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THE SPACE OF LOCALLY HÖLDER MAPS FROM A LOCALLY COMPACT METRIC SPACE TO A BANACH SPACE

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Abstract. T. Banakh, *The space of locally Hölder maps from a locally compact metric space to a Banach space*, Math. Stud. **2** (1993) 87–90.

For a separable locally compact metric space (X, d) and a separable Banach space Y, $C(X, Y)$ denotes the spaces of all continuous maps from X to Y, equipped with the compact-open topology. The linear subspace $H^\mu(X, Y) \subset C(X, Y)$, $\mu \in (0, 1]$, consisting of all locally μ-Hölder maps is considered. It is proved that the couple $(C(X, Y), H^\mu(X, Y))$ is homeomorphic either to (s, Σ) or to $(s \times s, \Sigma \times s)$ or to $(s^\omega, \Sigma^\omega)$. Here $s = (-1, 1)^\omega$ is the pseudo-interior of the Hilbert cube and Σ is its radial interior.

1. Introduction

In the article the topology of the space consisting of locally Hölder maps from a separable locally compact metric space to a separable Banach space is studied. We show that it is homeomorphic to some well known infinite-dimensional model spaces.

Recall that $s = (-1, 1)^\omega$ is the pseudo-interior of the Hilbert cube $Q = [-1, 1]^\omega$ and $\Sigma = \{(x_i)_{i=1}^\infty \in Q \mid \sup\{|x_i| \mid i \in \mathbb{N}\} < 1\} \subset s \subset Q$ is its radial interior.

Let (X, d_X), (Y, d_Y) be metric spaces. The space of all continuous functions from X to Y, equipped with the compact-open topology, is denoted by $C(X, Y)$. For $0 \le \mu \le 1$ let $H^\mu(X, Y) = \{f \in C(X, Y) \mid \forall x \in X\ \exists U,\ x \in U^{open} \subset X$, such that $\sup\{d_Y(f(x), f(x'))/(d_X(x, x'))^\mu \mid x, x' \in U\} < \infty\}$ be the linear subspace in $C(X, Y)$ consisting of all locally μ-Hölder maps. Note that $H^0(X, Y) = C(X, Y)$ and $H^1(X, Y) = L(X, Y)$ is the set of all locally Lipshchitz maps.

It follows from Ascoli-Arzela Theorem and [1, Theorem VIII.3.1] that the couple $(C(X, Y), L(X, Y))$ is homeomorphic to (s, Σ), provided X is a non-discrete compactum and $(Y, \|\ \|)$ is a finite-dimensional Banach space.

This result was generalized by K. Sakai and R. Wong [2]. They proved that $(C(X, Y), L(X, Y))$ is an (s, Σ)-manifold, provided X is a non-discrete compactum and Y is a separable locally-compact locally convex set in a normed linear space (see also [3]).

In this article we obtain another generalization of the above statement.

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For a topological space X by $X^{(1)} = \{x \in X \mid x \in \mathrm{cl}_X(X\setminus\{x\})\} \subset X$ we denote the set of all cluster points of X.

The main result of the paper is the following classification

Theorem. *Let (X,d) be a separable locally-compact non-discrete metric space and $(Y, \|\ \|)$ be a separable Banach space. Then for every $0 < \mu \le 1$ the couple $(C(X,Y), H^\mu(X,Y))$ is homeomorphic to*

1. *(s, Σ), provided X is compact and $\dim Y < \infty$;*
2. *$(s \times s, \Sigma \times s)$, provided $X^{(1)}$ is compact and either X is not compact or $\dim Y < \infty$;*
3. *$(s^\omega, \Sigma^\omega)$, provided $X^{(1)}$ is not compact.*

Assumptions. *Throughout the article $\mu \in (0,1]$, $(Y, \|\ \|)$ is a separable Banach space and (X,d) is a separable non-discrete locally-compact metric space. Let $X = \bigcup_{n=1}^\infty X_n$ where $X_n \subset \bar{X}_n \subset X_{n+1}$, $n \in \mathbb{N}$, are open sets with the compact closures $\bar{X}_n$.*

2. Borelian classification of subsets $H^\mu(X,Y) \subset C(X,Y)$.

Let X_0 be a closed subset in X. Let $C(X|X_0,Y) = \{f \in C(X,Y) \mid f|X_0 \equiv 0\} \subset C(X,Y)$ and $C_0(X,Y) = \{f \in C(X,Y) \mid \mathrm{supp}(f) = \mathrm{cl}_X(f^{-1}(Y\setminus\{0\}))$ is compact$\}$. Let $H^\mu(X|X_0,Y) = H^\mu(X,Y) \cap C(X|X_0,Y)$, $H_0^\mu(X,Y) = C_0(X,Y) \cap H^\mu(X,Y)$ and $H_0^\mu(X|X_0,Y) = H_0^\mu(X,Y) \cap H^\mu(X|X_0,Y)$.

Recall [4] that the topology on $C(X,Y)$ is generated by the pre-basis $\{\langle K,U\rangle = \{f \in C(X,Y) \mid f(K) \subset U\} \mid K$ is compact in X and U is open in $Y\}$. This implies that $C(X|X_0,Y)$ is closed in $C(X,Y)$.

