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ON THE AXIOM SYSTEM FOR BCI-ALGEBRAS

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BCI-algebras and related systems such as BCK-algebras, BCC-algebras, BCH-algebras etc. are motivated by implicational logic and by propositional calculi. We prove that the class of BCI-algebras forms a quasivariety of groupoids $(G, \bullet, 0)$ determined by the following independent axioms system: (1) $((xy)(xz))(zy) = 0$, (4) $xy = yx = 0$ implies $x = y$, (6) $x0 = x$. The class of all BCK-algebras (connected with BCC-logic) is a quasivariety defined by an independent axioms system: (1), (4), (6) and $0x = 0$. The class of BCK-algebras satisfying (9) $x(xy) = y(yx)$ is a variety defined by (1), (6) and (9).

In this note by an algebra (i.e. a groupoid) we mean a non-empty set G together with a binary multiplication "$\cdot$" (denoted by juxtaposition) and a some distinguished element 0. Such an algebra is denoted by $(G, \cdot, 0)$. Each such algebra will have certain equality axioms (including $x = x$) and the rule of substitution of equality as well as perhaps some other rules.

Many of such algebras were inspired by some logical systems (cf. [1, 6, 7, 9]). For example, so-called BCK-algebras are inspired by a BCK logic, i.e. an implicational logic based on modus ponens and the following axioms scheme (for detail see for example [1]):

Axiom B $\quad A \supset B. \supset .(C \supset A) \supset (C \supset B)$

Axiom C $\quad A \supset (B \supset C). \supset .B \supset (A \supset C)$

Axiom K $\quad A \supset (B \supset A)$

This inspiration is illustrated by the similarities between the names. We have BCK-algebra and BCK positive logic, BCI-algebra and BCI positive logic, positive implicative BCK-algebra and positive implicative logic, implicative BCK-algebras and implicative (classical) logic and so on. In many cases, the connection between such algebras and their corresponding logics is much stronger. In this case one

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can give a translation procedure which translates all well formed formulas and all theorems of a given logic L into terms and theorems of the corresponding algebra (cf. [1]). In some cases one can give also an inverse translation procedure. In this case we say that the logic L and its corresponding algebra are isomorphic.

Nevertheless the study of algebras motivated by known logics is interesting and very useful for corresponding logics, also in the case when these structures are not isomorphic. Such study (in general) gives a new axioms system of such algebras which is a new inspiration to the investigation of corresponding logics.

In this note we give axioms system for BCI-algebras and some their specifications.

Let $(G,\cdot,0)$ be an algebra of type $(2,0)$, i.e. a non-empty set G together with a binary operation $\cdot$ and a some distinguished element 0. In the sequel an operation $\cdot$ will be denoted by juxtaposition. We use dots only to avoid repetitions of brackets. In this convention the word $((xy)z)u$ will be written as $(xy\cdot z)u$.

As it is well-known an algebra $(G,\cdot,0)$ is called a *BCI-algebra* if for all $x,y,z\in G$ the following axioms are satisfied:

(1) $\quad (xy\cdot xz)\cdot zy=0,$

(2) $\quad (x\cdot xy)y=0,$

(3) $\quad xx=0,$

(4) $\quad xy=yx$ implies $x=y,$

(5) $\quad x0=0$ implies $x=0.$

The quasi-identity (5) may be replaced (see for example [4] or [5]) by the identity

(6) $\quad x0=x.$

A BCI-algebra in which the identity

(7) $\quad 0x=0$

holds is called a *BCK-algebra*.

In the theory of all BCI-algebras a very important role plays the identity

(8) $\quad xy\cdot z=xz\cdot y,$

which holds in all BCI-algebras. The simple proof of this identity is given in [5].

Note that the above axioms system is not independent. For example, putting $y=0$ in (1) and using (6) we obtain (2). Similarly (2) implies (3). In my talk during the Conference on Universal Algebra and its Applications (Opole-Jarłontówek 1985, Poland) it was presented the following result:

Proposition 1. *The class of all BCI-algebras is characterized by (1), (4) and (6).*

This result was also obtained by L.H.Shi in [11]. It is a consequence of the following general result.

Proposition 2. *The class of all BCI-algebras is defined by the independent axiom system: (1), (4), (6) and (7).*

Proof. We prove only that these axioms are independent.

It is clear that any non-trivial Boolean group satisfies (1), (4) and (6), but not (7). Hence that axiom (7) is independent.

