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NEARSTANDARD OPERATORS

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In the paper [1] by means of standard filling notion a structure of nearstandardness has been introduced on the set $\mathbb{T}$ finite in the sense of E. Nelson's IST [4,5]. Now we extend this structure on linear operators acting in the space $\mathbb{C}^{\mathbb{T}}$. The examples related to the discrete differentiation, discrete analogues of the Fourier transform and the Riemann-Lebesgue lemma are studied.

This article extends the investigations that are presented in the paper [1]. In what follows we use the same terminology and notation as that of [1].

1. Condition <nst> of nearstandardness.

Let (X, d_X), (Y, d_Y) be standard metric spaces. We endow the Cartesian product $Z = X \times Y$ with the distance $d_Z(z_1, z_2) = [d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2]^{1/2}$ where $z_i = (x_i, y_i), i = 1, 2$. A mapping $F : X \longrightarrow Y$ is called **nearstandard** if the shadow* $^\circ(\mathit{graf}\ F)$ of its graph is the graph of some mapping. If F is nearstandard then its shadow (the mapping $^\circ F$) is defined by the condition

$$\mathit{graf}(^\circ F) = {}^\circ(\mathit{graf}\ F). \tag{1.1}$$

1.1. Let F be nearstandard. Since the set $\mathit{dom}\ (^\circ F)$ is standard, then it is uniquely defined by its standard elements. Denote**

$$\mathit{dom}_{nst} F := \{x \in {}^{nst}\mathit{dom} F : \ F(x) \in {}^{nst}Y\}. \tag{1.2}$$

From (1.1) we can conclude that

$$(\forall x \in {}^{st}X)\ (x \in \mathit{dom}(^\circ F) \Longleftrightarrow (\exists \tilde{x} \in \mathit{dom}_{nst} F)(x \approx \tilde{x})). \tag{1.3}$$

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*Recall that the **shadow** $^\circ M$ of a set M in a standard metric space is defined as the *standard* set whose standard elements are *infinitesimally* close to M. Existence and uniqueness of the shadow $^\circ M$ are provided by the standartization principle .

**^{nst}E is the collection of all nearstandard $x \in E$.

1.2. Since $^{\circ}F$ is standard , then it is uniquely defined by its values at standard elements. We conclude directly from (1.1) that

$$(\forall x \in {}^{st}dom(^{\circ}F))\ (\forall \tilde{x} \in dom_{nst}F)\,(x \approx \tilde{x} \implies (^{\circ}F)(x) = {}^{\circ}(F(\tilde{x}))). \tag{1.4}$$

1.3. We say that a mapping F satisfies the <nst> condition (i.e. the shadow existence condition) if

$$(\forall x_1, x_2 \in dom_{nst}F)\ (x_1 \approx x_2 \implies F(x_1) \approx F(x_2)). \tag{1.5}$$

By direct check we can see that F is nearstandard **if and only if** it satisfies the <nst> condition. A simple consequence is

1.4. Any restriction of a nearstandard mapping is nearstandard.

However,

1.5. A standard mapping is not necessarily nearstandard. For example, if a function $F : \mathbb{R} \to \mathbb{R}$ has a discontinuity of first kind at some point $x_0 \in {}^{st}domF$, then $^{\circ}(graf\ F)$ contains different points with the same coordinate x_0.

It is easy to see, if a standard mapping F is nearstandard then it is the restriction of its shadow : $F \subseteq {}^{\circ}F$.

1.6. The notion of the **standartization** ^{S}F of a mapping F resembles the notion of shadow, but differs from it. Assume that $\forall x \in {}^{st}domF\ F(x) \in {}^{nst}Y$. Then we define ^{S}F as a standard mapping such that $(\forall x \in {}^{st}domF)\,(^{S}F)(x) = {}^{\circ}[F(x)]$. Existence and uniqueness of the mapping ^{S}F are again provided by the standartization principle.

2. The case of a linear operator.

Let X, Y be standard Banach spaces, T be a linear mapping $X \longrightarrow Y$. For $F = T$ the <nst> condition, as it is easy to see, has the following form:

$$(\forall x \in dom_{nst}T)\ (x \approx 0 \implies Tx \approx 0), \tag{2.1}$$

and the formulae describing the shadow $^{\circ}T$ of the nearstandard operator T as follows:

$$^{st}dom(^{\circ}T) = \{x \in {}^{st}X :\ (\exists \tilde{x} \in dom_{nst}T)(x \approx \tilde{x})\}, \tag{2.2}$$

$$(\forall x \in dom_{nst}T)\ (^{\circ}x \in dom(^{\circ}T) \wedge (^{\circ}T)x = {}^{\circ}(Tx)).$$

2.1. The <nst> condition (2.1) is somewhat similar to the closability condition for a linear operator. Therefore it is not surprising that the shadow $^{\circ}T$ of a nearstandard operator T is a **closed** operator.

□ The shadow of an internal set is closed (see for example, [2]). In particular, the shadow of a graph is closed ■

2.2. A standard closable linear operator is nearstandard and its shadow coincides with its closure. In particular, a standard closed linear operator is nearstandard, and coincides with its shadow.

□ The shadow of a standard set coincides with its closure ■

The next proposition shows that nearstandardness of an operator without imposing further restrictions is not too sapid notion.

2.3. Let T be an operator belonging to the Banach space $\mathcal{B}(X;Y)$ of bounded linear operators in Y^X. If $\|T\| \ll \infty$ then T is nearstandard ,

$$\forall x \in {}^{st}X \quad x \in dom(^\circ T) \Longleftrightarrow Tx \in^{nst} Y, \tag{2.4}$$

$$\forall x \in {}^{st}dom(^\circ T) \quad (^\circ T)x = {}^\circ(Tx), \tag{2.5}$$

$$\forall x \in dom(^\circ T) \quad \|(^\circ T)x\| \leq {}^\circ\|T\| \cdot \|x\|. \tag{2.6}$$

□ It follows from $\|T\| \ll \infty$ that $Tx \approx 0$ for $x \approx 0$, hence, the <nst> condition is satisfied. Let $x \in {}^{st}dom(^\circ T)$ so that $x \approx \tilde{x}$ for some $\tilde{x} \in X$ such that $T\tilde{x} \in^{nst} Y$ with $(^\circ T)x \approx T\tilde{x}$. As $\|T\| \ll \infty$, then $T\tilde{x} \approx Tx$; therefore $Tx \in {}^{nst}Y$ and (2.5) is satisfied. One can prove the rest in the same simple way. For example, let $x \in {}^{st}dom(^\circ T)$; then $\|(^\circ T)x\| \approx \|^\circ(Tx)\| = {}^\circ\|Tx\| \leq {}^\circ\|T\| \cdot \|x\|$. By the transfer principle, (2.6) is satisfied ■