Lemma 1. *$H_0^\mu(X,Y)$ is an F_σ-set in $C(X,Y)$. Moreover, if $\dim Y < \infty$, then $H_0^\mu(X,Y)$ is sigma-compact.*

Proof. Obviously, $H_0^\mu(X,Y) = \bigcup_{n=1}^\infty A_n$ where $A_n = \{f \in C(X,Y) \mid f|X\setminus X_n \equiv 0$ and $\|f(x) - f(x')\| \le n\, d(x,x')^\mu$ for $x, x' \in X\}$. It is easily seen that A_n, $n \in \mathbb{N}$, are closed subsets in $C(X,Y)$. If $\dim Y < \infty$, then, by Ascoli Theorem [4, 8.2.10], A_n, $n \in \mathbb{N}$, are compact.

Recall that for a topological space Z, $\mathcal{M}_2(Z)$ is the collection of subsets of Z that can be expressed as $\bigcap_{n=1}^\infty A_n$, where A_n, $n \in \mathbb{N}$, are F_σ-subsets in Z.

Corollary 1. *The set $H^\mu(X,Y)$ belongs to the class $\mathcal{M}_2(C(X,Y))$.*

Proof. Indeed, $H^\mu(X,Y) = \bigcap_{n=1}^\infty A_n$ where $A_n = \pi_n^{-1}(H^\mu(\bar{X}_n,Y))$, $n \in \mathbb{N}$. Here $\pi_n : C(X,Y) \to C(\bar{X}_n,Y)$ is the map defined by $\pi_n(f) = f|\bar{X}_n$. By Lemma 1, $H^\mu(\bar{X}_n,Y) = H_0^\mu(\bar{X}_n,Y)$ is an F_σ-subset in $C(\bar{X}_n,Y)$. This implies that $A_n = \pi_n^{-1}(H^\mu(\bar{X}_n,Y))$ is an F_σ-subset in $C(X,Y)$. Hence $H^\mu(X,Y) = \bigcap_{n=1}^\infty A_n \in \mathcal{M}_2(C(X,Y))$.

Lemma 2. *Assume that $X^{(1)}$ is compact. Then $H^\mu(X,Y)$ is an F_σ-subset in $C(X,Y)$.*

Proof. It is easily seen that $H^\mu(X,Y) = \bigcup_{n=1}^\infty A_n$ where $A_n = \{f \in C(X,Y) \mid \|f(x) - f(x')\| \le n(d(x,x'))^\mu$ for $x,x' \in O_d(X^{(1)}, 1/n)\}$, $n \in \mathbb{N}$. Here $O_d(X^{(1)}, 1/n)$ is the $\frac{1}{n}$-neighbourhood of $X^{(1)} \subset X$. Obviuosly, the set A_n is closed in $C(X,Y)$ for every $n \in \mathbb{N}$.

3. The topology of the couple $(C(X,Y), H^\mu(X,Y))$.

Note at first that $C(X,Y)$ is a Frechet space (i.e. a locally convex linear complete metric space): the topology on $C(X,Y)$ can be equivalently defined by the countable system of pseudonorms $\{\|\cdot\|_n\}_{n=1}^{\infty}$ where $\|f\|_n = \sup\{\|f(x)\| \mid x \in \bar{X}_n\}$, $f \in C(X,Y)$,
$n \in \mathbb{N}$.

The following is easy and can be proved by the standard methods.

Lemma 3. *Let $X_0 \subset X$ be a closed subset. Then the set $H_0^1(X|X_0,Y)$, and, consequently, $H_0^\mu(X|X_0,Y) \supset H_0^1(X|X_0,Y)$, is dense in $C(X|X_0,Y)$.*

Theorem 1. *Let X_0 be a closed subset in X. If $X\backslash X_0$ is non-discrete and $\dim L < \infty$ then the couple $(C(X|X_0,Y), H_0^\mu(X|X_0,Y))$ is homeomorphic to (s,Σ).*

Proof. By [1, Theorem VIII.3.1], it is sufficiently to prove that $H_0^\mu(X|X_0,Y)$ is a dense sigma-compact linear subset in $C(X|X_0,Y)$ which contains an infinite-dimensional convex compactum.

Since $\dim L < \infty$, by Lemmas 1 and 3, $H_0^\mu(X|X_0,Y) = H_0^\mu(X,Y) \cap C(X|X_0,Y)$ is a dense sigma-compact linear subspace in $C(X|X_0,Y)$.

Finally, since $X\backslash X_0$ is not discrete, there exists an infinite compactum $K \subset X\backslash X_0$. Obviously, $K \subset X_n$ for some $n \in \mathbb{N}$. The set $C = \{f \in C(X|X_0,Y) \mid f|X\backslash X_n \equiv 0,\ \|f(x) - f(x')\| \le d(x,x'),\ x,x' \in X\}$ is an infinite-dimensional convex compactum in $H_0^\mu(X|X_0,Y)$.

