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NEARSTANDARDNESS ON FINITE SET

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On the set $\mathbb{T}$, finite in the sense of the E.Nelson's Internal Set Theory, a measure ν is given. For "discrete integral" $\Sigma_{t\in\mathbb{T}}x(t)\nu\{t\}$, $x \in \mathbb{C}^{\mathbb{T}}$ the analogs of the classical theorems of Lebesgue integral theory are regarded. As the set $\mathbb{T}$ is nonstandard, for the measures and functions given on it, a direct definition of nearstandardness is impossible. Indirect way based on the embedding of the algebra $2^{\mathbb{T}}$ into the algebra $2^{\mathbf{T}}$ is used. Here, $\mathbf{T}$ is a standard set. Relations between nearstandard charges and their shadows, as well as nearstandard functions and their shadows appearing with such approach are investigated.

We use E. Nelson's IST (see [4,5]). This research is stimulated by the paper [6] where the simplicity was achieved due to the finiteness (in the sense of IST) of probability spaces in question. We also tend to simplicity dealing with the finite-dimensional (in the sense of IST) functional spaces.

Further all objects are supposed to be internal without additional specifications. Exceptions are possible if they are evident from the context. For instance, ${}^{\mathrm{st}}A$ denotes the class of all standard elements of the set A, i.e. external set. However, 2^A denotes (internal) set of all internal $B \subseteq A$.

1. Integration over finite set

Let $\mathbb{T}$ be a set finite in the sense of IST, and $\operatorname{card}\mathbb{T} \approx \infty$ (i.e. $\operatorname{card}\mathbb{T} \in \mathbb{N}\backslash{}^{st}\mathbb{N}$).

1.1. By $\mathcal{N}$ we denote the set of all charges (i.e. $\mathbb{C}$-valued additive functions) which are defined on $\mathbb{T}$. And by $\mathcal{N}_+$ we denote the set of $\nu \in \mathcal{N}$ which are measures (i.e. its values are nonnegative). We regard $\mathcal{N}$ as a Banach space defining the arithmetical operations naturally: $\forall E \in 2^{\mathbb{T}}$ $(\lambda_1\nu_1 + \lambda_2\nu_2)E = \lambda_1\nu_1 E + \lambda_2\nu_2 E$, and taking the norm $\|\nu\| := \sum_{t\in\mathbb{T}}|\nu_t|$ where $\forall t \in \mathbb{T}$ $\nu_t := \nu\{t\}$ is the value of ν on one-element set $\{t\}$. Obviously, $\operatorname{card}\mathcal{N} \in \mathbb{N}$.

1.2. To a charge $\nu \in \mathcal{N}$ and a set $E \in 2^{\mathbb{T}}$ there corresponds a functional ν_E defined on $\mathbb{C}^{\mathbb{T}}$ (B^A is the set of functions with domain A and taking values in B) by the formula

$$\forall x \in \mathbb{C}^{\mathbb{T}} \quad \nu_E x := \sum_{t\in E} x(t)\nu_t. \tag{1.1}$$

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This ν-integral is continuous in the semi-norm $\|\cdot\|_E = \max_{t\in E}|x(t)|$. Moreover, if the variation $\mathrm{var}_E\nu$ is finite in standard sense, then $\|x\|_E \approx 0 \Rightarrow \nu_E x \approx 0$.

1.3. Let $\nu \in \mathcal{N}_+,\ \tilde{\nu} \in \mathcal{N}$. We say that a charge $\tilde{\nu}$ is *ν-absolutely continuous* if

$$\forall E \in 2^{\mathbb{T}} \quad \nu E \approx 0 \Rightarrow \tilde{\nu}E \approx 0. \tag{1.2}$$

Obviously, from $\|\nu - \tilde{\nu}\| \approx 0$ it follows that charge $\tilde{\nu}$ is ν-absolutely continuous.

1.4. Let $\nu \in \mathcal{N}_+$; a function $x \in \mathbb{C}^{\mathbb{T}}$ is called *ν-locally integrable* if $1^\circ\ \forall E \in 2^{\mathbb{T}}\ \nu E \ll \infty$ and 2° the charge $E \mapsto \nu_E|x|$ is ν-absolutely continuous. (For $r \in \mathbb{R}$ the notation $|r| \ll \infty$ means that $(\exists n \in {}^{\mathrm{st}}\mathbb{N}) \quad (|r| < n))$. Moreover, if $\nu_{\mathbb{T}}|x| \ll \infty$ then we omit the adverb "locally".