2.4. the shadow of an operator $T \in \mathcal{B}(X;Y)$, with the norm $\|T\| \ll \infty$, can have the domain of definition which is not dense in X. For example, let $X = Y = H$ be a standard Hilbert space, $(e_n)_{n\in\mathbb{N}}$ be a standard orthonormal basis in H and $\omega \in \mathbb{N} \setminus {}^{st}\mathbb{N}$. Assigning $\forall x \in H \ Tx = \sum\limits_{k\leq\omega} (x|e_k)e_{k+\omega}$, we find $T \in \mathcal{B}(H)$, $\|T\| = 1$. If $x \in^{st} H \setminus \{0\}$ then $\sum\limits_{k\leq\omega} |(x|e_k)|^2 \approx \|x\|^2 \gg 0$. Therefore the vector Tx does not satisfy the nearstandardness condition : $\sum\limits_{n\in\mathbb{N}} c_n e_n \in^{nst} H \Longleftrightarrow (\forall \omega \in \mathbb{N} \setminus {}^{st}\mathbb{N}) \sum\limits_{n>\omega} |c_n|^2 \approx 0$. In such a manner, $(\forall x \in {}^{st}dom(^\circ T))\ (x = 0)$. By the transfer principle, $dom(^\circ T) = \{0\}$ ■

In the natural way we introduce the notions of **weak**, **strong** and **uniform** nearstandardness for operators in $\mathcal{B}(X;Y)$. For example, an operator $T \in \mathcal{B}(X;Y)$ is strong nearstandard if $\|T\| \ll \infty$ and there exists an operator $T_0 \in^{st} \mathcal{B}(X;Y)$ such that

$$\forall x \in^{st} X \quad \|(T - T_0)x\| \approx 0. \tag{2.7}$$

The condition $\|T\| \ll \infty$ ensures nearstandardness (in the sense of the definition by means of $graf\ T$) of a strong nearstandard operator T. It is easy to see, the operators $^\circ T$ and T_0 satisfying the (1.1) and (2.7) correspondingly are equal. The next sufficient condition holds:

2.5. Let $T \in \mathcal{B}(X;Y)$ and $\|T\| \ll \infty$. Assume that the shadow $^\circ T$ (in the sense of (1.1)) is densely determined. Then $^\circ T \in \mathcal{B}(X;Y)$, in particular, $dom(^\circ T) = X$ and T is strong nearstandard.

□ According to (2.6), the operator $^\circ T$ is closed ■

2.6. The condition $\|T\| \ll \infty$ in the definition of strong nearstandardness is essential. For example, let $X = Y = H$ be a standard Hilbert space and H_0 be a finite dimensional (in the sense of IST) subspace in H such that ${}^{st}H \subset H_0$ (s. for example, [3]). Consider orts e, f such that $e \in {}^{st}H$, $f \in H_0^\perp$ and a number $\omega \in \mathbb{N} \setminus {}^{st}\mathbb{N}$. Put $\forall x \in H \quad Tx = \omega(x|f)e$. Denote $x := \omega^{-1}f$; then $x \approx 0$ and $Tx = e \not\approx 0$. Consequently, T is not nearstandard. At the same time the condition (2.7) is satisfied with $T_0 = 0$.

2.7. Obviously, uniform nearstandardness (defined by the condition $^\circ T \in^{st} \mathcal{B}(X;Y)$,
$\|T -^\circ T\| \approx 0$) gives rise to strong nearstandardness. For a uniform nearstandard operator $\|^\circ T\| = {}^\circ\|T\|$, but in the same time one can state only $\|^\circ T\| \leq \|T\|$

for a strong nearstandard operator T. An operator adjoint to a strong nearstandard operator is not necessarily strong nearstandard. However, if T^* is strong nearstandard then $({}^\circ T)^* = {}^\circ(T^*)$. For a uniform nearstandard operator the latest equality is always satisfied. By the same token, if T is uniform nearstandard, $kerT^{-1} = \{0\}$, $imT = Y$ and $\|T^{-1}\| \ll \infty$, then T^{-1} is uniform nearstandard and $({}^\circ T)^{-1} = {}^\circ(T^{-1})$. We do not dwell on other ordinary relations between operations on operators and the operation of passing to limit.

3. Nearstandardness on $\mathcal{B}(\mathbb{H})$.

Let $\mathbb{T}$ be a finite set***, $(\mathbf{T}, Q, \lambda)$ be a standard filling of $\mathbb{T}$. Earlier (s. [1, c.2]) we assumed that $\forall t \in \mathbb{T} \quad \lambda Qt = h = const$. Replace this condition by the next one: $\forall t \in \mathbb{T} \quad \lambda Qt > 0$. In order to preserve all the statements fofmerly stated, we revise the formula [1,(2.9)], putting for each charge ν given on $\mathbb{T}$ and for each $\mathcal{E} \in \Lambda$ (Λ be a $\sigma-$algebra of measurable (with respect to a standard $\sigma-$additive measure λ) sets $\mathcal{E} \subseteq \mathbf{T}$ on the standard set $\mathbf{T}$):

$$Q\nu\mathcal{E} := \sum_{t\in\mathbb{T}} \lambda_{\mathcal{E}}(t)\nu_t, \quad \lambda_{\mathcal{E}}(t) := \lambda(\mathcal{E} \cap Qt)(\lambda Qt)^{-1}. \tag{3.1}$$

Restrict to case $\nu := \Pi\lambda$ (Π is an inductor at charges; cf. [1,(2.11)]) and denote by $\mathbb{H}$ the linear space $\mathbb{C}^{\mathbb{T}}$ with the inner product $(x|y) = \sum_{t\in\mathbb{T}} x(t)\overline{y(t)}\nu_t$. Since the algebra $\mathcal{B}(\mathbb{H})$ is nonstandard, we define nearstandardness on it indirectly.

3.1. *Embedding* $\mathfrak{Q}$ and *inductor* $\mathfrak{P}$. To each operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ we put in correspondence the operator $\mathfrak{Q}\mathbb{A} := Q\mathbb{A}\Pi$, and to each operator $\mathbf{A} \in \mathcal{B}(\mathbf{H})$ ($\mathbf{H}$ be the standard Hilbert space $L_2(\mathbf{T}, \Lambda, \lambda)$) we put the operator $\mathfrak{P}\mathbf{A} := \Pi\mathbf{A}Q$. From the properties of the embedding $Q : \mathbb{H} \longrightarrow \mathbf{H}$ and the inductor $\Pi : \mathbf{H} \longrightarrow \mathbb{H}$ (s. [1, c.3]) it follows that

3.2. The embedding $\mathfrak{Q}$ is a contractive injective mapping $\mathcal{B}(\mathbb{H}) \longrightarrow \mathcal{B}(\mathbf{H})$. The inductor $\mathfrak{P}$ is left inverse to the embedding $\mathfrak{Q}$.