Theorem 2. *Let X_0 be a closed subset in X. If $X\backslash X_0 \neq \emptyset$ and $\dim L = \infty$, then the couple $(C(X|X_0,Y), H_0^\mu(X|X_0,Y))$ is homeomorphic to $(s \times s, \Sigma \times s)$.*

Proof. By [5, Theorem 3.5], it is sufficient to prove that $H_0^\mu(X|X_0,Y)$ is a dense linear F_σ-subset in $C(X|X_0,Y)$ which contains a closed in $C(X,Y)$ convex non-locally compact subset.

By Lemmas 1 and 3, $H_0^\mu(X|X_0,Y)$ is a dense linear F_σ-subset in $C(X|X_0,Y)$.

It is easily seen that the convex set $C = \{f \in C(X|X_0,Y) \mid \|f(x) - f(x')\| \le d(x,x')$ for $x,x' \in X\}$ is closed in $C(X|X_0,Y)$ and C is not locally compact (recall that Y is not locally compact). This completes the proof of the theorem.

Theorem 3. *Assume that $X^{(1)}$ is compact and either X is not compact or $\dim L = \infty$. Then the couple $(C(X,Y), H^\mu(X,Y))$ is homeomorphic to $(s \times s, \Sigma \times s)$.*

Proof. By Lemmas 2 and 3, $H^\mu(X,Y)$ is a dense linear F_σ-subset in $C(X,Y)$. By [5, Theorem 3.5], it is sufficiently to show that $H^\mu(X,Y)$ contains a closed in $C(X,Y)$ convex non-locally compact subset.

Assume at first that X is not compact. Then for every $n \in \mathbb{N}$ the set $X\backslash\bar{X}_n$ is infinite. Since $X^{(1)}$ is compact, $X^{(1)} \subset X_n$ for some $n \in \mathbb{N}$. Then $X\backslash X_{n+1}$ is a discrete infinite closed subset in X. Obviously, $C = \{f \in C(X,Y) \mid f|\bar{X}_n \equiv 0\} \cong Y^{X\backslash\bar{X}_n} \cong s$ is a closed in $C(X,Y)$ convex non-locally compact subset in $H^\mu(X,Y)$.

Assume now that $\dim Y = \infty$, i.e. Y is not locally compact. Then $C = \{f \in C(X,Y) \mid f \equiv \text{constant} \in Y\}$ is a closed in $C(X,Y)$ convex non-locally compact set of $H^\mu(X,Y)$.

Theorem 4. *If $X^{(1)}$ is not compact, then the couple $(C(X,Y), H^\mu(X,Y))$ is homeomorphic to $(s^\omega, \Sigma^\omega)$.*

Proof. By [5, Proposition 4.2], if F is a separable Frechet space and $E \in \mathcal{M}_2(F)$ is a dense linear subset such that for some closed linear subspace $G \subset F$ the couple $(G, G \cap E)$ is homeomorphic to $(s^\omega, \Sigma^\omega)$, then the couple (F, E) is homeomorphic to $(s^\omega, \Sigma^\omega)$ as well.

Since $X^{(1)}$ is not compact, there exists a closed discrete countable set $\{x_n\}_{n=1}^\infty \subset X^{(1)}$. Let $\chi : X \to \mathbb{R}$ be a function with $\chi(x_n) = n$ for every $n \in \mathbb{N}$. For $n \in \mathbb{N}$ let $U_n = \chi^{-1}((n - \frac{1}{3}, n + \frac{1}{3})) \subset X$. Let $X_0 = X \setminus \bigcup_{n=1}^\infty U_n$.

Then, obviously,

$(C(X|X_0, Y), H^\mu(X|X_0, Y)) \cong \Pi_{n=1}^\infty (C(X|(X\setminus U_n), Y), H^\mu(X|(X\setminus U_n), Y))$.

By Theorems 1 and 2, for every $n \in \mathbb{N}$ the couple $(C(X|(X\setminus U_n), Y), H^\mu(X|(X\setminus U_n), Y))$ is homeomorphic either to (s, Σ) or to $(s \times s, \Sigma \times s)$. This implies that $(C(X|X_0, Y), H^\mu(X|X_0, Y)) \cong (s^\omega, \Sigma^\omega)$. Since $C(X|X_0, Y)$ is a closed linear subspace in $C(X, Y)$ and, by Lemma 2, $H^\mu(X, Y) \in \mathcal{M}_2(C(X, Y))$ is a dense linear subspace in $C(X, Y)$, the couple $(C(X, Y), H^\mu(X, Y))$ is homeomorphic to $(s^\omega, \Sigma^\omega)$ as well.

Question. For an open subset $U \subset Y$ let $C(X, U) = \{f \in C(X, Y) \mid f(X) \subset U\}$ and $H^\mu(X, U) = H^\mu(X, Y) \cap C(X, U)$. Assume that $X^{(1)}$ is not compact. Is the couple $(C(X, U), H^\mu(X, U))$ an $(s^\omega, \Sigma^\omega)$-manifold? (Note that $C(X, U)$ is not open in $C(X, Y)$).

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