1.4.1. If $\nu\mathbb{T} \ll \infty$ then each ν-locally integrable function is ν-integrable. If $\|x\|_{\mathbb{T}} \ll \infty$ then x is ν-locally integrable.

1.4.2. Let $\forall t \in \mathbb{T} \quad \delta_t \in \mathbb{C}^{\mathbb{T}}$ and $\forall s \in \mathbb{T} \quad \delta_t(s) := 0$ if $t \neq s$ and $\delta_t(t) := \nu_t^{-1}$. It is clear that $\nu_T x(\cdot)\delta_t = x(t)$, moreover discrete Dirac's delta δ_t is ν-integrable only in the case of $\nu_t \gg 0$ (i.e. the number ν_t is not infinitesimal). If $\nu_t \approx 0$, then the function $\sqrt{\nu_t}\delta_t$ is ν-integrable.

1.4.3. Let $\forall t \in \mathbb{T}\ \nu_t \approx 0$. If $x \in \mathbb{C}^{\mathbb{T}}$ and the charge $E \mapsto \nu_E|x|$ is ν-absolutely continuous, then the function x is ν-locally integrable.

□ Assume for some $E \in 2^{\mathbb{T}} \quad \nu E \ll \infty$, although $\nu_E|x| \approx +\infty$. Divide E in the disjoint parts E', $E"$ such that $\nu E' \approx \nu E" \approx \frac{1}{2}\nu E$. Then $\nu_{E'}|x| \geq \frac{1}{2}\nu_E|x|$ or $\nu_{E"} \geq \frac{1}{2}\nu_E|x|$. Hence, there exist sets E_n such that $E \supset E_1 \supset E_2 \supset \ldots$, with $\nu E_n|x| \leq 2^{-n}(\nu E + 1)$ and $\nu_{E_n}|x| \geq 2^{-n}\nu_E|x|$. Choose $n_0 \approx +\infty$ such that $2^{-n_0}\nu_E|x|$ is still infinite. Then $\nu E_{n_0} \approx 0$ and $\nu_{E_{n_0}}|x| \approx +\infty$ contrary to ν-absolute continuity of the charge $E \mapsto \nu_E|x|$ ■

1.4.4. A function $x \in \mathbb{C}^{\mathbb{T}}$ is called *ν-summable* if

$$(\forall \omega \in \mathbb{N}) \qquad (\omega \approx +\infty \Rightarrow \sum_{|x(t)|>\omega} |x(t)|\nu_t \approx 0). \tag{1.3}$$

In the same way as in [3], one can prove that a function $x \in \mathbb{C}^{\mathbb{T}}$ is ν-locally integrable if and only if it is ν-summable.

1.5. For ν-integral, analogs of the classical theorems of Lebesgue integral theory are almost trivial.

1.5.1. (Analogue of Nykodym-Radon's theorem). Let $\nu \in \mathcal{N}_+$, $\tilde{\nu} \in \mathcal{N}$, and $\forall E \in 2^{\mathbb{T}} \quad \nu E \ll \infty \Rightarrow |\tilde{\nu}E| \ll \infty$. Then charge $\tilde{\nu}$ is ν-absolutely continuous if and only if there exists ν-locally integrable function $x \in \mathbb{C}^{\mathbb{T}}$ such that $\forall E \in 2^{\mathbb{T}} \quad \tilde{\nu}E = \nu_E x$.

1.5.2. (Analogue of Fischer-Riesz's theorem). Let $x,\, y \in \mathbb{C}^{\mathbb{T}}, \nu \in \mathcal{N}_+$ and $\nu_{\mathbb{T}}|x - y| \approx 0$. If the function x is ν-integrable, then y is ν-integrable.

□ Since $\forall E \in 2^{\mathbb{T}} \quad |\nu_E x - \nu_E y| \leq \nu_{\mathbb{T}}|x - y|$, then

$$\nu_{\mathbb{T}}|x - y| \approx 0 \Rightarrow (\forall E \in 2^{\mathbb{T}})\,(\nu_E x \approx \nu_E y)\blacksquare \tag{1.4}$$

1.5.3. Let $p(t)$ be a sentence depending on a variable $t \in \mathbb{T}$. We say that $p(t)$ holds *ν-quasi everywhere* on E if there exists $E_0 \subseteq E$ such that $\nu E_0 \approx 0$ and $p(t)$ is true when $t \in E \backslash E_0$.

1.5.4. (Analogue of Lebesgue's theorem on majorable convergence).

