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A CONTINUUM OF NORMAL MONADS

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It is proved that there exists exactly continuum of nonisomorphic normal monads in the category of compact Hausdorff spaces.

A *monad* on a category $\mathcal{G}$ is defined to be a triple $\mathbb{T} = (T, \eta, \mu)$ consisting of an endofunctor $T : \mathcal{G} \to \mathcal{G}$ and of natural transformations $\eta : 1_{\mathcal{G}} \to T$ and $\mu : T^2 \to T$ such that $\mu \circ T\eta = 1_T = \mu \circ \eta T$ and $\mu \circ \mu T = \mu \circ T\mu$. A natural transformation $\psi : T \to T'$ is called a *morphism* from monad $\mathbb{T} = (T, \eta, \mu)$ into monad $\mathbb{T}' = (T', \eta', \mu')$ if $\psi \circ \eta = \eta'$ and $\psi \circ \mu = \mu' \circ \psi T' \circ T\psi$. A morphism of monads is called an *isomorphism*, provided each component ψ_X is a homeomorphism.

All spaces and mappings are taken from the category $\mathcal{C}omp$ of compacta (i.e. compact Hausdorff spaces), and all functors are covariant endofunctors acting in $\mathcal{C}omp$.

A monad $\mathbb{T} = (T, \eta, \mu)$ is called *normal* if so is its functorial part T. Recall that a functor is said to be normal if it is continuous, monomorphic and epimorphic and it preserves weight, interesections, preimages, singletons space, and the empty set [3].

It is known [1] that there exists exactly continuum of nonisomorphic normal functors. In this article we shall prove the analogous result for normal monads.

By exp we denote the hyperspace functor [2]. Recall that for a compactum X, $\exp X$ is the space of all non-empty closed subsets of X, equipped with the Vietoris topology which base consists of the sets

$$\langle U_1, ..., U_k \rangle = \{A \in \exp X \mid A \subset U_1 \cup ... \cup U_k, A \cap U_i \neq \varnothing \text{ for all } i\},$$

where U_i run over the topology of X. For a mapping $f : X \to Y$ the mapping $\exp f$ is defined by the formula $\exp f(A) = f(A) \in \exp Y$, $A \in \exp X$. Define the natural transformations $s : 1_{\mathcal{C}omp} \to \exp$ and $u : \exp^2 \to \exp$ as follows: $s_X(x) = \{x\}$ for each $x \in X$, $u_X(\mathcal{A}) = \cup\mathcal{A}$, $\mathcal{A} \in \exp^2 X$. Then $\mathbb{H} = (\exp, s, u)$ is normal monad [2].

Each order $\preceq$ on the set $\mathbb{N}$ determines the functor $F_{\preceq}$ as follows:

$$F_{\preceq}X = \{(A_i)_{i=1}^{\infty} \in (\exp X)^{\omega} \mid i \preceq j \implies A_i \subset A_j\}.$$

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If $f : X \to Y$ and $(A_i)_{i=1}^\infty \in F_\preceq X$ then $F_\preceq f((A_i)_{i=1}^\infty) = (f(A_i))_{i=1}^\infty$.

Since for epimorphism $f : X \to Y$ and any $B, B_i, B_j \in \exp Y$ $f(f^{-1}(B)) = B$; and $f^{-1}(B_i) \subset f^{-1}(B_j)$ whenever $B_i \subset B_j$, $F_\preceq$ is epimorphic. Therefore $F_\preceq$ is normal functor as a subfunctor of normal functor $(\exp -)^\omega$. (It easily follows from the results of [4]).

One can easily verify that the triple $\mathbb{E} = ((\exp -)^\omega, \eta, \mu)$ where $\eta_X = (s_X)_{i=1}^\infty$, $\mu_X = \prod_{i=1}^{\infty} (u_X \circ \exp \pi_i)$ forms a normal monad (here $\pi_i : (\exp X)^\omega \to \exp X$ is the projection onto the corresponding factor).

