Математичні студії
1994, випуск 3

SOME OPEN PROBLEMS IN THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

Problem 1 (M. Sheremeta). Let (n_p) be an increasing sequence of natural numbers, $n_0 = 0$. It is proved in [1] that in order that for every analytic in $\mathbb{D} = \{z : |z| < 1\}$ function f the univalence in $\mathbb{D}$ of all its derivatives $f^{(n_p)}$ would imply that f is an entire function it is necessary and sufficient that

$$\lim_{p\to\infty} \{\ln n_p - \frac{1}{n_p} \sum_{j=1}^{p} (n_j - n_{j-1}) \ln(n_j - n_{j-1})\} = +\infty.$$

Find a necessary and sufficient condition on (n_p) in order that for every analytic in $\mathbb{D}$ function f the univalence in $\mathbb{D}$ of all its derivatives $f^{(n_p)}$ would imply that f can be analytically extended onto the disc $\mathbb{D}_R = \{z : |z| < R\}$ with $R > 1$?

[1] Sheremeta M.M. *Disproving of a Shah hypothesis on univalent functions* // Matematychni Studiï. **2** (1993) 46–48.

Problem 2 (M. Sheremeta). Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be an increasing to infinity sequence of nonnegative numbers and let $S(\Lambda)$ be the class of entire (absolutely convergent in $\mathbb{C}$) Dirichlet series $F(s) = \sum_{n=0}^{\infty} a_n \exp(s\lambda_n)$, $s = \sigma + it$. Put $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$, $\mu(\sigma, F) = \max\{|a_n| \exp(\sigma\lambda_n) : n \geq 0\}$, and denote by $L_{\text{пз}}$ the class of nonnegative continuous increasing to ∞ functions φ such that $\varphi(2x) \sim \varphi(x)$, $x \to +\infty$.

Find a necessary and sufficient condition for Λ in order that for each function $F \in S(\Lambda)$ the relation $\varphi(\ln M(\sigma, F)) \sim \varphi(\ln \mu(\sigma, F))$ holds for $0 \leq \sigma \to +\infty$ outside of a set of finite measure where $\varphi \in L_{\text{пз}}$.

[For an information on results obtained in this direction see the article of M.M. Sheremeta contained in this issue].

Problem 3 (M. Sheremeta). Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be an increasing to ∞ sequence of nonnegative numbers and L be the class of continuous nonnegative increasing to ∞ on $[0, +\infty)$ functions. Denote by $S_\psi(\Lambda)$ the class of entire (absolutely conversgent in $\mathbb{C}$) Dirichlet series $F(s) = \sum_{n=0}^{\infty} a_n \exp(s\lambda_n)$, $s = \sigma + it$ such that $|a_n| \leq \exp\{-\lambda_n \psi(\lambda_n)\}$, when $n \geq n_0$, $\psi \in L$. We say that $\psi \in L_{\text{пз}}$ if $\psi \in L$ and $\psi(2x) \sim \psi(x)$, when $x \to +\infty$.

In [1] it is proved the following

Theorem. *Let* $\psi \in L$, $\limsup_{n\to\infty} (\ln n / \lambda_n \psi(\lambda_n)) = q < 1$, *and let a function* f *be such that* $\varphi(\ln x) \in L_{\text{пз}}$ *and the function* $\ln \varphi(x)$ *in concave. In order that for every function* $F \in S_\psi(\Lambda)$ *the relation* $\varphi(\ln M(\sigma, F)) \sim \varphi(\ln \mu(\sigma, F))$ *holds for* $\sigma \to +\infty$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX *it is necessary and sufficient that*

$$\lim_{n\to\infty}\frac{\varphi(\ln n+\psi(\lambda_n)\gamma(\psi(\lambda_n)))}{\varphi(\psi(\lambda_n))\gamma(\psi(\lambda_n))}=1.$$

Here, $M(\sigma,F)=\sup\{|F(\sigma+it)|:t\in\mathbb{R}\}$ *and* $\mu(\sigma,F)=\max\{|a_n|\exp(\sigma\lambda_n):n\geq 0\}$.

It is also shown in [1] that the condition $\varphi(\ln x)\in L_{\text{пз}}$ is essential in this Theorem.

Is the condition of concavity of the function $\ln\varphi(x)$ essential?

[1] Sheremeta M.N. *On relationships between the maximal term and the maximum modulus of an entire Dirichlet series* // Mat. Zametki. 1992. V.51, № 5. P.141–148.

Problem 4 (M. Sheremeta). Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be an entire transcendent function with entire coefficients, (γ_k) be a sequence of sign changes of the sequence (a_n), and $M_f(r)=\max\{|f(z)|:|z|=r\}$.