Let $\nu \in \mathcal{N}_+$ and x, y are $\nu -$ integrable functions from $\mathbb{C}^{\mathbb{T}}$. If $\nu E \ll \infty$ and $x(t) \approx y(t)$ holds ν-quasi everywhere on E then

$$\nu_E x \approx \nu_E y. \tag{1.5}$$

1.5.5. Let $\nu \in \mathcal{N}_+$, $E \in 2^{\mathbb{T}}$. We call the set qker $\nu_E\{x \in \mathbb{C}^{\mathbb{T}} : \nu_E|x| \approx 0\}$ *E-quasikernel* of ν-integral. E-quasikernel is trivial in the sense that $\nu_E|x| \approx 0$ implies $x(t) \approx 0$ quasi everywhere on E.

□ For arbitrary $r \in \mathbb{R}$ we denote $E_r := \{t \in E : |x(t)| > r\}$. As $\nu_E|x| > r\nu E_r$, then $(\forall n \in^{\text{st}} \mathbb{N})(\nu E_{1/n} \approx 0)$ for x from E-quasikernel. By Robinson's lemma, $\nu E_r \approx 0$ for some $r \approx 0$, and for $t \in E \backslash E_r$ we have $|x(t)| \leq r$■

1.5.6. Corollary. Let $\nu \in \mathcal{N}_+, \tilde{\nu} \in \mathcal{N}$. If $\|\nu - \tilde{\nu}\| \approx 0$ then $\tilde{\nu}_t/\nu_t \approx 1$ quasi everywhere on $\mathbb{T}$. In particular, the function $t \mapsto \tilde{\nu}_t/\nu_t$ is ν-locally integrable.

2. Nearstandard charges

As $\mathbb{T}$ itself is nonstandard, one can define the nearstandardness on $\mathbb{T}$ only indirectly.

2.1. We postulate the existence of a standard set $\mathbf{T}$ and a mapping Q which satisfy the conditions:

1° a mapping Q transforms each one-element set $\{t\} \in \mathbb{T}$ into a set $Qt \in \mathbf{T}$ so that

$$t_1 \neq t_2 \quad \Rightarrow Qt_1 \cap Qt_2 = \emptyset; \tag{2.1}$$

2° by the definition

$$\forall E \in 2^{\mathbb{T}} \qquad QE := \bigcup_{t \in E} Qt; \tag{2.2}$$

3° a set $\mathbf{T}$ coincides with the standartization of $Q\mathbb{T}$, $\mathbf{T} = {}^{\text{s}}(Q\mathbb{T})$, i.e.

$$\mathbf{T} \text{ is standard and } {}^{\text{st}}\mathbf{T} = {}^{\text{st}}(Q\mathbb{T}). \tag{2.3}$$

2.1.1. Obviously, a mapping Q is an embedding of the algebra $2^{\mathbb{T}}$ into the algebra $2^{\mathbf{T}}$ which preserves the theoretic-set operations.

2.1.2. We obtain an evident example taking as $\mathbb{T}$ a "*discrete interval*" $\overset{\vdash}{ab}$:

$$\overset{\vdash}{ab} := \{a, a+h, ..., b-2h, b-h\} \tag{2.4}$$

where a, b, $h \in \mathbb{R}$, $\quad a < b$, $\quad h > 0$ and $h \approx 0$, and as $\mathbf{T}$ the standartization of the "solid" interval $[a, b[= \{r \in \mathbb{R} : a \leq r < b\}$. We define the embedding Q: $2^{\mathbb{T}} \to 2^{\mathbf{T}}$ by the formula

$$\forall t \in \overset{\vdash}{ab} \quad Qt = [t, t+h[. \tag{2.5}$$

For the condition (2.3) to be true in the case when $|a| < \infty$ ($|b| < \infty$), one requires ${}^\circ a \leq a$ (${}^\circ b \geq b$).

Let $\nu \in \mathcal{N}$ be a charge. We transfer it from $\mathbb{T}$ onto $\mathbf{T}$ putting

$$(\forall E \in 2^{\mathbb{T}}) \quad (\mathcal{E} = QE \Rightarrow Q\nu\mathcal{E} := \nu E). \tag{2.6}$$

However, the domain $Q2^{\mathbb{T}}$ of the charge $Q\nu$ is not sufficiently rich. Therefore we extend a little our constructions. Denote by $\mathcal{M}$ the set of all regular (relatively to fixed standard topology on $\mathbf{T}$) $\mathbb{C}$-valued σ-additive charges on $\mathbf{T}$, and by $\mathcal{M}_+$ those of them which take nonnegative values (they are the measures). We treat the set $\mathcal{M}$ as a normed space with the usual arithmetical operations and with the norm

$$\forall \mu \in \mathcal{M} \quad \|\mu\| := \text{var}\ \mu\mathbb{T} \tag{2.7}$$

where var is the total variation.