Note that $\forall \mathbf{A} \in \mathcal{B}(\mathbf{H}) \ \forall x, y \in \mathbb{H}$

$$(\mathfrak{P}\mathbf{A}x|y) = (\mathbf{A}Qx|Qy).$$

3.3. We define the *quasikernel of the inductor* $\mathfrak{P}$ by the formula $qker\mathfrak{P} := \{\mathbf{A} \in \mathcal{B}(\mathbf{H}) : \|\mathfrak{P}\mathbf{A}\| \approx 0\}$. It is said to be *trivial* if ${}^{st}qker\mathfrak{P} = \{0\}$. Assigning to an operator $\mathbf{A} \in \mathcal{B}(\mathbf{H})$ the matrix $\mathbf{A}(t, s) := (\mathbf{A}Q\delta_s|Q\delta_t)$, $t, s \in \mathbb{T}$ (δ_t is the discrete Dirac delta concentrated at the point $t \in \mathbb{T}$), we can conclude that $qker\ \mathfrak{P}$ is trivial iff

$$(\forall \mathbf{A} \in {}^{st}\mathcal{B}(\mathbf{H}))\ (\forall t, s \in \mathbb{T})\ (\mathbf{A}(t, s) \approx 0 \implies \mathbf{A} = 0). \tag{3.2}$$

In the case $qker\mathfrak{P}$ is trivial, we introduce the following definition.

3.4. An operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ is called **standard** if there exists an operator $\mathbf{A} \in {}^{st}\mathcal{B}(\mathbf{H})$ such that $\mathbb{A} = \mathfrak{P}\mathbf{A}$. In this case $\mathbf{A}$ is called the **standardized image** of the operator $\mathbb{A}$. The triviality of $qker\mathfrak{P}$ provides the uniqueness of the standardized image.

3.5. We call an operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ **nearstandard** if the operator $\mathfrak{Q}\mathbb{A}$ is nearstandard (in the sense of c.1). In this case the shadow ${}^\circ(\mathfrak{Q}\mathbb{A})$ is also called the

***We are interested first and foremost in the case $card\ T \in \mathbb{N} \setminus {}^{st}\mathbb{N}$.

shadow of the operator $\mathbb{A}$ and is denoted by $^\circ\mathbb{A}$. It is not difficult to reformulate the <nst> condition. We define for an operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$

$$dom_{nst}\mathbb{A} := \{x \in {}^{nst}\mathbb{H} : \ \mathbb{A}x \in {}^{nst}\mathbb{H}\}. \tag{3.2}$$

Then $\mathbb{A}$ satisfies the condition <nst> of nearstandardness if

$$\forall x \in dom_{nst}\mathbb{A} \quad (x \approx 0 \implies \mathbb{A}x \approx 0). \tag{3.3}$$

If $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ then $^\circ\mathbb{A}$ is a standard closed operator $\mathbf{H} \longrightarrow \mathbf{H}$, with

$$(\forall \xi \in {}^{st}\mathbf{H})(\xi \in dom^\circ\mathbb{A}) \Longleftrightarrow (\exists x \in dom_{nst}\mathbb{A})(\xi =^\circ x) \tag{3.4}$$

and moreover,

$$(\forall \xi \in {}^{st}dom(^\circ\mathbb{A}))(\forall x \in dom_{nst}\mathbb{A})(\xi \approx^\circ x \implies (^\circ\mathbb{A})\xi =^\circ (\mathbb{A}x)). \tag{3.5}$$

The notions of weak, strong and uniform nearstandardness gain grounds in an obvious way for operators $\mathbb{A} \in \mathcal{B}(\mathbb{H})$. For example, a nearstandard operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ is **strong nearstandard** if $^\circ\mathbb{A} \in^{st} \mathcal{B}(\mathbb{H})$ and

$$\forall \xi \in^{st} \mathbf{H} \quad \|(\mathfrak{Q}\mathbb{A} -^\circ \mathbb{A})\xi\| \approx 0.$$

Now we proceed to several examples which, as we hope, will convince us that the constructions related to standard filling notion are sensible.

4. $2\ell-$periodic discrete axis.

Consider the finite set

$$\mathbb{T} = \overset{\circ\!-}{2\ell} = \{-\ell, -\ell + h, ...0, ..., \ell - h\} \tag{4.1}$$

where $\ell \in \mathbb{R}, h \in \mathbb{R}, \ell > 0, h > 0$ and $h \approx 0$. This set interpreted as an additive group, with the usual addition $mod\ 2\ell$, is called the 2ℓ-periodic discrete axis ; in particular, $(\ell - h) + h = -\ell$. Assume that the number $\omega := \frac{1}{2}card\mathbb{T} = \ell/h$ is of the form $\omega = \omega_0!$, where $\omega_0 \in \mathbb{N} \setminus {}^{st}\mathbb{N}$. For simplicity in the case $\ell \ll \infty$ we consider $\ell \in^{st} \mathbb{R}$. When $\ell \ll \infty$ we put $\mathbf{T} = [-\ell, \ell]$, and if $\ell \approx \infty$ we define $\mathbf{T} = \mathbb{R}$. For λ we take the standard Lebesgue measure on $\mathbf{T}$, and we define the embedding $Q : \mathbb{T} \longrightarrow \Lambda$ (Λ be an algebra of $\lambda-$measurable sets $\mathcal{E} \subset \mathbf{T}$) by the condition $\forall t \in \mathbb{T}\ Qt = [t, t+h[$. In the same way as in [1,2.2.3] and [1,3.3.3] we can convince, if such a choice of the standard filling $(\mathbf{T}, Q, \lambda)$ of the 2ℓ-periodic discrete axis, then the inductor $\Pi : \mathbf{H} \longrightarrow \mathbb{H}$ is exact and the projector $P : \mathbf{H} \longrightarrow Q\mathbb{H}$ is a quasi-unity of the algebra $\mathcal{B}(\mathbf{H})$. In this case, $\mathbb{H}$ is the linear space $\mathbb{C}^{\mathbb{T}}$ with the inner product $(x|y) = \sum_{t\in\mathbb{T}} x(t)\overline{y(t)}h$ and $\mathbf{H}$ is the standard Hilbert space $L_2(\mathbf{T})$.

4.1. Define the **shift** U on the space $\mathbb{H}$ by the formula : $\forall x \in \mathbb{H}\ \forall t \in \mathbb{T}\ Ux(t) = x(t + h)$, in particular, $Ux(\ell - h) = x(-\ell)$. Obviously, U maps $\mathbb{H}$ unitarily onto $\mathbb{H}$. Its eigenvalues are the $\zeta-$roots of the equation $\zeta^{2\omega} = 1$, so the spectrum of U is the set $\sigma(U) = \{e^{-i\pi\frac{k}{\omega}}\}_{k=-\omega}^{\omega-1}$. To the eigenvalue ζ there corresponds an eigenfunction $e(\cdot, \zeta)$, where

$$\forall t \in \mathbb{T} \quad e(t, \zeta) = \zeta^{t/h}. \tag{4.2}$$

4.2. *Discrete Fourier transform.* Denote

$$\hat{h} := \frac{\pi}{\ell},\ \hat{\ell} := \omega\hat{h},\ \hat{\mathbb{T}} := \{-\hat{\ell}, -\hat{\ell}+\hat{h}, ..., 0, ..., \hat{\ell}-\hat{h}\}. \tag{4.3}$$