Lemma 1. *For any $X \in \mathcal{C}omp$ we have $\eta_X(X) \subset F_\preceq X$.*$\square$

Let $(\mathcal{A}_i)_{i=1}^\infty \in (\exp(\exp X)^\omega)^\omega$. Note that every element $\alpha \in \mathcal{A}_i$ can be identified with sequence $(A_n^\alpha(i))_{n=1}^\infty \in (\exp X)^\omega$. Denote by $\mathfrak{A}_i$ the set $\cup\{A_i^\alpha(i) \mid \alpha \in \mathcal{A}_i\}$. Then

$$\mu_X((\mathcal{A}_i)_{i=1}^\infty) = ((u_X \circ \exp \pi_i)(\mathcal{A}_i))_{i=1}^\infty = (\mathfrak{A}_i)_{i=1}^\infty.$$

Lemma 2. *For any $X \in \mathcal{C}omp$ we have $\mu_X(F_\preceq^2 X) \subset F_\preceq X$.*

Proof. Let $(\mathcal{A}_i)_{i=1}^\infty \in F_\preceq^2 X$. Then $\mu_X((\mathcal{A}_i)_{i=1}^\infty) = (\mathfrak{A}_i)_{i=1}^\infty$. It is enough to prove that $\mathfrak{A}_i \subset \mathfrak{A}_j$ whenever $i \preceq j$. Let $i \preceq j$ are given and let $x \in \mathfrak{A}_i$. Then $x \in A_i^{\alpha_0}(i)$ for some $\alpha_0 \in \mathcal{A}_i$. Since $\mathcal{A}_i \subset \mathcal{A}_j$, we have $(A_k^{\alpha_0}(i))_{k=1}^\infty \in \mathcal{A}_j$ and hence $A_k^{\alpha_0}(i) = A_k^{\alpha'}(j)$ for some $\alpha' \in \mathcal{A}_j$ $(k = 1, \ldots, \infty)$. Furthermore $x \in A_i^{\alpha_0}(i) \subset A_j^{\alpha_0}(i) = A_j^{\alpha'}(j) \subset \mathfrak{A}_j$.$\square$

It is easy to obtain the following from Lemmas 1 and 2.

Proposition 1. *The triple $\mathbb{F}_\preceq = (F_\preceq, \eta, \mu \mid F_\preceq^2)$ forms a normal monad for any order $\preceq$ on $\mathbb{N}$.*$\square$

Let $(\mathbb{N}, \preceq_1)$ be an ordered set and let $\alpha\mathbb{N} = \{0\} \cup \mathbb{N}$ be the Alexandrov compactification of $\mathbb{N}$. We let $0 \preceq_1 i$ for any $i \in \mathbb{N}$. Denote by N_i the set $\{j \in \alpha\mathbb{N} \mid j \preceq_1 i\} \in \exp \alpha\mathbb{N}$. For a space X under X^* we understand the product $X \times \{0, 1\}$. For $A \subset X$ let ${}^1A$ and ${}^2A$ denote the sets $A \times \{0\}$ and $A \times \{1\}$ respectively.

Lemma 3. *$i \preceq_1 j$ if and only if $N_i^* \subset N_j^*$.*$\square$

Now let $(\mathbb{N}, \preceq_2)$ be an ordered set and denote by F_1, F_2 the corresponding functors $F_{\preceq_1}$ and $F_{\preceq_2}$, and by $\mathbb{F}_1$ and $\mathbb{F}_2$ the monads $\mathbb{F}_{\preceq_1}$ and $\mathbb{F}_{\preceq_2}$.

In the sequel we shall assume that orders $\preceq_1$ and $\preceq_2$ are linear. Assume that monads $\mathbb{F}_1$ and $\mathbb{F}_2$ are isomorphic. Our aim is to prove that the orders $\preceq_1$ and $\preceq_2$ are isomorphic.

For, consider $F_1(\alpha\mathbb{N}^*)$ and the point $\mathcal{N} = (N_i^*)_{i=1}^\infty \in F_1(\alpha\mathbb{N}^*)$. Let $\psi = \{\psi_X\}_{X \in \mathcal{C}omp}$ be an isomorphism of the monads $\mathbb{F}_1$ and $\mathbb{F}_2$, and let $\psi_{\alpha\mathbb{N}^*}(\mathcal{N}) = \mathcal{M} = (M_i)_{i=1}^\infty$.

Lemma 4. *If $i^* \cap M_k \neq \varnothing$ for some $i \in \alpha\mathbb{N}$, then $i^* \subset M_k$.*

Proof. It's enough to remark that the mapping $f : \alpha\mathbb{N}^* \to \alpha\mathbb{N}^*$ which permutates the points in every pair i^* $(i \in \alpha\mathbb{N})$ is continuous and $F_1(f)(\mathcal{N}) = \mathcal{N}$.$\square$

Lemma 5. *If $i^* \subset M_k$ for some $i \in \mathbb{N}$, then $N_i^* \subset M_k$.*