Fix some standard measure $\lambda \in {}^{st}\mathcal{M}_+$ such that

$$\forall t \in \mathbb{T} \quad Qt \in \Lambda \text{ and } \lambda Qt = h \tag{2.8}$$

where Λ is an algebra of λ-measurable sets $\mathcal{E} \subseteq \mathbf{T}$, and h is some fixed positive number.

2.1.3. Definition. A triple $(\mathbf{T}, Q, \lambda)$, satisfying the conditions stated above, is called *a standard filling* of the finite set $\mathbb{T}$. A charge $Q\nu$ (where $\nu \in \mathcal{N}$) is extended from the algebra $Q2^{\mathbb{T}}$ onto ν-algebra Λ putting

$$\forall \mathcal{E} \in \Lambda \quad Q\nu\mathcal{E} := \sum_{t\in\mathbb{T}} \lambda(\mathcal{E} \cap Qt)\nu_t h^{-1}. \tag{2.9}$$

2.1.4. Note that $Q\nu(\mathbf{T}\backslash Q\mathbb{T}) = 0$. It is easy to see that the mapping $\nu \mapsto Q\nu$ isometrically transforms $\mathcal{N} \longrightarrow \mathcal{M}$.

2.1.5. In the case $\mathbb{T} = \overset{\vdash}{ab}$ (s. 2.1.2) and $\mathbf{T}= {}^{s}[a,b[$, and Q being defined by the formula (2.5), we take standard Lebesgue measure on $\mathbf{T}$ as λ, so that $Qt = h$ is the step of the discrete interval (2.4).

Henceforth, $\mathbb{T}$ and its standard filling are given and fixed.

2.1.6. Definition. We call a charge $\nu \in \mathcal{N}$ *relatively standard* (and we write $\nu \in {}^{\mathrm{rst}}\mathcal{N}$) if there exists a charge $\mu \in {}^{\mathrm{st}}\mathcal{M}$ such that

$$\forall \mathcal{E} \in \Lambda \quad Q\nu(\mathcal{E} \cap Q\mathbb{T}) = \mu(\mathcal{E} \cap Q\mathbb{T}). \tag{2.10}$$

In this case we call μ *a standardized image* of the charge ν.

2.2. We introduce a mapping Π, "conjugated" to embedding $Q : \mathcal{N} \longrightarrow \mathcal{M}$. By definition,

$$\forall \mu \in \mathcal{M} \quad \forall E \in 2^{\mathbb{T}} \quad \Pi\mu E := \mu Q E, \tag{2.11}$$

in particular, $\forall t \in \mathbb{T} \quad (\Pi\mu)_t = \mu Qt$. We call a transform Π an inductor $\mathcal{M} \longrightarrow \mathcal{N}$.

2.2.1. It is easy to see that inductor Π is the left inverse to Q, namely $\forall \nu \in \mathcal{N}\ \forall E \in 2^{\mathbb{T}} \quad \Pi Q\nu E = \nu E$. If $\nu \in {}^{\mathrm{rst}}\mathcal{N}$ and μ is a standardized image of ν, then $\Pi\mu = \nu$. Inductor Π is an unstretching transform, i.e. $\forall \mu \in \mathcal{M} \quad \|\Pi\mu\| \leq \|\mu\|$.

2.2.2. We call the set

$$\text{qker } \Pi := \{\mu \in \mathcal{M} : \quad (\forall E \in 2^{\mathbb{T}})\, (\Pi\mu E \approx 0)\} \tag{2.12}$$

the *quasikernel* of inductor Π. An inductor Π is called *exact* if ${}^{\mathrm{st}}$qker $\Pi = \{0\}$ where 0 is the charge with all values equal to zero.

2.2.3. For example, let $\mathbb{T} := \overset{\vdash}{ab}$ (s.2.1.2) be a discrete interval containing each standard rational number from the "solid" interval $[a,b[$. For this purpose, it is sufficient to suppose that the infinitesimal h has a form $h = (n!)^{-1}$ for some $n \in \mathbb{N}\backslash{}^{\mathrm{st}}\mathbb{N}$. Corresponding inductor Π is exact in this case.

