, $\widehat{\Omega}_\alpha$, and $\widehat{\Pi}_i$ "in nature", e.g., in the hyperspace theory or in the theory of function spaces.*

The arguments of section 2 cannot be applied to the case $\mathcal{C} = \mathcal{M}_2 = F_{\sigma\delta}$.

Question 5. *Are there $\mathcal{M}_2$(c.d.)-absorbing sets?*

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