Математичні студії
1993, випуск 2

УДК 517.547.3

BLASCHKE PRODUCTS OF GIVEN QUANTITY INDEX

M.O. Ghirnyk, A.A. Kondratyuk

Abstract. M. Ghirnyk, A. Kondratyuk, *Blaschke products of given quantity index*, Math. Stud. **2** (1993) 49–52.

Quantity index $p[f]$ of a function meromorphic in the disc and of bounded Nevanlinna characteristic is defined by the equality $p[f]=p-1$ where $p^{-1}+q^{-1}=1$ and $q=\sup\{s\ge 1\colon \|\ln|f(re^{i\varphi})|\|_{L^s[-\pi;\pi]}=O(1),\ r\to 1\}$.

Theorem. For each $p\in[1,+\infty]$ there exists a Blaschke product $B(z)$ such that $p[B]=p-1$.

Let $\{a_n\}$ be a sequence of complex numbers such that $0<|a_n|<1$ and $\sum_{n=1}^{\infty}(1-|a_n|)<\infty$. The Blaschke product

$$B(z):=\prod_{n=1}^{\infty}\frac{a_n-z}{1-\bar a_n z}\,\frac{\bar a_n}{|a_n|}$$

generated by the sequence $\{a_n\}$ is analytic function in the unit disc $\mathbb{D}:=\{z\colon |z|<1\}$ with zeros $\{a_n\}$. The inequality $|B(z)|<1$ is true and the Nevanlinna characteristic $T(r,B)$ is bounded [1, Lemma 6.6].

Therefore, for studying behavior of the Blaschke product $B(z)$ as $|z|\to 1$ it is convenient to use the integral means

$$m_s(r,B):=\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|\log|B(r\,e^{i\varphi})|\big|^s\varphi\Big)^{1/s},$$

where $s\in[1,\infty[,\ r\in[0,1[$.

It follows from the Jensen formula [1, § 1.1] that for any Blaschke product, $m_1(r,B)$ is bounded. In [2] L.R. Sons considered the Blaschke product with zeros $a_n=1-(n+1)^{-4/3}$ for which $m_2(r,B)\to\infty$ as $r\to 1$. In view of it we define the quantity index $p(B)$ of a Blaschke product $B(z)$ as follows $p(B):=p-1$, where $\frac{1}{p}+\frac{1}{q}=1$ and

$$q:=\sup\{s\in[1,+\infty[\colon m_s(r,B)=O(1),\ r\to 1\}.$$

1991 *Mathematics Subject Classification.* 30D35, 30D50.

Theorem. *For any* $p \in [1, +\infty]$ *there exists a Blaschke product* $B(z)$ *of quantity index* $p(B) = p - 1$.

Proof. We consider three cases.

1. $p = +\infty$. Let $\rho(r) \in C^1[0,1]$ be a function such that:

i) $\rho(r) \geq 0$; ii) $\rho(r) \to 1$ as $r \to 1$; iii) $\rho'(r)(1-r)\log(1-r) \to 0$ as $r \to 1$;

iv) $\int^1 (1-t)^{-\rho(t)} dt < \infty$ (for example, $\rho(r) = 1 + \log(\log^2(1-r))/\log(1-r)$).

Let $D(z; R) = \{w \in \mathbb{C} : |w - z| < R\}$, M_j be positive constants. We take the Blaschke product $B(z)$ with positive zeros $\{a_n\}$ such that the number $n(r)$ of a_n in the disc $\{z : |z| \leq t\}$ counted according to multiplicity satisfies

$$n(r) \sim \Delta(1-r)^{-\rho(r)}, \quad \Delta \in]0; +\infty[,$$

as $r \to 1$.

Then

$$\log|B(z)| = \mathrm{Re}\big(-2\Delta(1-z)^{-1}\int_{1-|z|}^{1}(1-t)^{-\rho(t)}dt)(1+\gamma(z)\big), \tag{1}$$

where $\gamma(z) \to 0$ uniformly as $z \to 1$, $z \notin \bigcup_{n=1}^{\infty} D(a_n; r_n)$,

$$\sum_{n:\, a_n + r_n \geq r} r_n = o(1-r), \quad r \to 1. \tag{2}$$

The asymptotic formula (1) follows from [3]. It should be mentioned that in [3] the product of zero genus was not considered but the conclusion of the theorem from [3] and the proof remain to be true for this case.