Note that the step $\hat{h}$ is infinitesimal only if $\ell \approx +\infty$. The linear space $\mathbb{C}^{\hat{\mathbb{T}}}$, with the inner product $(\hat{x}|\hat{y}) = \sum_{\hat{t}\in\hat{\mathbb{T}}} \hat{x}(\hat{t})\overline{\hat{y}(\hat{t})}\hat{h}$, is denoted by $\hat{\mathbb{H}}$. Let

$$\forall x \in \mathbb{H} \quad \forall \hat{t} \in \hat{\mathbb{T}} \quad \mathfrak{E}x(\hat{t}) := \frac{1}{\sqrt{2\pi}} \sum_{t\in\mathbb{T}} x(t)e^{-i\hat{t}t}h. \tag{4.4}$$

4.2.1. The discrete Fourier transform $\mathfrak{E}$ is a unitary mapping of $\mathbb{H}$ onto $\hat{\mathbb{H}}$, with the inversion formula

$$\forall \hat{x} \in \hat{\mathbb{H}} \quad \forall t \in \mathbb{T} \quad \mathfrak{E}^{-1}\hat{x}(t) = \frac{1}{\sqrt{2\pi}} \sum_{\hat{t}\in\hat{\mathbb{T}}} \hat{x}(\hat{t})e^{i\hat{t}t}\hat{h}. \tag{4.5}$$

□ The proof is evident, but it demonstrates usefulness of the employed notation. Mark the eigenvalues of the shift U with points $\hat{t} = k\hat{h},\ k \in \{-\omega, ..., \omega-1\}$:

$$\zeta_{\hat{t}} = e^{i\hat{t}h} = e^{ik\hat{h}h} = e^{ik\frac{\pi}{\omega}}. \tag{4.6}$$

The eigenfunction of the shift U corresponding to $\zeta_{\hat{t}}$ is $e_{\hat{t}}$, where (s. (4.2))

$$e_{\hat{t}}(t) := e(t, \zeta_{\hat{t}}) = (\zeta_{\hat{t}})^{t/h} = e^{i\hat{t}t}. \tag{4.7}$$

Taking it into account that $|\zeta| = 1 \implies \|e(\cdot,\zeta)\| = \sqrt{2\ell}$ and U is a unitary operator, we can conclude that

$$\left(\frac{1}{\sqrt{2\ell}} e_{\hat{t}}\right)_{\hat{t}\in\hat{\mathbb{H}}} \tag{4.8}$$

is an orthonormal basis for the space $\mathbb{H}$. The corresponding Parseval's equality can be written as

$$\forall x, y \in \mathbb{H} \quad (x|y)_{\mathbb{H}} = (\mathfrak{E}x|\mathfrak{E}y)_{\hat{\mathbb{H}}} \ ■ \tag{4.9}$$

4.3. *Discrete Riemann-Lebesgue lemma.* By the formula

$$D := \frac{1}{ih}(U - \mathbb{I}_{\mathbb{H}}) \tag{4.10}$$

we define the discrete differentiation operator that is given at functions $x \in \mathbb{C}^{\mathbb{T}}$, where $\mathbb{T}$ is the 2ℓ-periodic discrete axis. This operator is normal, but not self-adjoint: $D^* = DU^{-1}$. It has the spectrum

$$\sigma(D) = \{\lambda_{\hat{t}}\}_{\hat{t}\in\hat{\mathbb{T}}}, \quad \lambda_{\hat{t}} := \frac{1}{ih}(e^{i\hat{t}h} - 1). \tag{4.11}$$

Evidently, this spectrum lies in the circle with radius $\frac{1}{h} \approx +\infty$ and centre at the point $\frac{i}{h}$. Note that shadow of the circle is the real axis. From (4.11) and the trigonometric inequality

$$\forall \sigma \in [-\pi, +\pi] \quad \frac{2}{\pi}|\sigma| \le |e^{i\sigma} - 1| \le |\sigma| \tag{4.12}$$

it follows that the next estimation for the eigenvalues $\lambda_{\hat{t}}$ of the operator D is valid:

$$\forall \hat{t} \in \hat{\mathbb{T}} \quad \frac{2}{\pi}|\hat{t}| \leq |\lambda_{\hat{t}}| \leq |\hat{t}|. \tag{4.13}$$

For $x \in \mathbb{C}^{\mathbb{T}}$ and $\xi \in L_1(\mathbf{T})$ we denote

$$\|x\|_1 := \sum_{t \in \mathbb{T}} |x(t)|h, \quad \|\xi\|_1 = \int_{\mathbf{T}} |\xi(\tau)|\, d\tau. \tag{4.14}$$

It is easy to see, the embedding $Q: \ \mathbb{C}^{\mathbb{T}} \to L_1(\mathbf{T})$ is isometric with regard to the norms (4.14).

4.3.1. Let $x \in \mathbb{C}^{\mathbb{T}}$ and $\|Dx\|_1 \ll \infty$; then if $|\hat{t}| \approx +\infty$ then

$$(x|e_{\hat{t}}) \approx 0. \tag{4.15}$$

□ From $(Dx|e_{\hat{t}}) = \lambda_{\hat{t}}(x|e_{\hat{t}})$ and $|e_{\hat{t}}(t)| \equiv 1$ it follows that $|\lambda_{\hat{t}}| \cdot |(x|e_{\hat{t}})| \leq \|Dx\|_1$, hence by (4.13)

$$|(x|e_{\hat{t}})| \leq \frac{\pi\|Dx\|_1}{2|\lambda_{\hat{t}}|} \quad \blacksquare \tag{4.15'}$$

The next statement is an analogue of the classic Riemann-Lebesgue lemma.

4.3.2. Let $x \in^{nst} \mathbb{C}^{\mathbb{T}}$ in the sense that $\|x - \Pi\xi\|_1 \approx 0$ for some function $\xi \in^{st} L_1(\mathbf{T})$. Then if $|\hat{t}| \approx +\infty$ then 4.15 holds.

First we prove two auxiliary statements.

4.3.3. (Relation between discrete and ordinary derivative) Let $\eta \in^{st} \mathcal{C}_0^{(2)}(\mathbf{T})$ and $y := \Pi\eta$; then

$$\|Dy - \frac{1}{i}\Pi\frac{d\eta}{d\tau}\|_1 \approx 0. \tag{4.16}$$

□ From the definitions of the operators D and Π it follows that $Dy(t) - \frac{1}{ih}\Pi\eta'(t) = \frac{1}{ih^2}\int\limits_t^{t+h}[\eta(\tau+h)-\eta(\tau)]d\tau - \frac{1}{ih}\int\limits_t^{t+h}\eta'(\sigma)\,d\sigma = \frac{1}{ih^2}\int\limits_t^{t+h}d\tau\left[\int\limits_\tau^{\tau+h}\eta'(\sigma)d\sigma - \int\limits_t^{t+h}\eta'(\sigma)\,d\sigma\right]$ $= \frac{1}{ih^2}\int\limits_t^{t+h}d\tau\int\limits_t^{t+h}[\eta'(\sigma+\tau-t) - \eta'(\sigma)]\,d\sigma = \frac{1}{ih^2}\int\limits_t^{t+h}d\tau\int\limits_t^{t+h}d\sigma\int\limits_\sigma^{t+h}\eta''(\rho)\,d\rho$. Consequently,

$$\forall\ t \in \mathbb{T} \quad |Dy(t) - \frac{1}{i}\Pi\eta'(t)| \leq \int\limits_t^{t+h}|\eta''(\rho)|\,d\rho. \tag{4.16'}$$