Proof. Indeed, assume that $j \in N_i$ and $j^* \not\subset M_k$. Consider the retraction $p : \alpha\mathbb{N}^* \to \alpha\mathbb{N}^*\backslash i^*$ such that $p(i^*) = {}^1j$. The mapping p is continuous and $F_1p(\mathcal{N})$ is symmetric with respect to the permutation of ${}^1j$ and ${}^2j$. In the meantime $F_2p(\mathcal{M})$ is nonsymmetric.□

Lemmas 4 and 5 imply that each coordinate M_k can be expressed in the form

$$M_k = \cup\{N_i^* \mid i^* \subset M_k\}.$$

Lemma 6. *For any $i \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $i^* \subset M_k$.*

Proof. Indeed, otherwise consider the mapping $g : \alpha\mathbb{N}^* \to 0^* \cup i^*$ which is the identity on i^* and such that $g({}^n\alpha\mathbb{N}\backslash{}^ni) = {}^n0$, $n = 1, 2$. Then g is continuous and $(0^*, \dots, 0^*, \dots) = F_2g \circ \psi_{\alpha\mathbb{N}^*}(\mathcal{N}) = \psi_{o^*\cup i^*} \circ F_1g(\mathcal{N})$, which is impossible, because $F_1g(\mathcal{N}) \neq (0^*, \dots, 0^*, \dots)$ and for any $X \in \mathcal{C}omp$ and a closed subset $A \subset X$ we have $\psi_X((A, \dots, A, \dots)) = (A, \dots, A, \dots)$.□

It follows from the Lemmas 5 and 6 that for any $i \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $N_i^* \subset M_k$. In fact, a stronger result is valid.

Proposition 2. *For any $i \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $N_i^* = M_k$.*

Proof. Consider $N' = \alpha\mathbb{N} \sqcup \{x\}$, where x is an isolated point satisfying $i \prec_1 x \prec_1 j$ for any $j \succ_1 i$. Consider the point $\mathcal{N}' = ({N'_m}^*)_{m=1}^{\infty} \in F_1(N'^*)$, where

$$N'_m = \begin{cases} N_m, & \text{for } m \preceq_1 i \\ N_m \cup \{x\}, & \text{for } m \succ_1 i. \end{cases}$$

Let

$$\psi_{N'^*}(\mathcal{N}') = \mathcal{M}' = (M'_k)_{k=1}^{\infty}.$$

We shall show that N_i^* must be among M'_k. Indeed, otherwise considering the mapping $p : N'^* \to \alpha\mathbb{N}^*$ identical on $\alpha\mathbb{N}^*\backslash i^*$ and such that $p(i^*) = {}^1i$, $p(x^*) = {}^2i$ we shall obtain that $F_2p(\mathcal{M}')$ is symmetric with respect to the permutation ${}^1i$ and ${}^2i$, but $F_1p(\mathcal{N}')$ is nonsymmetric.

Now consider the retraction $f : N'^* \to \alpha\mathbb{N}^*$ such that $f({}^nx) = {}^n0$, $n = 1, 2$. We shall obtain that $F_1f(\mathcal{N}') = \mathcal{N}$, therefore $F_2f(\mathcal{M}') = \mathcal{M}$ and then N_i^* must be among M_k.□

Below we shall show that the sequences $\mathcal{M} = (M_k)_{k=1}^{\infty}$ and $\mathcal{N} = (N_k)_{k=1}^{\infty}$ are equal up to coordinate permutation. To prove this we need the following

Proposition 3. *For any $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $M_k = N_m^*$.*

Proof. At first we consider the case when there exists the minimal element $y \in (\mathbb{N}, \preceq_1)$. Denote by $\mathbb{N}'$ the set $\mathbb{N} \sqcup \{x\}$ where x is an isolated point. We let $x \prec_2 i$ for any $i \in \mathbb{N}$. Let $\alpha(\mathbb{N} \times \mathbb{N}') = \{0\} \sqcup (\mathbb{N} \times \mathbb{N}')$ be the Alexandrov compactification of $\mathbb{N} \times \mathbb{N}'$. Denote by $N_i^{(2)}$ the set $\{j \in \mathbb{N}' \mid j \preceq_2 i\}$. Define the set $R_k \in \exp(\alpha(\mathbb{N} \times \mathbb{N}')^*)$ for any $k \in \mathbb{N}$ as follows

$$R_k = 0^* \cup [(M_k\backslash 0^*) \times N_k^{(2)}].$$

Obviously $\mathcal{R} = (R_k)_{k=1}^{\infty} \in F_2(\alpha(\mathbb{N}\times\mathbb{N}')^*)$. Let

$$\mathcal{P} = \psi^{-1}_{\alpha(\mathbb{N}\times\mathbb{N}')^*}(\mathcal{R}) = (P_k)_{k=1}^{\infty}.$$