Now we obtain the lower estimate for $m_s(r, B)$ using the method from [4] (see also [5, p. 20-21]). Choose $\arg(1-z)$ in the domain $\mathbb{C}\backslash[1;\infty[$ such that $\arg 1 = 0$ and denote by $A = A(r)$ the set

$$\{\varphi : \arg(1 - re^{i\varphi}) \in \big[-\frac{3\pi}{4}, -\frac{2\pi}{4}\big] = [\varphi_1(r); \varphi_2(r)] \subset [0, \pi/2],$$

where $\varphi_j(r) = \arcsin(r^{-1}\sin\alpha_j) - \alpha_j$, $j = 1, 2$, $\alpha_1 = \frac{\pi}{4}$, $\alpha_2 = \frac{\pi}{3}$. It is easy to see that for r sufficiently near one the intersection of the set $\{re^{i\varphi} : \varphi \in A\}$ with an exceptional set satisfying (2) is empty.

By (1) we get

$$(m_s(r, B))^s \geq \frac{1}{2\pi}\int_A \big|\log|B(re^{i\varphi})|\big|^s d\varphi \geq$$

$$\geq \frac{1}{2\pi}(\Delta\int_r^1 (1-t)^{-\rho(t)}dt)^s \int_A |\mathrm{Re}(1-re^{i\varphi})^{-1}|^s d\varphi. \tag{3}$$

Let us continue the estimate (3)

$$\int_A |\mathrm{Re}(1-re^{i\varphi})^{-1}|^s d\varphi = \int_A |1-re^{i\varphi}|^s\,|\cos\arg(1-re^{i\varphi})|^s d\varphi \geq$$

$$\geq (\cos\frac{\pi}{3})^s \int_A |1-re^{i\varphi}|^{-s} d\varphi. \tag{4}$$

Further, $\min\{|1-re^{i\varphi}|^{-s}\colon \varphi\in A=[\varphi_1(r);\varphi_2(r)]\}=|1-re^{i\varphi_2(r)}|^{-s}$. Thus

$$\int_{\varphi_1(r)}^{\varphi_2(r)}|1-re^{i\varphi}|^{-s}d\varphi\ge|1-re^{i\varphi_2(r)}|^{-s}(\varphi_2(r)-\varphi_1(r))\sim$$
$$\sim(1-r)^{-s}(\cos\frac{\pi}{3})^s(1-r)(\operatorname{tg}\frac{\pi}{3}-\operatorname{tg}\frac{\pi}{4})\quad(5)$$

as $r\to1$. By (3)–(5)

$$m_s(r,B)\ge(2\pi)^{-s}\frac{\Delta}{4}\big(\frac{\sqrt3}{2}-1\big)^{1/s}(1-r)^{-1+1/s}\int_r^1(1-t)^{-\rho(t)}dt\to\infty$$

as $r\to1$.

2. $p\in]1;\infty[$. Let us consider the Blaschke product $B(z)$ formed with positive zeros $\{a_n\}$ such that $n(r)\sim\Delta(1-r)^{-\rho}$, $0<\Delta<\infty$, $\rho=1-\frac1p$, as $r\to1$.

An integration by parts shows

$$\log B(z)=\int_0^1\log\frac{t-z}{1-tz}dn(t)=\int_0^1\frac{(1-z^2)n(t)}{(t-z)(1-tz)}dt.\quad(6)$$

Here $\log B(z)$ is chosen in the domain $\mathbb{D}\backslash[a_1;1[$ to satisfy the condition $\log B(0)<0$.

Further, applying the inequality $|1-tz|\ge|1-z|/2$, $t\in[0,1]$, $z\in\mathbb{D}$, and cutting the tails of the integral we get that

$$|\log|B(z)||\le M_1\int_0^1\frac{(1-t)^{-\rho}}{|t-z|}dt=M_1\int_{a(z)}^{b(z)}\frac{(1-t)^{-\rho}}{|t-z|}dt+o(|1-z|^{-\rho}),\ r\to1,\quad(7)$$

where $a(z)=1+|1-z|\log|1-z|$, $b(z)=1+|1-z|(\log|1-z|)^{-1}$.