Therefore

$$\|Dy - \frac{1}{i}\Pi\eta'\|_1 \leq h\|\eta''\|_1 \tag{4.16''}$$

and, in particular, (4.16) holds $\blacksquare$

4.3.4. (*Density lemma.*) Let $x \in^{\mathrm{nst}} \mathbb{C}^{\mathbb{T}}$ (see (4.3.2)); then there exists a sequence of functions $y_n \in \mathbb{C}^{\mathbb{T}}$ such that

$$\forall n \in^{st} \mathbb{N} \quad \|Dy_n\|_1 \ll \infty \ \wedge\ \|x - y_n\|_1 < \frac{1}{n}. \tag{4.17}$$

□ Denote by ξ such standard function in $L_1(\mathbf{T})$ that $\|x-\Pi\xi\|_1 \approx 0$. Construct a *standard* sequence of functions $\eta_n \in \mathcal{C}_0^{(2)}(\mathbf{T})$ such that $\forall n \in \mathbb{N} \quad \|\xi-\eta_n\|_1 \leq \frac{1}{2n}$. Denote $y_n := \Pi\eta_n$. Since the embedding Q is isometric and $Q\Pi = P$ is a quasi-unity, then $\|x-y_n\| = \|Qx - Q\Pi\eta_n\| \approx \|\xi - \eta_n\|$, therefore $\|x-y_n\| < \frac{1}{n}$. Now we verify that $\forall n \in^{st} \mathbb{N} \quad \|Dy_n\| \ll \infty$. We have $\|Dy_n\|_1 \leq \|Dy_n - \frac{1}{i}\Pi\eta_n'\|_1 + \|\Pi\eta_n'\|$. By (4.16) the first addend is infinitesimal. Since $\forall\, n \in^{st} \mathbb{N}$ the function η_n' is standard, then $\|\Pi\eta_n'\|_1 = \sum\limits_{t\in\mathbb{T}} |\frac{1}{h}\int\limits_t^{t+h} \eta'(\sigma)\,d\sigma| \cdot h \leq \|\eta'\|_1 < \infty$ ■

The proof of the proposition 4.3.2.

□ Let $x \in^{nst} \mathbb{C}^{\mathbb{T}}$, and $(y_n)_{n\in\mathbb{N}}$ be the sequence constructed in lemma 4.3.4. We have $(x|e_{\hat{t}}) = (y_n|e_{\hat{t}}) + (x-y_n|e_{\hat{t}})$ and according to (4.15') and (4.17) $|(x|e_{\hat{t}})| \leq \frac{1}{n} + \frac{\pi\|Dy_n\|_1}{2|\hat{t}|}$, therefore $\forall n \in^{st} \mathbb{N} \quad |(x|e_{\hat{t}})| \leq \frac{1}{n} + \frac{1}{\sqrt{\hat{t}}}$, i.e. (4.15) holds ■

For the future references it is useful to consider another proof of the proposition 4.3.2 assuming $\ell \ll \infty$.

□ Define

$$\forall \hat{t} \in \hat{\mathbb{T}} \quad \forall \tau \in T \quad e_{\hat{t}}(\tau) = e^{i\hat{t}\tau}. \tag{4.18}$$

Then $\forall t \in \mathbb{T} \quad \Pi e_{\hat{t}}(t) = \frac{1}{h}\int\limits_t^{t+h} e^{i\hat{t}\tau} d\tau = \frac{e^{i\hat{t}h}-1}{i\hat{t}h} e^{i\hat{t}t}$. Thus

$$\forall \hat{t} \in \hat{\mathbb{T}} \quad e_{\hat{t}} = \gamma_{\hat{t}} \Pi \epsilon_{\hat{t}}, \text{ where } \gamma_{\hat{t}} := \frac{i\hat{t}h}{e^{i\hat{t}h}-1}. \tag{4.19}$$

From (4.12) it follows that

$$\forall \hat{t} \in \hat{\mathbb{T}} \quad 1 \leq |\gamma_{\hat{t}}| \leq \frac{\pi}{2}. \tag{4.20}$$

Consider an arbitrary function $x \in^{nst} \mathbb{C}^{\mathbb{T}}$. Let $\xi \in^{st} L_1(\mathbf{T})$ and $\|Qx - \xi\|_1 \approx 0$. Considering the isometrisity of Q and the equalities $Q\Pi = P, \quad P^* = P, \quad PQ = Q$, by (4.19) we find $(x|e_{\hat{t}}) = (Qx|Qe_{\hat{t}}) = \overline{\gamma_{\hat{t}}}(Qx|P\epsilon_{\hat{t}}) = \overline{\gamma_{\hat{t}}}(Qx|\epsilon_{\hat{t}})$, hence

$$(x|e_{\hat{t}}) \approx \overline{\gamma_{\hat{t}}}(\xi|\epsilon_{\hat{t}}). \tag{4.21}$$

But according to the classic Riemann-Lebesgue lemma and to the esimation (4.20) the equality (4.15) holds ■

5. Nearstandardness criterion.

Let $\ell \in^{st} \mathbb{R}$, hence $\mathbf{H} = L_2(\mathbf{T}) \subseteq L_1(\mathbf{T})$. We restrict the study to this case. Formulate the criterion for nearstandardness in terms of the discrete Fourier transform .

5.1. Let $\ell \in^{st} \mathbb{R},\ x \in \mathbb{H}$; then $x \in {}^{nst}\mathbb{H}$ iff (cf. with [3]) $\|x\| \ll \infty$ and

$$\forall n \in \mathbb{N} \quad n \approx +\infty \implies \sum_{|\hat{t}|>n} |(x|e_{\hat{t}})|^2 \approx 0. \tag{5.1}$$

□ Since $\ell \in^{st} \mathbb{R}$, $\mathbf{H} := L_2(-\ell,\ell)$ is standard. Assume that $x \in {}^{nst}\mathbb{H}$, i.e. $Qx \in^{nst} \mathbf{H}$. Denote $\forall \hat{t} \in \hat{h}\mathbb{N} \quad (Qx)_{\hat{t}} = \frac{1}{2\ell}\sum\limits_{|\hat{s}|>\hat{t}} (Qx|\epsilon_{\hat{s}})\epsilon_{\hat{s}}$. Since $(\frac{1}{\sqrt{2\ell}}\epsilon_{\hat{t}})_{\hat{t}\in\hat{h}\mathbb{Z}}$ is a standard orthonormal basis in $\mathbf{H}$, we have $(Qx)_{\hat{t}} \approx 0$ when $\hat{t} \approx +\infty$. However,

$$\|(Qx)_{\hat{t}}\|^2 = \frac{1}{2\ell}\sum_{|\hat{s}|>\hat{t}} |(Qx|\epsilon_{\hat{s}})|^2 = \frac{1}{2\ell}\sum_{|\hat{s}|>\hat{t}} |\gamma_{\hat{s}}|^{-2}|(x|e_{\hat{s}})|^2 \geq \frac{1}{\pi\ell}\sum_{|\hat{s}|>\hat{t}} |(x|e_{\hat{s}})|^2, \tag{5.2}$$

because $(Qx|\epsilon_{\hat{s}}) = (x|\Pi\epsilon_{\hat{s}})$, $\Pi\epsilon_{\hat{s}} = \overline{\gamma}_{\hat{s}}^{-1}e_{\hat{s}}$ and $\overline{\gamma}_{\hat{s}}^{-1} \geq \frac{2}{\pi}$.

