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EXTENDING METRICS IN COMPACT PAIRS

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A regular (= linear positive with unit norm) extension operator $T: C(X \times X) \to C(Y \times Y)$ preserving (pseudo)metrics is constructed for every pair (Y, X) of metric compacta such that $X \subset Y$ contains more than one point.

1. PREFACE.

This work is inspired by the articles of C. Bessaga [1] and T. Banakh [2] which deal with the question of extending metrics. It is proved that every metrizable compact pair admits a *regular* (i.e. linear, positive, and having norm equal to 1) operator extending metrics.

Further in this article all spaces are assumed to be metrizable compacta and the question of extending metrics concerns pairs (Y, X) where X consists of at least two points. $Metr(X)$ means the set of all admissible (=generating the original topology) metrics on X, $PMetr(X)$ is the set of all continuous pseudometrics on X.

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2. SQUEEZED JOIN CONSTRUCTION.

For the sake of completeness we include some definitions of [2]. For metric space (X, d) denote by $SJ(X)$ the quotient set $X \times X \times [0,1]/\sim$ (where $\sim$ stands for the equivalence relation generated by the set $A = \{((x,y,0),(x,y',0))|x,y,y' \in X\} \cup \{((x,y,1),(x',y,1))|\ x,x',y \in X\} \cup \{((x,x,t),(x,x,t'))|x \in X, t,t' \in [0,1]\}$). $SJ(X)$ is regarded as a metric space with the metric $sj(d)$:

$$sj(d)(u,v) = \min\{1-t, 1-t'\}d(x,x') + \max\{0, t'-t\}d(x,y') + \\ + \max\{0, t-t'\}d(y,x') + \min\{t,t'\}d(y,y').$$

X is naturally embeddable into $SJ(X)$ by mapping which sends x to $[x,x,0]$. For convenience, we always consider X as subset of $SJ(X)$ due to the above mentioned embedding.

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In obvious way by iteration we define the metric space $SJ^\infty(X)=(\bigcup_{n=1}^\infty SJ^n(X), sj^\infty(d))$.

It is proved that the topology of the space $(SJ^\infty(X), sj^\infty(d))$ does not depend on the choice of bounded admissible metric d on X and that the operator sj^∞ is a regular extension operator [2, Lemmas 1–9, Theorems 1 and 2].

3. The case of compact pair.

First of all, we will reduce the problem.

Lemma 1. *If for every pair (Y, X) where X is a continuum there exists a regular operator extending metrics, then such an operator exists for every pair.*

Proof. If we have pair (Y, X) we can create new pair (Y', X') by letting $X' = SJ(X)$ and $Y' = Y \bigcup_X SJ(X)$. The space Y' is metrizable, its topology can be induced for example by the metric

$$\bar{d}(z_1, z_2) = \begin{cases} sj(d)(z_1, z_2), & \text{if } z_1, z_2 \in SJ(X) \\ d(z_1, z_2), & \text{if } z_1, z_2 \in Y' \backslash SJ(X) \\ \inf\{sj(d)(z_1, x) + d(x, z_2) | x \in X\}, & \text{if } z_1 \in SJ(X), z_2 \in Y' \backslash SJ(X) \end{cases}$$

The easy verification is left to the reader. X' is a continuum since $SJ(X)$ is linearly connected: for every x_1,$x_2 \in SJ(X)$ there exist a,b,c,$d \in X$ such that $x_1 \in [a, b]_1$, $x_2 \in [c, d]_1$ and if we "move" from a to b by $[a, b]_1$, them from b to c by $[b, c]_1$, and from c to d by $[c, d]_1$ we will get in x_1 and x_2.

Let $T' : Metr(X') \to Metr(Y')$ be a regular extention operator. We define T by the formula: $T(d) = T'(sj(d))|_X$, $d \in Metr(X)$. Since T',sj, and a restriction operator are regular so is T. Lemma 1 is proved.

Lemma 2. *For every pair (Y, X), where X is a continuum in a compact metric space (Y, d) and for every $n \in \mathbb{N}$ there is a continuous map $f_n : Y \to SJ^\infty(X)$ such that $f_n|_X = 1_X$ and for arbitrary two points $y_1, y_2 \in Y$ such that $d(y_1, y_2) > 1/n$ $f_n(y_1) \neq f_n(y_2)$.*

Proof. Let $\{\lambda_w : Y\backslash X \to [0, 1] | w \in W\}$ be a countable locally finite partition of unity subordinate to the cover $\mathcal{V} = \{O_d(y, \min\{d(y, X)/2, 1/(2n+1)\} \mid y \in Y\backslash X\}$. Let $U_w = \lambda_w^{-1}((0, 1])$, $w \in W$, $\mathcal{U} = \{U_w | w \in W\} \in cov(Y\backslash X)$, and $N(\mathcal{U})$ be its nerve [3].

Define a map $g : Y\backslash X \to N(\mathcal{U})$ in the following way: $g(y) = \sum_{w\in W} \lambda_w(y) \cdot U_w$, $y \in Y\backslash X$. For every $w \in W$ find $a_w \in X$ such that $d(a_w, \lambda_w^{-1}(0, 1]) < 2d(X, \lambda_w^{-1}(0, 1])$ and all points a_w are different. This is possible, because X is a nondegenerate continuum. $K^{(n)}$ stands for n-skeleton of $N(\mathcal{U})$.