□ Let $\mu \in {}^{\mathrm{st}}$qker Π, i.e. $\mu \in {}^{st}\mathcal{M}$ and $\forall E \subseteq \overset{\vdash}{ab} \quad \Pi\mu E \approx 0$. We have to prove $\mu = 0$. Considering transfer principle and regularity (relatively the standard topology on $\mathbb{R}$) of a charge μ, it is sufficient to prove that $\mu[\alpha,\beta[= 0$ for each standard interval $[\alpha,\beta[\subseteq {}^{s}[a,b[$. Consider the numbers α, $\beta \in {}^{\mathrm{st}}[a,b[, \quad \alpha < \beta$ and standard sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ of the rational numbers such that $\alpha_n \nearrow \alpha$, $\beta_n \nearrow \beta$ as $n \to \infty$. As a charge μ is countably-additive, then $\mu[\alpha_n, \beta_n[\to \mu[\alpha, \beta[$ if $n \to \infty$. By the nonstandard criterion of limit,

$$n \approx +\infty \quad \Rightarrow \quad \mu[\alpha_n, \beta_n[\approx \mu[\alpha, \beta[. \tag{2.13}$$

Let $n \in {}^{\mathrm{st}}\mathbb{N}$. Due to the condition ${}^{\mathrm{st}}\mathbb{Q}\cap[a,b[\subseteq \overset{\vdash}{ab}$ $\;[\alpha_n, \beta_n[= \sum_{t\in[\alpha_n,\beta_n[} \mu[t, t+h[= \sum_{t\in[\alpha_n,\beta_n[}(\Pi\mu)_t = \Pi\mu\{t \in \overset{\vdash}{ab} : \alpha_n \le t \le \beta_n\} = \Pi\mu E$, $\;E \in 2^{\mathbb{T}}$. In consequence, $\forall n \in {}^{\mathrm{st}}\mathbb{N}$ $\;\mu[\alpha_n, \beta_n[\approx 0$. By Robinson's lemma, $\mu[\alpha_{n_0}, \beta_{n_0}[\approx 0$ for some $n_0 \approx +\infty$. Then by (2.13) $\mu[\alpha, \beta[\approx 0$. As number $\mu[\alpha, \beta[$ is standard, we obtain $\mu[\alpha, \beta[= 0$ ■

Henceforward, we assume that an inductor P in question is exact.

2.2.4. Corollary. Let $\nu \in {}^{\mathrm{rst}}\mathcal{N}$ and $\|\nu\| \approx 0$; then $\nu = 0$.

2.2.5. Corollary. Standardized image of the charge $\nu \in {}^{\mathrm{rst}}\mathcal{N}$ is unique.

Remark. At first sight it seems naturally to define relative standardness of a charge $\nu \in \mathcal{N}$ demanding $Q\nu \in {}^{\mathrm{st}}\mathcal{M}$. However in general case, when the set $Q\mathbb{T}$ is nonstandard, no charge of the form $Q\nu$ is standard.

2.3. We call a charge $\nu \in \mathcal{N}$ *nearstandard* (and we write $\nu \in {}^{\mathrm{nst}}\mathcal{N}$) if $\|\nu - \Pi\mu\| \approx 0$ for some $\mu \in {}^{\mathrm{st}}\mathcal{M}$. In this case we denote ${}^{\circ}\nu := \mu$, $\;{}^{\bullet}\nu = \Pi\mu$. Obviously, the shadows ${}^{\circ}\nu$ and ${}^{\bullet}\nu$ are uniquely determined.

2.3.1. Let $\nu \in {}^{\mathrm{nst}}\mathcal{N}$; then $\|\nu - {}^{\bullet}\nu\| \approx 0$, and by 1.3 the charge ${}^{\bullet}\nu$ is ν-absolutely continuous. Therefore ${}^{\bullet}\nu_t/\nu_t \approx 0$ quasi everywhere on $\mathbb{T}$ relative to $\operatorname{var}\nu$.

2.3.2. Each charge $\mu \in {}^{\mathrm{st}}\mathcal{M}$ is a shadow of some charge $\nu \in \mathcal{N}$, e.g. $\mu = {}^{\circ}(\Pi\mu)$.

2.3.3. Example. Let $\nu \in \mathcal{N}$ and $\sum_{t\in\mathbb{T}} |\nu_t - h| \approx 0$, then $\nu \in {}^{\mathrm{nst}}\mathcal{N}$ and ${}^{\circ}\nu = \lambda$.

2.3.4. The definition of nearstandardness of a charge $\nu \in \mathcal{N}$ by requiring $\|Q\nu - \mu\| \approx 0$ for some $\mu \in {}^{\mathrm{st}}\mathcal{M}$ is more particular, because $\|\nu - \Pi\mu\| \le \|Q\nu - \mu\|$.