Conversely, let $\|x\| \ll \infty$ and (5.1) holds. Note that $|\frac{1}{\overline{\gamma}_{\hat{t}}}(x|e_{\hat{t}})| \ll \infty$ and denote by $(c_{\hat{t}})_{\hat{t}\in\hat{h}\mathbb{Z}}$ *the standard extension* of the sequence $({}^{\circ}(\frac{1}{\overline{\gamma}_{\hat{t}}}(x|e_{\hat{t}}))_{\hat{t}\in^{st}(\hat{h}\mathbb{Z})}$. From (4.20) and (4.21) it follows that $\forall n \in^{st} \mathbb{N}$ $\sum\limits_{|\hat{t}|\leq n} |c_{\hat{t}}|^2 \leq \sum\limits_{|\hat{t}|\leq n} |(x|e_{\hat{t}})|^2 + 1 \leq 2\ell\|x\|^2 + 1$. Consequently, the (standard) series $\sum\limits_{\hat{t}\in\hat{h}\mathbb{Z}} |c_{\hat{t}}|^2$ converges. Denote $\xi := \sum\limits_{\hat{t}\in\hat{h}\mathbb{Z}} c_{\hat{t}}\epsilon_{\hat{t}}$. Then $\xi \in^{st} \mathbf{H}$ and $\forall n \in \mathbb{N}$ $\|Qx - \xi\| \leq (\frac{1}{2\ell}\sum\limits_{|\hat{t}|\leq n} |(Qx-\xi|\epsilon_{\hat{t}})|^2)^{1/2} +$
$(\frac{1}{2\ell}\sum\limits_{|\hat{t}|>n} |(Qx|\epsilon_{\hat{t}})|^2)^{1/2} +$
$(\frac{1}{2\ell}\sum\limits_{|\hat{t}|>n} |(\xi|\epsilon_{\hat{t}})|^2)^{1/2}$. Hence it follows $\|Qx - \xi\| \leq (\frac{1}{2\ell}\sum\limits_{|\hat{t}|\leq n} |(\overline{\gamma_{\hat{t}}}(x|e_{\hat{t}}) - c_{\hat{t}}|^2)^{1/2} +$
$(\frac{1}{2\ell}\sum\limits_{|\hat{t}|>n} |\frac{1}{\overline{\gamma}_{\hat{t}}}(x|e_{\hat{t}})|^2)^{1/2} + (\frac{1}{2\ell}\sum\limits_{|\hat{t}|>n} |c_{\hat{t}}|^2)^{1/2}$. With $n \in^{st} \mathbb{N}$ the first right-hand addend is infinitesimal, by (4.21). By the Robinson lemma, it is still infinitesimal for some $n \approx +\infty$. For such n the second addend is infinitesimal by (5.1); and the third is infinitesimal due to convergence of the standard series $\sum\limits_{\hat{t}\in\hat{h}\mathbb{Z}} |c_{\hat{t}}|^2$ ■

5.2. *A formula for the shadow.* Assume that $\ell \in^{st} \mathbb{R}$ and $x \in {}^{nst}\mathbb{H}$, then

$$^{\circ}x = \frac{1}{2\ell}\sum_{\hat{t}\in\hat{h}\mathbb{Z}} c_{\hat{t}}\epsilon_{\hat{t}} = \frac{1}{2\pi}\sum_{\hat{t}\in\hat{h}\mathbb{Z}} c_{\hat{t}}\epsilon_{\hat{t}}\hat{h}, \tag{5.3}$$

where $(c_{\hat{t}})_{\hat{t}\in\hat{h}\mathbb{Z}}$ is the standard extension of the sequence $({}^{\circ}(x|e_{\hat{t}}))_{\hat{t}\in\hat{h}^{st}\mathbb{Z}}$.

□ Since $(\frac{1}{\sqrt{2\ell}}\epsilon_{\hat{t}})_{\hat{t}\in\hat{h}\mathbb{Z}}$ is an orthonormal basis for the space $\mathbf{H}$, we have ${}^{\circ}x = \frac{1}{2\ell}\sum\limits_{\hat{t}\in\hat{h}\mathbb{Z}} ({}^{\circ}x|\epsilon_{\hat{t}})\epsilon_{\hat{t}}$. Because of $\|{}^{\circ}x - Qx\| \approx 0$, $\forall\hat{t} \in \hat{h}\mathbb{Z}$ $({}^{\circ}x - Qx|\epsilon_{\hat{t}}) \approx 0$. From (4.19) we conclude $\gamma_{\hat{t}} \approx 0$ when $\hat{t} \in \hat{h}^{st}\mathbb{Z}$. Therefore, according to (4.21), the following formula is true:

$$\forall\hat{t} \in \hat{h}^{st}\mathbb{Z} \quad {}^{\circ}(x|e_{\hat{t}}) = ({}^{\circ}x|\epsilon_{\hat{t}}). \ \blacksquare \tag{5.4}$$

6. Shadow of the discrete derivative.

Using 5.1 and 5.2, it is not difficult to prove that the shadow of the shift U (s. 4.1) is the identical transformation of the space $\mathbf{H}$.

6.1. The discrete differentiation operator D (s. (4.10)) is nearstandard, and its shadow ${}^{\circ}D$ is the "usual" differentiation, i.e.

(*) $dom({}^{\circ}D)$ consists of those $\xi \in \mathbf{H}$ which are absolutely continuous and satisfy the conditions:

$$\frac{d\xi}{d\tau} \in \mathbf{H}, \quad \textit{and if } \ell \ll \infty \textit{ then } \xi(-\ell) = \xi(\ell). \tag{6.1}$$

(**) $\forall\xi \in dom({}^{\circ}D)$ $\quad {}^{\circ}D\xi = \frac{d\xi}{d\tau}$.