We shall construct inductively the map $h : N(\mathcal{U}) \to SJ^\infty(X)$ as follows: for $x \in K^{(0)}$ let $h(x) = a_w$, where $x = U_w \in \mathcal{U}$. Assume that the map h is defined on the $(k-1)$-skeleton $K^{(k-1)}$ and $h(K^{(k-1)}) \subset SJ^{k-1}(X)$. Let $\sigma \in K^{(k)}$ be a k-simplex and $b \in \sigma$ be its barycenter. Let $h(b) = h(v)$, where $v \in K^{(0)}$ is any of vertices of σ. Each point $x \in \sigma\backslash\{b\}$ can be expressed in unique way as $x = (1 - t)y + tb$ where $t \in [0, 1]$ and $y \in \partial\sigma$ ($\partial\sigma$ is the boundary of σ). Let $h(x) = [(h(y), h(b), t)]_k \in SJ^k(X)$.

Proceeding inductively, we shall construct the map $h : N(\mathcal{U}) \to SJ^\infty(X)$. Define the map $f_n : Y \to SJ^\infty(X)$ by the formula

$$f_n(y) = \begin{cases} y, & \text{if } y \in X \\ h \circ g(y), & \text{if } y \in Y \setminus X. \end{cases}$$

This constructrion is like that of T. Banakh [2, Lemma 10], except it differs slightly when we choose the cover $\mathcal{V}$. He gave the proof of continuity of f_n, which fits (even without any changes) here. The interested reader should see [2].

Let $y_1, y_2 \in Y$, $d(y_1, y_2) > 1/n$. Let σ_1 and σ_2 be symplexes containing $g(y_1)$ and $g(y_2)$ respectively. These symplexes do not have common vertices, otherwise there exist u_1, $u_2 \in \mathcal{U}$, such that $u_1 \cap u_2 \neq \varnothing$ and $y_1 \in u_1$, $y_2 \in u_2$. But this implies that the distance between y_1 and y_2 is less than $1/n$. Now we have to remark that $h(\sigma_1) \cap h(\sigma_2) = \varnothing$. For if $k = \max\{\dim(\sigma_1), \dim(\sigma_2)\}$ then first see that if $A, B \subset X$ are disjoint so are $SJ(A), SJ(B) \subset SJ(X)$, then $SJ^2(A), SJ^2(B) \subset SJ^2(X)$ and so on. At kth step we will receive the required property.

Hence $f(y_1) \neq f(y_2)$. Lemma is proved.

Remark. Additionally we may request that for arbitrary $x \in X$ and $y \in Y \setminus X$ either f_1 or f_2 separates x and y. Indeed, suppose that f_1 is already constructed and while processing we picked set $A_1 = \{a_w | w \in W_1\} \subset X$. Then, constructing f_2 we will choose such set $A_2 = \{a_w | w \in W_2\}$ that $A_1 \cap A_2 = \varnothing$. The point x cannot belong to both A_1 and A_2 at the same time. Suppose that $x \notin A_1$. Then $f_1(Y) \cap \{x\} = \varnothing$, what implies $f_1(y) \neq f_1(x)$.

Let $T_n : PMetr(X) \to PMetr(Y)$ $n \in \mathbb{N}$ be a sequence of regular extention operators. We can define the operator $T = \sum_{i=1}^{\infty} T_i/2^i$. It is clear that T is a linear, positive extention operator. Moreover, in assumptions stated above the following Lemma is true.

Lemma 3. *For every continuous pseudometric p on X the function $T(p)$ is a continuous pseudometric on Y. And $T : PMetr(X) \to PMetr(Y)$ is a regular extention operator.*

Proof. Obviously, $T(p)$ is a pseudometric. $T(p)$ is continuous as $|T(p)(y_1, y_2) - T(p)(z_1, z_2)| \leq |\sum_{i=1}^{N}(T_i(p)(y_1, y_2)/2^i - T_i(p)(z_1, z_2)/2^i)| + 2\sum_{i=N+1}^{\infty} \sup(p(Y \times Y))/2^i$, and both parts can be made smaller than arbitrary ε (by choosing the proper N first and then the suitable neighbourhood of (y_1, y_2)).

The norm of T is bounded by $\sum_{i=1}^{\infty} \|T_i\|/2 = 1$. Lemma is proved.

Theorem 1. *Every compact pair (Y, X) admits a regular operator extending metrics.*

Proof. Without loss of generality we assume that X is a continuum consisting of at least two points (see Lemma 1). For every $n \in \mathbb{N}$ we construct the mapping f_n described in Lemma 2. We additionally assume that f_1 and f_2 satisfy conditions of Remark to Lemma 2. The operator $T_n : Metr(X) \to PMetr(Y)$ is defined in the following way: $T_n(d)(y_1, y_2) = sj^{\infty}(d)(f_n(y_1), f_n(y_2))$. As a consequence of Lemma 3 we obtain that $T = \sum_{i=1}^{\infty} T_i/2^i$ is a regular operator.

Take two different points y_1 and y_2 in Y. If $y_1, y_2 \in X$ then for every $i \in \mathbb{N}$ holds $f_i(y_1) = y_1 \neq y_2 = f_i(y_2)$. If $y_1 \in Y \setminus X$ and $y_2 \in X$ then either $f_1(y_1) \neq f_1(y_2)$ or $f_2(y_1) \neq f_2(y_2)$. If $y_1, y_2 \in Y \setminus X$ then for $n > 1/d(y_1, y_2)$ $f_n(y_1) \neq f_n(y_2)$. In all the cases there exists f_i separating these two points. For $sj^{\infty}(d)$ is metric $T_i(d)(y_1, y_2) > 0$ as well as $T(d)(y_1, y_2) > 0$. Since the last inequality holds for arbitrary two points of Y, the the function $T(d)$ is a metric.

Since every continuous metrics on compact space generates the original topology, $T(d)$ is admissible. Theorem is proved.

References

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