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GROWTH OF FUNCTIONS ANALYTIC IN THE UNIT DISC WHICH ARE GENERATING FUNCTIONS OF PÓLYA FREQUENCY SEQUENCES

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For any integer $r \geq 2$ and any positive number ρ, an example of the function $f(z)$ is constructed with the following properties: (i) $f(z)$ is analytic in the unit disc, (ii) $f(z)$ is a generating function of the Pólya frequency sequence of the order r, (iii) the order of the growth of $f(z)$ in the unit disc is equal to ρ.

The Pólya frequency sequences had been introduced at the beginning of 20th century by M.Fekete in the connection with the study of distribution of zeros of polynomials. Later, these sequences found a lot of applications in Analysis (see, e.g. [1]; concerning applications to the theory of Padé approximation see [2], ch.5). Recall, that the sequence $\{c_k\}_{k=0}^{\infty}$ is called Pólya frequency sequence of order $r \leq \infty$ if all minors of order $\leq r$ (in the case $r = \infty$, all minors) of the infinite matrix

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \dots \\ 0 & c_0 & c_1 & c_2 & \dots \\ 0 & 0 & c_0 & c_1 & \dots \\ 0 & 0 & 0 & c_0 & \dots \\ . & . & . & . & \dots \end{pmatrix}$$

are non-negative. The class of all Pólya frequency sequences of order r is denoted usually by PF_r. The class of the corresponding generating functions

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \tag{1}$$

we shall also denote by PF_r. We hope, it does not lead to any ambiguity. Note that for $r \geq 2$ the radius of convergence of (1) is positive ([1], p.394). Further we shall suppose, without loss of generality, that $c_0 = 1$.

In 1953, the problem of the description of the class PF_∞ was exhaustively solved in [3]:

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Theorem [3]. *The class PF_∞ consists of the functions of the form*

$$f(z) = e^{\gamma z} \prod_{k=1}^{\infty} (1 + \alpha_k z)/(1 - \beta_k z),$$

where $\gamma \geq 0,\ \alpha_k \geq 0,\ \beta_k \geq 0,\ \sum(\alpha_k + \beta_k) < \infty$.

In 1955, I.J.Schoenberg [4] set up the problem of the description of the classes PF_r for finite values of $r \geq 2$ and obtained the deep results concerning distribution of zeros of polynomials belonging to these classes. For entire transcendental functions, this problem has been investigated later in [5, 6]. In these papers, some essential distinctions of the properties of entire functions of PF_∞ from those of $PF_r, r < \infty$ are discovered. The main of them are the following:

(i) While zeros of an entire function of PF_∞ are negative and their exponent of convergence is not greater than one, the zero-set of an entire function of $PF_r,\ r < \infty$, can be much more complicated.

(ii) While growth of an entire function of PF_∞ is not greater than of order one and of normal type, an entire function of $PF_r,\ r < \infty$, can possess a rather arbitrary growth.

The present paper deals with the study of the growth of functions of the classes $PF_r,\ 2 \leq r < \infty$, which are analytic in the unit disc $\mathbf{D}$. From the theorem of [3] quoted above, it follows that the function of PF_∞ analytic in $\mathbf{D}$ either possesses a pole at the point $z = 1$ or does not have any singularity on $\partial\mathbf{D}$. The following theorem, which is the main result of this paper, shows that a function of any class $PF_r, r < \infty$ can possess on $\partial\mathbf{D}$ an essential singularity and the growth of such a function at this singularity can be rather fast.

Theorem A. *Suppose that an integer $r \geq 2$ and a positive number ρ are given. There exists a function $g(z) \in PF_r$ analytic in $\mathbf{D}$ and possessing an essential singularity at $z = 1$ such that*

$$\lim_{x \to 1} \frac{\log\log M(x, g)}{\log(1/(1-x))} = \rho,$$

where $M(x, g) = \max\{|g(z)| : |z| = x\},\ x > 0$.

To prove the theorem we need some lemmas.

Lemma 1. *Denote by*

$$E_\rho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + k/\rho)}, \quad 0 < \rho < \infty,$$

the classical Mittag-Leffler function. For any $r = 2, 3, \ldots,$ there exists an integer $s \geq 0$ such that the function $(1 + z)^s E_\rho(z)$ belongs to PF_r.