2.4. (Analogue of Helly's theorem). Let ν, $\tilde{\nu} \in \mathcal{N}$, $x \in \mathbb{C}^{\mathbb{T}}$, $E \in 2^{\mathbb{T}}$, a function x is ν-integrable and $\tilde{\nu}$-integrable on the set E with $\nu E \ll \infty$. Then

$$\|\nu - \tilde{\nu}\| \quad \approx \quad 0 \quad \Rightarrow \quad \nu_E x \quad \approx \quad \tilde{\nu}_E x. \tag{2.14}$$

In particular, if $\nu \in {}^{\mathrm{nst}}\mathcal{N}$, $\nu E \ll \infty$, then for ν-integrable and ${}^{\bullet}\nu$-integrable function x we have

$$\nu_E x \quad \approx \quad {}^{\bullet}\nu_E x. \tag{2.15}$$

□ Denote $E_n := \{t \in E : \; |x(t)| \ge n\}$. Then $\forall n \in {}^{\mathrm{st}}\mathbb{N}$ $\;\sum_{t\in E\setminus E_n} |x(t)| \cdot |\nu_t - \tilde{\nu}_t| \approx 0$. By Robinson's lemma, $\sum_{t\in E\setminus E_n} |x(t)| \cdot |\nu_t - \tilde{\nu}_t| \approx 0$ for some $n_0 \approx +\infty$. But an integrable function is summable, so $\sum_{t\in E_{n_0}} |x(t)| \cdot (|\nu_t| + |\tilde{\nu}_t|) \approx 0$. Therefore $(\nu - \tilde{\nu})_E x \approx 0$ ■

2.4.1. The condition $\|\nu - \tilde{\nu}\| \approx 0$ does not ensure the equivalence of ν-integrability and $\tilde{\nu}$-integrability. For example, let $t_0 \in \mathbb{T}$ and $\forall t \in \mathbb{T}\setminus\{t_0\}$ $\;\nu_t = h$ and $\nu_{t_0} = h + \sqrt{h}$. Then it can be easily seen that $\nu \in {}^{\mathrm{nst}}\mathcal{N}$ and ${}^{\circ}\nu = \lambda$. Denoting $\tilde{\nu} := \Pi\lambda$, we find $\forall t \in \mathbb{T}$ $\;\tilde{\nu}_t = h$, consequently $\|\nu - \tilde{\nu}\| = \sqrt{h} \approx 0$. Define $x \in \mathbb{C}^{\mathbb{T}}$ by the formula $x(t) = 0$ when $t \ne t_0$ and $x(t_0) = h^{-1/2}$. Then $\tilde{\nu}_{\mathbb{T}} x = \sqrt{h} \approx 0$, so the function x is $\tilde{\nu}$-integrable. However, if $E \in 2^{\mathbb{T}}$ and $t_0 \in \mathbb{T}$, then $\nu_E x = \frac{h+\sqrt{h}}{\sqrt{h}} \not\approx 0$, although $\nu\{t_0\} \approx 0$. We can see x is not ν-integrable ■

3. Nearstandard functions.

We consider some connections between functions of discrete and continuous arguments.

3.1. To a measure ν we set in correspondence a Hilbert space $\mathbb{H}_\nu$ containing the functions $x \in \mathbb{C}^{\mathbb{T}}$ with the equality definition: $((x = y) \equiv (\forall t \in \mathbb{T})\ (\nu_t > 0 \Rightarrow x(t) = y(t))$, with the arithmetical operations which hold pointwise, and with the inner product $(x|y) := \nu_{\mathbb{T}} x\bar{y} := \sum_{t\in\mathbb{T}} x(t)\overline{y(t)}\nu_t$, and with the norm $\|x\| = (x|x)^{\frac{1}{2}}$. Obviously, $\dim \mathbb{H}_\nu = \operatorname{card}\{t \in \mathbb{T} : \nu_t > 0\} \le \operatorname{card}\mathbb{T}$. Moreover,

$$\forall x \in \mathbb{H}_\nu \quad x = \sum_{t\in T}(x|\delta_t)\delta_t\nu_t, \tag{3.1}$$

$(\sqrt{\nu_t}\delta_t)$ being an orthonormal basis in $\mathbb{H}_\nu$ (see 1.4.2). Let $\mu \in \mathcal{M}_+$; by $\mathbf{H}_\mu$ we denote a Hilbert space $L_{2,\mu}(\mathbf{T})$ with the usual inner product $(\xi|\eta) := \int_{\mathbf{T}} \xi(\tau)\overline{\eta(\tau)}\mu(d\tau)$ and with the norm $\|\xi\| = (\xi|\xi)^{1/2}$. By the assumption of exactness of the inductor Π, the subalgebra $\{QE : \ E \in 2^{\mathbb{T}}\}$ of the σ-algebra Λ is complete in the following sense.