□ Assume that $\ell \ll \infty$. Make use of the expansion

$$Dx = \frac{1}{2\pi}\sum_{\hat{t}} \lambda_{\hat{t}}(x|e_{\hat{t}})e_{\hat{t}}\hat{h} \tag{6.2}$$

where $\lambda_{\hat{t}} \in \sigma(D)$ (s. (4.11)). According to (4.13), $|\lambda_{\hat{t}}| \le |\hat{t}|$. Therefore $\frac{1}{2\ell} \sum_{|\hat{t}| \le n} |\lambda_{\hat{t}}(x|e_{\hat{t}})|^2 \le n^2 \|x\|^2$. Consequently, if $x \approx 0$ then $\forall n \in^{st} \mathbb{N}$ $\sum_{|\hat{t}| \le n} |\lambda_{\hat{t}}(x|e_{\hat{t}})|^2 \approx 0$. By the Robinson lemma, the last is still true for some $n \in \mathbb{N} \setminus {}^{st}\mathbb{N}$. But when $Dx \in {}^{nst}\mathbb{H}$, according to 5.1, for all $n \approx +\infty$ we have $\sum_{|\hat{t}| > n} |\lambda_{\hat{t}}(x|e_{\hat{t}})|^2 \approx 0$. Therefore from $x \approx 0$ and $Dx \in {}^{nst}\mathbb{H}$ it follows that $Dx \approx 0$. This means that D satisfies the <nst> condition, so D is nearstandard (s. 3.5). Let $\xi \in {}^{st}dom({}^{\circ}D)$ and $x := \Pi\xi$, so $\xi =^{\circ} x$ and $({}^{\circ}D)\xi =^{\circ} (Dx)$. According to (5.3) ${}^{\circ}(Dx) = \frac{1}{2\pi} \sum_{\hat{t} \in \hat{h}\mathbb{Z}} c_{\hat{t}} \epsilon_{\hat{t}} \hat{h}$, where $(c_{\hat{t}})_{\hat{t} \in \hat{h}\mathbb{Z}}$ is a standard extension of the sequence $({}^{\circ}(\lambda_{\hat{t}}(x|e_{\hat{t}}))_{\hat{t} \in \hat{h}^{st}\mathbb{Z}}$. From (4.11) it follows that $\lambda_{\hat{t}} \approx \hat{t}$ when $\hat{t} \in \hat{h}^{st}\mathbb{Z}$. And according to (5.4) $(x|e_{\hat{t}}) \approx (\xi|\epsilon_{\hat{t}})$ when $\hat{t} \in \hat{h}^{st}\mathbb{Z}$. Thus

$$\forall \xi \in dom({}^{\circ}D) \quad ({}^{\circ}D)\xi = \frac{1}{2\pi} \sum_{\hat{t} \in \hat{h}\mathbb{Z}} \hat{t}(\xi|\epsilon_{\hat{t}})\epsilon_{\hat{t}}\hat{h}, \tag{6.3}$$

by the transfer principle. From (6.3) it is clear that ξ, $\frac{d\xi}{d\tau} \in \mathbf{H}$ and $\xi(-\ell) = \xi(\ell)$.

Conversely, let a function ξ satisfies the conditions just stated. Then the series $\sum_{\hat{t} \in \hat{h}\mathbb{Z}} |\hat{t}(\xi|\epsilon_{\hat{t}})|^2$ converges. Put

$$x := \frac{1}{2\pi} \sum_{\hat{t} \in \hat{h}\mathbb{Z}} \overline{\gamma_{\hat{t}}}(\xi|\epsilon_{\hat{t}})e_{\hat{t}}\hat{h}. \tag{6.4}$$

Since $|\gamma_{\hat{t}}| \le \frac{\pi}{2}$, we have $\sum_{|\hat{t}| > n\hat{h}} |\overline{\gamma_{\hat{t}}}(\xi|\epsilon_{\hat{t}})|^2 \approx 0$ when $n \approx +\infty$. Therefore (s. 5.1) $x \in {}^{nst}\mathbb{H}$. Taking it into account that $\forall \hat{t} \in \hat{h}^{st}\mathbb{Z} \quad \gamma_{\hat{t}} \approx 1$ and $(\xi|\epsilon_{\hat{t}}) \in^{st} \mathbb{C}$, in virtue of 5.2 we can conclude ${}^{\circ}x = \frac{1}{2\pi} \sum_{\hat{t} \in \hat{h}\mathbb{Z}} (\xi|\epsilon_{\hat{t}})\epsilon_{\hat{t}}\hat{h} = \xi$. Therefore $({}^{\circ}D)\xi =^{\circ} (Dx)$. But from (6.4) it follows that

$$Dx = \frac{1}{2\pi} \sum_{\hat{t} \in \hat{h}\mathbb{Z}} \lambda_{\hat{t}}\overline{\gamma_{\hat{t}}}(\xi|\epsilon_{\hat{t}})\epsilon_{\hat{t}}. \tag{6.4'}$$

As $\forall \hat{t} \in^{st} (\hat{h}\mathbb{Z}) \quad \lambda_{\hat{t}} \approx \hat{t}$, then it is clear from (6.4') that $Dx \in {}^{nst}\mathbb{H}$. So, the function ξ satisfies the condition (3.4) where $\mathbb{A} := D$. Therefore $\xi \in dom({}^{\circ}D)$. We are exempted from assumption on the standardness of ξ due to the transfer principle. This proves the proposition (*). The proposition (**) follows directly from (6.3) ■

To examine the case $\ell \approx +\infty$ we need a corollary of the inductor exactness.

6.2. *Lemma.* Let $\xi \in {}^{st}\mathbf{H}$, $\tilde{\xi} \in^{st} \mathcal{C}(\mathbf{T})$ and $\Pi\xi(t) \approx \tilde{\xi}(t)$ quasi everywhere when $t \in \mathbb{T}$; then $\xi = \tilde{\xi}$.

□ Let $E_0 \in 2^{\mathbb{T}}$, $\nu E_0 \approx 0$ and $\forall t \in \mathbb{T} \setminus E_0 \quad \Pi\xi(t) \approx \tilde{\xi}(t)$. Denoting $\alpha(t) := \Pi\xi(t) - \tilde{\xi}(t)$ and $\alpha := max_{t \in \mathbb{T} \setminus E_0} |\alpha(t)|$, we have $\alpha \approx 0$. Taking it into account that $\lambda QE_0 = \nu E_o \approx 0$ and ξ is a standard function, we find $\int_{QE} \xi(\tau)d\tau \approx$

$\int\limits_{Q(E\setminus E_0)} \xi(\tau)d\tau$ $=$ $\sum\limits_{s\in E\setminus E_0} \frac{1}{h}\int\limits_{s}^{s+h} \xi(\tau)d\tau \cdot h = \sum\limits_{s\in E\setminus E_0} \Pi\xi(s)h = \sum\limits_{t\in E\setminus E_0} [\tilde{\xi}(t) + \alpha(t)]h$. However, since $\xi \in^{st} \mathcal{C}(\mathbf{T})$, then $\sum\limits_{t\in E\setminus E_0} \tilde{\xi}(t)h \approx \int\limits_{QE} \tilde{\xi}(\tau)d\tau$. Moreover, $\sum\limits_{t\in E} \alpha(t)h \approx 0$ under the condition $hcardE \ll \infty$. Thus $\forall E \in 2^{\mathbb{T}}\ \nu E \ll \infty \implies \int\limits_{QE} [\xi(\tau) - \tilde{\xi}(\tau)]d\tau \approx 0$. However, a set E can be chosen in such a way that QE be an interval with arbitrary rational endpoints given in advance. Therefore, the functions ξ and $\tilde{\xi}$ are equal as the elements of $\mathbf{H}$ ■

Now we prove proposition 6.1 without the assumption $\ell \ll \infty$.