Validity of this lemma for $\rho \leq 1/2$ and $\rho = 1$ is an immediate corollary of the theorem of [3] quoted above. Indeed, all zeros of $E_\rho(z)$ are negative for $\rho \leq 1/2$ (see [6]) and $E_1(z) = e^z$. Validity of the lemma for the remaining values of ρ is a corollary of the following result of [7]:

Theorem[7]. *Let $E(z)$, $E(0) = 1$, be an entire function of the normal type of the finite positive order ρ and of completely regular growth in Levin-Pfluger's sense. Suppose that the function $E(z)$ is positive on the positive ray and is non-vanishing in an angle $\{z : |\arg z| \le \theta_0\}$. Moreover, suppose that the indicator*

$$h(\theta, E) = \overline{\lim_{r\to\infty}}\, r^{-\rho} \log |E(re^{i\theta})|$$

satisfies the following conditions:
(i) $h(\theta, E) = h(0, E)\cos \rho\theta, \quad |\theta| \le \theta_0$;
(ii) $h(\theta, E) < h(0, E)$ *for all θ belonging to the open interval $(0, 2\pi)$. Then, for any $r = 2, 3, \ldots$, there exists an integer $s \ge 0$ such that the function $(1 + z)^s E(z)$ belongs to PF_r.*

The well-known asymptotic expression (see, e.g.[8, p.114])

$$E_\rho(z) = \begin{cases} e^{z^\rho} + O(|z|^{-1}), & |\arg z| \le \pi/(2\rho), \\ -\{z\Gamma(1 - (1/\rho))\}^{-1} + O(|z|^{-2}), & \pi/(2\rho) < |\arg z| \le \pi, \end{cases}$$

yields that the function $E_\rho(z)$, $\rho > 1/2$, satisfies the conditions of the quoted theorem of [7] and therefore Lemma 1 is valid.

Lemma 2. *Suppose that $\{c_n\}_{n=0}^\infty \in PF_r$, $r \ge 2, 0 < \sum c_n < \infty \ldots$. Set*

$$d_k = \sum_{n=0}^{\infty} \frac{c_n k^{n+r-1}}{\Gamma(n + r)}, \quad k = 0, 1, 2, \ldots \tag{2}$$

Then $\{d_k\}_{k=0}^\infty \in PF_r$.

This lemma can be derived from the following theorem due to S.Karlin ([1, p.107]):

Theorem [1]. *Suppose that $\{c_n\}_{n=0}^\infty \in PF_r$, $r \ge 2$, $0 < \sum c_n < \infty$, $\alpha > r - 2$. Set*

$$f_\alpha(x) = \begin{cases} \sum_{n=0}^\infty c_n x^{n+\alpha}/\Gamma(n + \alpha + 1), & x \ge 0, \\ 0, & x < 0. \end{cases}$$

Then $f_\alpha(x)$ is a Pólya frequency function of the order r.

Recall that, by the definition of the Pólya frequency function of the order r, for any $n \le r$ and for any system of numbers $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \ldots y_n$, we have

$$\det \|f_\alpha(x_i - y_j)\|_{i,j=1}^n \ge 0.$$

Setting $\alpha = r - 1$, $d_k = f_{r-1}(k)$, $k = 0, \pm1, \pm2, \ldots$ and taking $x_i, y_j \in \mathbf{Z}$, we see that any minor of the matrix

$$\begin{pmatrix} d_0 & d_1 & d_2 & d_3 & \ldots \\ 0 & d_0 & d_1 & d_2 & \ldots \\ 0 & 0 & d_0 & d_1 & \ldots \\ 0 & 0 & 0 & d_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

can be written in the form $\det \|f_{r-1}(x_i - y_j)\|_{i,j=1}^n$. Therefore Lemma 2 is valid.

Lemma 3. *Set*

$$(1+z)^s E_\rho(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{3}$$

where s is chosen according to Lemma 1. Then the sequence $\{d_k\}_{k=0}^{\infty}$ defined by the equation (2) belongs to PF_r.

The lemma is an immediate corollary of Lemma 1 and Lemma 2.

Lemma 4. *The function*

$$f_{r-1}(x) = \sum_{n=0}^{\infty} \frac{c_n x^{n+r-1}}{\Gamma(n+r)},$$

where the coefficients c_n are defined by (3), satisfies the following asymptotic relation

$$\log f_{r-1}(x) = Cx^{\rho/(\rho+1)}(1+o(1)), \quad x \to \infty,$$

where C is a positive constant not depending on x.