It is easy to see (compare [1, § 4.6.4]) that

$$\int_{a(z)}^{b(z)}\frac{(1-t)^{-\rho}}{|t-z|}dt\le(1-b(z))^{-\rho}\int_{a(z)}^{b(z)}\frac{dt}{|t-z|}\le$$
$$\le M_2|1-z|^{-\rho}\log^\rho\frac{2}{|1-z|}\big(\log\log\frac{8}{|1-z|}+\log\frac{4}{|\arg(1-z)|}\big)\le$$
$$\le M_3|1-z|^{-\rho-\varepsilon}\log\frac{4}{|\arg(1-z)|},\quad(8)$$

where $\arg(1-z)$ is chosen as in the part 1 and a number ε satisfies $\rho<\rho+\varepsilon<1$. Let $E(r)=\{\varphi\colon|\arg(1-re^{i\varphi})|\le1-r\}$. Therefore, we get that for s satisfying the inequality $(\rho+\varepsilon)s<1$,

$$m_s^s(r,B)\le M_4\int_{-\pi}^{\pi}|1-re^{i\varphi}|^{-(\rho+\varepsilon)s}\log^s\frac{4}{|\arg(1-re^{i\varphi})|}d\varphi=$$
$$=M_4\left(\int_{E(r)}+2\int_{[0,\pi]\backslash E(r)}\right).\quad(9)$$

Further, we have

$$\int_{E(r)} |1 - re^{i\varphi}|^{-(\rho+\varepsilon)s} \log^s \frac{4}{|\arg(1 - re^{i\varphi})|} d\varphi \leq$$
$$\leq M_5(1-r)^{-(\rho+\varepsilon)s} \int_0^{1-r} \log^s \frac{4}{\psi} d\psi(1-r) = O(1) \quad (10)$$

as $r \to 1$.

The second integral is estimated as follows

$$\int\limits_{[0,\pi]\setminus E(r)} |1 - re^{i\varphi}|^{-(\rho+\varepsilon)s} \log^s \frac{4}{|\arg(1 - re^{i\varphi})|} d\varphi \leq$$
$$\leq M \int_{1-r}^{\pi/4} (1-r)^{-(\rho+\varepsilon)s} \psi^{-(\rho+\varepsilon)} \log^s \frac{4}{\psi} d\psi(1-r) + O(1) = O(1),\ r \to 1. \quad (11)$$

Combining (10) and (11) we obtain $p(B) \leq p - 1$. It is shown in [6] that

$$\log|B(z)| = \mathrm{Re}((1-z)^{-\rho} M(\rho, \operatorname{sgn} \operatorname{Im} z))(1 + o(1)),\ z \to 1,\ |\arg(1-z)| \geq \delta, \quad (12)$$

where $M(\rho, \operatorname{sgn} \operatorname{Im} z)$ is a constant depending on ρ and $\operatorname{sgn} \operatorname{Im} z$. In the similar way as in the part 1 we can prove that if $s > 1/\rho$ then $m_s(r, B) \to \infty$ as $r \to 1$. Therefore, on the other hand we get $p(B) \geq p - 1$ and thus $p(B) = p - 1$.

3. $p = 1$. In this case we take Blaschke product with counting function $n(t)$ of slow growth.

REFERENCES

1. Хейман У.К. Мероморфные функции. М.: Мир, 1966. 287 с.
2. Sons L.R. Zeros of functions with slow growth in the unit disc // Math. Japonica. 1979. V.24, N.3. P.271–282.
3. Гирнык М.А. Об асимптотических свойствах некоторых канонических произведений // Сиб. мат. ж. 1974. Т.15, N.5. С.1036–1048.
4. Крутинь В.И. О величинах дефектов Р. Неванлинны для мероморфных при $|z| < 1$ функций // Изв. АН АрмССР. Матем. 1973. Т.8, N.5. С.347–358.
5. Гирнык М.А. Аналог теоремы Линделёфа о типе канонических произведений // Теория функций, функ. ан. и их прилож. 1978, вып. 29. С.16–24.
6. Галоян Р.С. Об асимптотических свойствах функции $\pi_p(z; z_k)$ // Доклады АН АрмССР. 1974. Т.59, N.2. С.65–71

Department of Mathematics, Commerce and Trade Institute, 68 Ivan Franko str., Lviv, 290011, Ukraine

e-mail: root@ litech.lviv.ua

Department of Mechanics and Mathematics, Lviv University, Universytetska 1, Lviv, 290000, Ukraine

Received 17.02.1993