Denote by $\widehat{k}$ the set $(M_k\backslash 0^*)\times\{k\}\subset R_k$ for each $k\in\mathbb{N}$. By analogy with the proof of Lemmas 4-6 and Proposition 2 one can show that the following conditions are satisfied:

(a) if $\widehat{k}\cap P_m\neq\varnothing$ then $\widehat{k}\subset P_m$ (additionally, if $t^*\cap P_m\neq\varnothing$ for some $t\in\alpha(\mathbb{N}\times\mathbb{N}')$ then $t^*\subset P_m$);

(b) if $\widehat{k}\subset P_m$ then $R_k\subset P_m$;

(c) for any $k\in\mathbb{N}$ there exists $m\in\mathbb{N}$ such that $\widehat{k}\subset P_m$;

(d) for any $k\in\mathbb{N}$ there exists $m\in\mathbb{N}$ such that $R_k = P_m$.

(Among the statements (a)-(d) the last one requires some explanation. For its proof we need the equality $P_m = \cup\{R_k \mid \widehat{k}\subset P_m\}$, $m\in\mathbb{N}$ which follows from (a), (b) and the following reasoning. Let $c=(i,k)\in P_m\backslash\widehat{k}$. Assume that $i\in M_l$. Then $R_l\subset P_m$. Indeed, otherwise it follows from (a) and (b) that $\widehat{l}\cap P_m=\varnothing$. Consider the retraction $p:\alpha(\mathbb{N}\times\mathbb{N}')^*\to\alpha(\mathbb{N}\times\mathbb{N}')^*\backslash c$ which moves the point c into the point (i,l). It is continuous and $F_2p(\mathcal{R})$ is invariant with respect to permutations of points of $\widehat{l}$, but $F_1p(\mathcal{P})$ is not invariant. This is a contradiction.

Furthermore, let $\mathbb{N}''=\mathbb{N}'\sqcup\{z\}$, where the point z is isolated. We let $k\prec_2 z\prec_2 n$ for all $n\succ_2 k$. Let

$$\widetilde{N}^{(2)}_m = \begin{cases} N^{(2)}_m, & \text{for } m\preceq_2 k \\ N^{(2)}_m\cup\{z\}, & \text{for } m\succ_2 k, \end{cases}$$
$$\widetilde{R}_m = 0^*\sqcup((M_k\backslash 0^*)\times\widetilde{N}^{(2)}_m),\quad \widetilde{\mathcal{R}} = (\widetilde{R}_m)_{m=1}^{\infty},$$
$$\widetilde{\mathcal{P}} = \psi^{-1}_{\alpha(\mathbb{N}\times\mathbb{N}'')^*}(\widetilde{\mathcal{R}}) = (\widetilde{P}_m)_{m=1}^{\infty}.$$

Then R_k must be among $\widetilde{P}_m$. Indeed, otherwise consider the mapping $r:\alpha(\mathbb{N}\times\mathbb{N}'')^*\to\alpha(\mathbb{N}\times\mathbb{N}'')^*\backslash(y,z)^*$ which moves $(y,z)^*=\{({}^1y,z),({}^2y,z)\}$ into $({}^1y,k)$ and $({}^1y,k)$ into $({}^2y,k)$, and it is the identity elsewhere. It is continuous and $F_1r(\widetilde{\mathcal{P}})$ is a symmetric with respects to permutation $({}^1y,k)$ and $({}^2y,k)$, but $F_2r(\widetilde{\mathcal{R}})$ is not.

To complete the proof of (d) it is enough to consider the retraction $p:\alpha(\mathbb{N}\times\mathbb{N}'')^*\to\alpha(\mathbb{N}\times\mathbb{N}')^*$ such that $p(({}^ni,z))={}^n0$ $(i\in\mathbb{N},\ n=1,2)$.)

Now consider the mapping $f:\alpha(\mathbb{N}\times\mathbb{N}')^*\to\alpha\mathbb{N}^*$ such that $f(x,{}^ni)={}^ni$; $f(l,{}^ni)={}^n0$, $l\neq x$; $f({}^n0)={}^n0$ $(i\in\mathbb{N},\ n=1,2)$. It is continuous and $F_2f(\mathcal{R})=\mathcal{M}$. Then $F_2f(\mathcal{P})=\mathcal{N}$ and therefore there exists $m\in\mathbb{N}$ such that $M_k=N^*_m$.