In [1] it was made an effort to prove the following

Theorem. *If* $\sum_{k=1}^{\infty}1/\gamma_k<\infty$, *then*

$$\limsup_{x\to+\infty}(\ln|f(x)|/\ln M_f(x))=1.$$

Later, it turned out that in the proof of this theorem [1] is an essential gap which the author can't remove. We leave this as an open problem.

[1] Sheremeta M.N. *On a property of entire functions with real Taylor coefficients* // Mat. Zametki. 1975. V.18, № 3. P.395–402.

Problem 5 (M. Sheremeta). Let $\Lambda=(\lambda_n)_{n=0}^{\infty}$ be an increasing up to ∞ sequence of nonnegative numbers and $S(\Lambda)$ be the class of entire (absolutely convergent in $\mathbb{C}$) Dirichlet series $F(s)=\sum_{n=0}^{\infty}a_n\exp(s\lambda_n)$, $s=\sigma+it$. Put $M(\sigma,F)=\sup\{|F(\sigma+it)|:t\in\mathbb{R}\}$ and $\mu(\sigma,F)=\max\{|a_n|\exp(\sigma\lambda_n):n\geq 0\}$. O.B. Skaskiv [1] has shown the following. In order that for every function $F\in S(\Lambda)$ the relationship $M(\sigma,F)\sim\mu(\sigma,F)$ holds as $0\leq\sigma\to+\infty$ out of a set of finite measure it is necessary and sufficient that

$$\sum_{n=0}^{\infty}\frac{1}{\lambda_{n+1}-\lambda_n}<\infty. \tag{1}$$

Under this condition we have $\ln M(\sigma,F)-\ln\mu(\sigma,F)\to\infty$ and, moreover,

$$\sqrt{\ln M(\sigma,F)}-\sqrt{\ln\mu(\sigma,F)}\to 0 \tag{2}$$

as $0\leq\sigma\to+\infty$ out of a set of finite measure. It can be shown that the condition (1) is not necessary to satisfy (2) for every function $F\in S(\Lambda)$.

What is a necessary and sufficient condition in order that for every function $F\in S(\Lambda)$ the relationship (2) holds as $0\leq\sigma\to+\infty$ out of a set of finite measure.

[1] Skaskiv O.B. *Maximum modulus and maximal term of an entire Dirichlet series* // Dop. AN URSR, ser. A. 1984, № 11. P.22–24.

PROBLEM 6 (M. ZABOLOTSKYĬ). Let f be a meromorphic transcendent function, $\rho(f(z))$
$= |f'(z)|/(1+|f(z)|^2)$ be its spherical derivative, $\mu(r,f) = \max\{\rho(f(z)) : |z| = r\}$, $T(r,f)$ be Nevanlinna characteristic function and Φ be a convex with respect to logarithm on $[1,+\infty)$ function such that $\Phi(2r) = O(\Phi(r))$, $r \to \infty$. Denote by $W(f,\Phi)$ the class of meromorphic functions f such that

$$\mu(r,f) = O(\Phi'(r)), \quad r \to \infty,$$
$$T(r,f) \neq O(\Phi(r)), \quad r \to \infty.$$

It is proved in [1] that the functions of the class $W(f,\Phi)$ fail to have Nevanlinna exeptional values.

Conjecture. *There exists a meromorphic transcendent function f of the class $W(f,\Phi)$ such that $\Delta(\infty,f) > 0$ where $\Delta(a,f)$ is the Valiron defect of f in the point a, $\Phi(r)/\ln r \to \infty$, $r \to \infty$.*

Remark. In the case $\Phi(r) = \ln r$ the class $W(f,\Phi)$ coincides with that of exeptional in the sense of Julia functions. The functions of this class fail to have Valiron (and, therefore, Navanlinna) exeptional values.

[1] Zabolotskyĭ M.V. *A generalization of a theorem of Anderson and Clunie* // Matematychni Studiï. **1** (1991) 61–66.

PROBLEM 7 (M. ZABOLOTSKYĬ). It is proved in [1] that, if f_1, f_2 are meromorphic functions of zero genus with positive zeros and negative poles, $N(r,0,f_1) \geq N(r,0,f_2)$, $N(r,\infty,f_1) \geq N(r,\infty,f_2)$, then $T(r,f_1) \geq T(r,f_2)$. Here $N(r,a,f)$, $T(r,f)$ are the standard denotations from Nevanlinna theory.

Is the analogous statement valid for functions of higher growth category? In particular, for f_1, f_2 of order 1 and middle type.

More precisely, it can be formulated as follows. Find the least growth category of functions f_1 and f_2 for which the above mentioned statement is valid.

[1] Zabolotskyĭ M.V. *Some relationships for Nevanlinna characteristics of meromorphic functions of zero genus* // Ukr. mat. zh. 1981. V.33, № 6. P.805–810.