3.1.1. Let $\mu \in {}^{\mathrm{st}}\mathcal{M},\ \xi \in {}^{\mathrm{st}}\mathbf{H}_\mu$; then

$$(\forall E \in 2^{\mathbb{T}}) \quad (\int_{QE} \xi(\tau)\mu(d\tau) \approx 0 \quad \Rightarrow \quad \xi = 0). \tag{3.2}$$

□ Denote

$$\forall \mathcal{E} \in \Lambda \quad \mu_\xi \mathcal{E} := \int_{\mathcal{E}} \xi(\tau)\mu(d\tau). \tag{3.3}$$

Obviously, $\mu_\xi \in {}^{\mathrm{st}}\mathcal{M}$. Suppose that $\forall E \in 2^{\mathbb{T}} \quad \mu_\xi QE \approx 0$, i.e. the lefthand side of (3.2) holds. According to (2.11) this means that $\forall E \in 2^{\mathbb{T}} \quad \Pi\mu E \approx 0$, i.e. $\mu_\xi \in \operatorname{qker}\Pi$. By (2.20), $\forall \mathcal{E} \in \Lambda \quad \mu_\xi\mathcal{E} \approx 0$, i.e. $\int_{\mathcal{E}} \xi(\tau)\mu(d\tau) = 0$, so $\xi(\tau) = 0 \quad \mu-$ a.e. ■

3.2. Show that the embedding $Q : \mathcal{N} \longrightarrow \mathcal{M}$ has a natural extension $\mathbb{H}_\nu \longrightarrow \mathbf{H}_\mu$.

Let $\nu \in \mathcal{N},\ x \in \mathbb{C}^{\mathbb{T}}$. Denote

$$\nu_x E := \nu_E x = \sum_{t\in E} x(t)\nu_t. \tag{3.3'}$$

3.2.1. Let $\nu \in \mathcal{N}_+$. Then to each function $x \in \mathbb{H}_\nu$ there corresponds a unique function $Qx \in \mathbf{H}_\mu$ such that $Q(\nu_x) = (Q\nu)_{Qx}$ (s.(3.3)). Namely,

$$\forall t \in \mathbb{T} \quad \forall \tau \in \mathbf{T} \quad Qx(\tau) = \begin{cases} x(t), & \text{when } \tau \in Qt, \\ 0, & \text{when } \tau \in \mathbf{T}\backslash Q\mathbb{T}. \end{cases} \tag{3.4}$$

□ Let $\mathcal{E} \in \Lambda$. As $(\nu_x)_t = \nu_x\{t\} = x(t)\nu_t$, then according to (2.9), $Q(\nu_x)\mathcal{E} = \sum_{t\in\mathbb{T}} \lambda(\mathcal{E} \cap Qt)(\nu_x)_t h^{-1} = \sum_{t\in\mathbb{T}} \lambda(\mathcal{E}\cap Qt)x(t)\nu_t h^{-1}$. Therefore

$$\mathcal{E} \subseteq Qt \quad \Rightarrow \quad Q(\nu_x)\mathcal{E} = \lambda(\mathcal{E})x(t)\nu_t h^{-1}. \tag{3.5}$$

On the other hand, according to (3.3), $(Q\nu)_{Qx}\mathcal{E} = \int_{\mathcal{E}} Qx(\tau)Q\nu(d\tau)$. But according to (2.9), $Q\nu(d\tau) = \lambda(d\tau)\nu_t h^{-1}$ on the set $\mathcal{E} \subset Qt$. Thus if $\mathcal{E} \subseteq Qt$, then

$(Q\nu)_{Qx}\mathcal{E} = \int_{\mathcal{E}} Qx(\tau)\lambda(d\tau)\nu_t h^{-1}$. Now from $Q(\nu_x) = (Q\nu)_{Qx}$ and (3.5) we conclude that $\forall t \in \mathbb{T} \quad \forall \mathcal{E} \in \Lambda$

$$\mathcal{E} \subseteq Qt \quad \Rightarrow \quad \lambda(\mathcal{E})x(t) = \int_{\mathcal{E}} Qx(\tau)\lambda(d\tau). \tag{3.6}$$

Thus $Qx(\tau) = x(\tau) \quad \lambda$- a. e. for $\tau \in Qt$. By 2.1.4, $\quad Q(\nu_x)\mathcal{E} = 0$ when $\mathcal{E} \subseteq \mathbf{T}\backslash Q\mathbb{T}$ ■

3.2.2. The embedding Q defined on sets is related to the embedding Q defined on functions by the relationship

$$\forall E \in 2^{\mathbb{T}} \quad Q{\chi_E}^{\mathbb{T}} = {\chi_{QE}}^{\mathbf{T}} \tag{3.7}$$

where ${\chi_A}^B(b) = 1$ if $b \in A \cap B$ and ${\chi_A}^B(b) = 0$ if $b \in B\backslash A$.