Now we consider three groupoids $(G,\cdot)$ defined on the set $G=\{0,1,2\}$ as follows:

$\cdot$	0	1	2
0	0	0	0
1	2	0	0
2	2	2	0

Table 1

$\cdot$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Table 2

$\cdot$	0	1	2
0	0	0	0
1	1	0	2
2	2	2	0

Table 3

The groupoid defined by Table 1 satisfies (1), (4) and (7). Indeed, if $x=y$, then $(xx\cdot xz)\cdot zx=(0\cdot xz)\cdot zx=0\cdot zx=0$. If $x=z$, then $(xy\cdot xx)\cdot xy=(xy\cdot 0)\cdot xy=0$, because $xy=0$ or $xy=2$. The case $y=z$ is obvious. If all elements x,y,z are different, then $x=0$, $y=0$ or $z=0$ and direct computation shows that in these cases (1) holds, too. Conditions (4) and (7) are obvious. In this groupoid $10=2\neq 1$, which proves that (6) is independent.

In the same manner we prove that the groupoid defined by Table 2 satisfies (1), (6) and (7). Moreover, $21=12=0$, but $1\neq 2$. Hence the axiom (4) is independent.

Finally, in the groupoid defined by Table 3, we have (4), (6), (7), but $(12\cdot 10)\cdot 02=21\cdot 0=2$, i.e. (1) is independent, which completes the proof.

Corollary 1. *Axioms given in Proposition 1 are independent.*

A BCI-algebra $(G,\cdot,0)$ is called *commutative*, if

(9) $\qquad x\cdot xy=y\cdot yx$

for all $x,y\in G$. *A commutative BCI-algebra is a commutative BCK-algebra*, because $0\cdot 0x=x\cdot x0=xx=0$ and $0=(0\cdot 0x)x=0x$ by (2). Commutative BCK-algebras form a variety. This variety has (3), (6), (8) and (9) as its equational base (see [13]). At the end of [2], it was observed that these identities are independent. This variety is 2-based, but it is not 1-based (cf. Lemma 2 in [3]).

Proposition 3. *A variety of commutative BCK-algebras is defined by (1), (6) and (9). These identities are independent.*

Proof. Obviously every commutative BCK-algebra satisfies (1), (6) and (9). On the other hand, (1), (6) and (9) imply (4) and (7). Indeed, if $xy=yx=0$, then $x=y$ because $x=x0=x\cdot xy=y\cdot yx=y0=y$. Putting $y=0$ in (1) and using (6), we obtain $(x\cdot xz)z=0$, which gives $xx=(x\cdot x0)0=0$. Moreover, $0=(0\cdot 0z)z=(z\cdot z0)z=zz\cdot z=0z$ by (9), i.e. (7) holds, too. Hence (Proposition 2) a commutative BCK-algebra is defined by (1), (6) and (9).

To prove that these conditions are independent observe that the groupoid defined by Table 4

·	0	1
0	0	1
1	1	1

Table 4

·	0	1
0	0	0
1	0	0

Table 5

satisfies (6) and (9), but not (1). Thus (1) is independent. Also (6) is independent because the groupoid defined by Table 5 satisfies (1) and (6), but $2 \cdot 21 \neq 1 \cdot 12$, which shows that (9) is independent. This finishes the proof.

Corollary 2. *A variety of commutative BCK-algebras is defined by (6), (7) and*

(10) $\quad xy \cdot xz = zy = zx,$

or by (3), (6) and (10).

Proof. A commutative BCK-algebra satisfies (10) since $xy \cdot xz = (x \cdot xz)y = (z \cdot zx)y = zy \cdot zx$ by (9). Conversely, putting $x = 0$ in (10) we obtain $zy \cdot z = 0$ by (6) and (7). This together with (10) gives (1). Condition (9) follows from (10) and (6).

The proof of the second part is similar.

As it is well-known (cf.[9]) every commutative BCK-algebra is a lower semilattice with respect to $\wedge$, where $x \wedge y = y \cdot yx$. A commutative directed BCK-algebra, i.e. commutative BCK-algebra in which for every two elements x, y there exists an element z such that $xz = xy = 0$, is a distributive lattice with respect to $\wedge$ and $\vee$, where $x \vee y$ is defined as $c(cx \wedge cy)$ and c is any upper bound for x and y (cf. [12]). Moreover (cf. [10]), a commutative directed BCK-algebra is a Łukasiewicz algebra, i.e. an algebra $(G, \cdot, 0)$ of type $(2, 0)$ satisfying (1), (6), (9),

(11) $\quad xy = xy \cdot yx,$

(12) $\quad xy \cdot x = 0.$

Note that the converse is not true. There are Łukasiewicz algebras which are not directed (as BCK-algebras). Obviously every Łukasiewicz algebra is a commutative BCK-algebra with the identity $xy \cdot (xy \cdot yx) = 0$. Thus comparing results obtained in [8] with the above remarks, we have the following characterization of Łukasiewicz algebras.

Proposition 4. *Every Łukasiewicz algebra is a subdirect product of totally ordered BCK-algebras.*

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