In general case (when the minimal element in $(\mathbb{N},\preceq_1)$ not nessesary exists) consider the space $\widetilde{\mathbb{N}}=\alpha\mathbb{N}\cup\{y\}$, where y is an isolated point. We let $0\prec_1 y\prec_1 i$ for any $i\in\mathbb{N}$. Now let

$$\widetilde{\mathcal{N}} = (\widetilde{N}^*_i)_{i=1}^{\infty}\in F_1\widetilde{\mathbb{N}}^*,\quad \widetilde{N}_i = N_i\sqcup\{y\},\quad \widetilde{\mathcal{M}} = \psi_{\widetilde{\mathbb{N}}}(\widetilde{\mathcal{N}}) = (\widetilde{M}_i)_{i=1}^{\infty}.$$

Then there exists $m\in\mathbb{N}$ such that $\widetilde{M}_k=\widetilde{N}^*_m$. Now consider the retraction $r:\widetilde{\mathbb{N}}^*\to\alpha\mathbb{N}^*$ such that $r({}^ny)={}^n0$ $(n=1,2)$. It is continuous and $F_1r(\widetilde{\mathcal{N}})=\mathcal{N}$, therefore $F_2r(\widetilde{\mathcal{M}})=\mathcal{M}$ and hence $M_k=N^*_m$.$\square$

Remark 1. Let $k_1, k_2 \in \mathbb{N}$ be distinct. Then $R_{k_1} \neq R_{k_2}$. Now if $P_{m_1} = R_{k_1}$ and $P_{m_2} = R_{k_2}$, then $m_1 \neq m_2$.

Remark 2. It follows from Proposition 3 and Remark 1 that all the components of $\mathcal{M}$ are of the type N_i^*, and moreover, since all the coordinates of $\mathcal{N}$ are mutually distinct, so are the coordinates of $\mathcal{M}$. Hence, we have shown that the point $\mathcal{M}$ is formed by a coordinate permutation of $\mathcal{N}$.

This allows us to state the following.

Proposition 4. *Monads $\mathbb{F}_1$ and $\mathbb{F}_2$ are isomorphic if and only if the orders $\preceq_1$ and $\preceq_2$ are isomorphic.*

Proof. ($\Longleftarrow$) Given an isomorphism $d : (\mathbb{N}, \preceq_1) \to (\mathbb{N}, \preceq_2)$, the isomorphism $\psi : \mathbb{F}_1 \to \mathbb{F}_2$ is formed in such a way:

$$\psi_X((A_i)_{i=1}^{\infty}) = (A_{d^{-1}(i)})_{i=1}^{\infty}, \quad (A_i)_{i=1}^{\infty} \in F_1 X, \quad X \in \mathcal{C}omp.$$

($\Longrightarrow$) Let $\psi : \mathbb{F}_1 \to \mathbb{F}_2$ be an isomorphism. Construct the mapping $d : (\mathbb{N}, \preceq_1) \to (\mathbb{N}, \preceq_2)$ in such a way: $d(i) = k$ where k is such that $M_k = N_i^*$. Remark 2 implies that d is defined correctly and is a bijection.

Suppose that $d(i) \preceq_2 d(j)$. Then $N_i^* = M_{d(i)} \subset M_{d(j)} = N_j^*$ and by Lemma 3 $i \preceq_1 j$. Thus d^{-1} is a homomorphism. Now isomorphness of d follows from linearity of orders $\preceq_1$ and $\preceq_2$.□

Problem (M. Zarichnyi). *Is Proposition 4 valid for arbitrary (not necessarily linear) orders $\preceq_1$ and $\preceq_2$?*

Theorem. *There exists exactly continuum nonisomorhic normal monads.*

Proof. The upper estimate follows from the known fact stating that there is exactly continuum of normal nonisomorphic functors and for any two T_1 and T_2 of them there is no more than continuum of natural transformations from T_1 into T_2 (see [1] for details). The lower estimate follows from proposition 4, for there exists a continuum nonisomorphic linear orders on $\mathbb{N}$. (The last can be proved by the following arguments due to M. Zarichnyi. For any sequence $\mathbf{x} = \{x_n\}_{n \in \mathbb{N}}$, where $x_n \in \{0, 1\}$ consider the set $L(\mathbf{x}) = \{n + \dfrac{(-1)^{x_n}}{2^{k+1}} \mid k, n \in \mathbb{N}\} \subset \mathbb{R}$ equipped with the linear order inherited from $\mathbb{R}$. One can easily verify that the ordered sets $L(\mathbf{x})$ and $L(\mathbf{x}')$ are not isomorphic whenever $\mathbf{x} \neq \mathbf{x}'$.)□

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R E F E R E N C E S

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