The definition (3) of c_n's yields that

$$c_n = \sum_{j=0}^{s} \frac{C_s^j}{\Gamma(1+(n-j)/\rho)}, \quad n \geq s,$$

whence

$$\frac{1}{\Gamma(1+(n-s)/\rho)} < c_n < \frac{2^s}{\Gamma(1+(n-s)/\rho)}, \quad n \geq s.$$

Setting

$$F(x) = \sum_{n=s+1}^{\infty} \frac{x^n}{\Gamma(n+r)\Gamma(1+(n-s)/\rho)}, \tag{4}$$

we obtain

$$x^{r-1}F(x) < f_{r-1}(x) < 2^s x^{r-1} F(x) + O(x^{r+s}), \quad x \to +\infty,$$

whence

$$\log f_{r-1}(x) = \log F(x) + O(\log x), \quad x \to \infty. \tag{5}$$

To obtain the asymptotic expression of $F(x)$, we utilize the following result of the theory of entire functions which is an immediate consequence of the well-known theorems related to connection between growth of the entire function and decrease of its coefficients (see, e.g., [9, pp.11-12]).

Theorem. *Let $F(z) = \sum_{n=0}^{\infty}$ be an entire function. If for some $\lambda > 0$ the finite limit exists*

$$\lim_{n\to\infty} n|a_n|^{\lambda/n}, \tag{6}$$

then the following finite limit exists also

$$\lim_{x\to\infty} x^{-\lambda} \log M(x,F). \tag{7}$$

Apply this result to the function $F(z)$ defined by (4). Utilizing the Stirling formula, we obtain

$$\log a_n = -\log\Gamma(n+r) - \log\Gamma(1+(n-s)/\rho)$$
$$= -\frac{\rho+1}{\rho} n\log n + \frac{\rho+1}{\rho} n + O(\log n), \quad n\to\infty.$$

Therefore the limit (6) exists for $\lambda = \rho/(\rho+1)$. Denoting the limit (7) by C and noting that $M(x,F) = F(x)$, we conclude that

$$\log F(x) = (C+o(1))x^{\rho/(\rho+1)}, \quad x\to\infty.$$

Substituting into (5), we obtain the assertion of the lemma.

Further, we shall need in the following facts related to the functions

$$h(z) = \sum_{k=0}^{\infty} a_k z^k \tag{8}$$

analytic in the unit disc **D**.

Theorem(Wiegert, see, e.g.[10],p.394). *In order that a function $h(z)$ given by the series* (8) *have its only singularity at $z=1$ and be zero at infinity, it is necessary and sufficient that there be an entire function $G(z)$ not greater than of order 1 and of minimal type such that $a_k = G(k), k = 0, 1, \ldots$*

Theorem(see [11], [12]). *For the function* (8)*, set*

$$\lambda = \overline{\lim_{x\to 1}} \frac{\log\log M(x,h)}{\log(1/(1-x))}.$$

Then the following equality is valid

$$\frac{\lambda}{\lambda+1} = \overline{\lim_{k\to\infty}} \frac{\log^+ \log^+ |a_k|}{\log k}. \tag{9}$$

Proof of Theorem A. Set

$$g(z) = \sum_{k=0}^{\infty} d_k z^k,$$

where the coefficients d_k are defined by (2). By Lemma 3 we have $g(z) \in PF_r$. By Lemma 4 the function $f_{r-1}(z)$ is an entire function of the order $\rho/(\rho+1) < 1$. Since $d_k = f_{r-1}(k)$, $k = 0, 1, \ldots$, the Wiegert theorem is applicable to the function $g(z)$ and shows that it can be extended analytically to the whole complex plane without the point $z = 1$. By Lemma 4 we have also

$$\log d_k = (C+o(1))k^{\rho/(\rho+1)}, \quad k\to\infty.$$

Applying to the function $g(z)$ the formula (9) with $a_k = d_k$, we see that $\lambda = \rho$ and hence

$$\overline{\lim_{x\to 1}} \frac{\log\log M(x,g)}{\log(1/(1-x))} = \rho > 0.$$

Note that the point $z = 1$ must be an essential singularity since otherwise the number ρ could not be positive. The theorem is thus proved.

Remark. The question about the exhaustive description of the possible sets of singularities of functions belonging to $PF_r, 2 \leq r < \infty$, remains to be open.

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