□ Let $x \approx 0$ and $Dx \in {}^{nst}\mathbb{H}$, $\eta :=^\circ (Dx)$, so $\eta \in^{st} \mathbf{H}$ and $\|QDx - \eta\| \approx 0$. Consider arbitrary $t_0, t \in \mathbb{T}$ such that $0 < t - t_0 \ll \infty$; we have $|\int\limits_{t_0}^{t} [QDx(\tau) - \eta(\tau)]d\tau| \leq \sqrt{t - t_0}\|QDx - \eta\| \approx 0$. Consequently, $\int\limits_{t_0}^{t} \eta(\tau)d\tau \approx \int\limits_{t_0}^{t} QDx(\tau)d\tau = \sum\limits_{s=t_0}^{t-h} \frac{1}{ih}[x(s+h) - x(s)]h = \frac{1}{i}[x(t) - x(t_0)]$. From $\|x\| \approx 0$ it follows that $x(t) \approx 0$ quasi everywhere on $\mathbb{T}$. Thus, with the exception of a set of infinitesimal $\nu-$measure ($\nu_t \equiv h$), we have $\int\limits_{t_0}^{t} \eta(\tau)d\tau \approx 0$. But the function η is standard; therefore, using the absolute continuity property of the Lebesgue integral and the transfer principle, we find $\int\limits_{\tau_0}^{\tau} \eta(\sigma)d\sigma = 0$ for all $\tau_0, \tau \in T$. So, $\eta = 0$, therefore $Dx \approx 0$, i.e. the <nst> condition is satisfied for D.

Let $\xi \in {}^{st}dom({}^\circ D)$, $x := \Pi\xi$ so that $\xi =^\circ x$, $({}^\circ D)\xi =^\circ (Dx)$. Denote $\eta :=^\circ (Dx)$, then $\|QDx - \eta\| \approx 0$. Since η is standard, $\eta \approx P\eta$ and $\|QDx - P\eta\| \approx 0$, hence $\|Dx - \Pi\eta\| \approx 0$. Consider any $t_0, t \in \mathbb{T}$ such that $0 < t - t_0 \ll \infty$; we have $|\sum\limits_{s=t_0}^{t-h} [Dx(s) - \Pi\eta(s)]h| \leq \sqrt{t - t_0}\|Dx - \Pi\eta\| \approx 0$. Consequently, $x(t) - x(t_0) \approx i\sum\limits_{s=t_0}^{t-h} \Pi\eta(s)h = i\int\limits_{t_0}^{t} \eta(\tau)d\tau$. Thus,

$$\Pi\xi(t) \approx {}^\circ\left[x(t_0) + i\int_{{}^\circ t_0}^{t} \eta(\tau)d\tau \right]. \tag{6.5}$$

Hence it follows, by 6.2, that almost everywhere with $\tau \in \mathbb{R}$

$$\xi(\tau) = {}^\circ\left[x(t_0) + i\int_{{}^\circ t_0}^{\tau} \eta(\sigma)d\sigma \right]. \tag{6.6}$$

But this means that the function ξ is absolutely continuous and $({}^\circ D)\xi = \frac{1}{i}\frac{d\xi}{d\tau} \in \mathbf{H}$. Conversely, let $\xi \in^{st} \mathbf{H}$, ξ is absolutely continuous and $\frac{d\xi}{d\tau} \in \mathbf{H}$. Denote $\eta := \frac{1}{i}\frac{d\xi}{d\tau}$. Now we show that $\xi \in dom({}^\circ D)$ and $({}^\circ D)\xi = \eta$. According to 3.5, to this end one needs verify that $\xi =^\circ x$ for some $x \in dom_{nst}D$ and $\eta =^\circ (Dx)$. Define

$\forall t \in \mathbb{T}\ x(t) := \xi(t)$; this definition is correct because ξ is continuous. Now we show that $x \in {}^{nst}\mathbb{H}$ and ${}^{\circ}x = \xi$. Taking it into account that $Qx(\tau) = 0$ when $|\tau| > \ell$, and $\int\limits_{|\tau|>\ell} |\xi(\tau)|^2 d\tau \approx 0$, we find $\|Qx - \xi\|^2 = \int\limits_{|\sigma|<\ell} |Qx(\sigma) - \xi(\sigma)|^2 d\sigma + \alpha$ where $\alpha \approx 0$. Therefore $\|Qx - \xi\|^2 \approx \sum\limits_{t=-\ell}^{\ell-h} \int\limits_{t}^{t+h} |\xi(t) - \xi(\sigma)|^2 d\sigma \leq \sum\limits_{t=-\ell}^{\ell-h} \int\limits_{t}^{t+h} d\sigma\ (\sigma - t) \int\limits_{t}^{\sigma} |\eta(\tau)|^2 d\tau \leq \frac{1}{2} h^2 \|\eta\|^2 \approx 0$, because $\|\eta\| \ll \infty$. Thus ${}^{\circ}x = \xi$. Denote $y := D$. It remains to verify ${}^{\circ}y = \eta$. We have $\|Qy - \eta\| \approx \int\limits_{|\sigma|<\ell} |QDx(\sigma) - \eta(\sigma)|^2 d\sigma =$
$\sum\limits_{t=-\ell}^{\ell-h} \int\limits_{t}^{t+h} \left| \frac{1}{ih}[\xi(t+h) - \xi(t)] - \eta(\sigma) \right|^2 d\sigma = \sum\limits_{t=-\ell}^{\ell-h} \int\limits_{t}^{t+h} \left| \frac{1}{h} \int\limits_{t}^{t+h} \eta(\tau) d\tau - \eta(\sigma) \right|^2 d\sigma =$
$\sum\limits_{t=-\ell}^{\ell-h} \int\limits_{t}^{t+h} |\Pi\eta(t) - \eta(\sigma)|^2 d\sigma = \sum\limits_{t=-\ell}^{\ell-h} \int\limits_{t}^{t+h} |Q\Pi(\sigma) - \eta(\sigma)|^2 d\sigma = \|P\eta - \eta\|^2 \approx 0$, because $\eta \in {}^{st}\mathbf{H}$ and P is a quasiunity ■

REFERENCES

1. Kudryk T., Lyantse V. and Chuiko G. Nearstandardness on finite set// Matematychni Studii, V.2. 1993. - P. 25–34.
2. Albeverio S., Fenstad J., Hoegh-Krohn T., Lindstrom T. Nonstandard methods in stochastic analysis and mathematical physics. – New York: Academic Press, 1986.
3. Davis M. Applied nonstandard analysis. A Wiley - Interscience publication. John Wiley & Sons. New Jork–London–Sydney–Toronto, 1977.
4. Nelson E. Internal set theory: a new approach to nonstandard analysis // Bull. Amer. Math. Soc. 1977. V.83, N.6. P.1165–1198.
5. Nelson E. Radically elementary probability theory. – Princeton, New Jersey: Princeton University Press, 1987. 97 p.

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