3.3. Projector P. Let $\mu \in \mathcal{M}_+$. The set $Q\mathbb{C}^{\mathbb{T}}$ is a finite-dimensional subspace in $\mathbf{H}_\mu$: $\dim Q\mathbb{C}^{\mathbb{T}} \leq \operatorname{card} \mathbb{T}$. Denote by P_μ (in abbreviated form P) the orthoprojector $\mathbf{H}_\mu \longrightarrow Q\mathbb{C}^{\mathbb{T}}$.

3.3.1. The orthoprojector P is an averaging operator, namely
$\forall \xi \in \mathbf{H}_\mu \quad \forall t \in \mathbb{T} \quad \forall \tau \in \mathbf{T}$

$$P\xi(\tau) = \begin{cases} 1/\nu_t \int_{Qt} \xi(\sigma)\mu(d\sigma), & \text{when } \tau \in Qt, \\ 0, & \text{when } \tau \in \mathbf{T}\backslash Q\mathbb{T}, \end{cases} \tag{3.8}$$

where $\nu_t := \mu Qt$. In particular, if $\mu = Q\nu$ for some $\nu \in \mathcal{N}_+$, then

$$P\xi(\tau) = \begin{cases} h^{-1} \int_{Qt} \xi(\sigma)\lambda(d\sigma), & \text{when } \tau \in Qt, \\ 0, & \text{when } \tau \in \mathbf{T}\backslash Q\mathbb{T}. \end{cases} \tag{3.9}$$

□ The set $(\sqrt{\nu_t}Q\delta_t)_{t\in\mathbb{T}}$ is an orthonormal basis of the space $Q\mathbb{C}^{\mathbb{T}}$. By means of this basis we find $P\xi = \sum_{t\in\mathbb{T}}(\xi|Q\delta_t)Q\delta_t\nu_t$. Thus $\forall \tau \in Qt \quad P\xi(\tau) = (\xi|Q\delta_t)$ being the same as (3.8). According to (2.9),

$$\mu = Q\nu \quad \Rightarrow \quad \mu(d\tau)|_{Qt} = \nu_t h^{-1}\lambda(d\tau). \tag{3.10}$$

Thus if $\mu = Q\nu$, then (3.9) follows from (3.8) ■

3.3.2. Definition. Let $\mu \in {}^{\mathrm{st}}\mathcal{M}_+$, so that Hilbert space $\mathbf{H}_\mu$ is standard. A projector $P = P_\mu$ is called *a quasi-unity* (of the algebra $\mathcal{B}(\mathbf{H}_\mu)$) if $\forall \xi \in {}^{\mathrm{st}}\mathbf{H}_\mu \ \|P\xi - \xi\| \approx 0$.

3.3.3. Note that in the case when $\mathbb{T}$ is a discrete interval $\overset{\vdash}{ab}$, $\mathbf{T}= {}^{\mathrm{s}}[a,b[$ and λ is Lebesgue measure on $\mathbf{T}$, the projector P is a quasi-unity.

3.3.4. Henceforth, we suppose that the orthoprojector P in question is a quasi-unity. Thus the functions $\xi \in {}^{\mathrm{st}}\mathbf{H}_\mu$ are within an arbitrary infinitesimal precision approximated by the functions from $Q\mathbb{C}^{\mathbb{T}}$.

However, in some sense $Q\mathbb{C}^{\mathbb{T}}$ is "more ample" than ${}^{\mathrm{st}}\mathbf{H}_\mu$. Namely, if $\operatorname{card}\{t \in \mathbb{T} : \mu Qt > 0\} \approx +\infty$, then $(\exists \eta \in Q\mathbf{C}^{\mathbb{T}}\backslash\{0\}) \quad (\forall \xi \in {}^{\mathrm{st}}\mathbf{H}_\mu) \quad ((\xi|\eta) = 0)$. The proof follows easy from the idealization principle.

3.4. We extend inductor Π, given on charges by the relationship $\Pi\mu E = \mu QE$, to the functions through the condition

$$\Pi(\mu_\xi) = (\Pi\mu)_{\Pi\xi}.$$

3.4.1. The next formula is true:

$$\Pi = Q^{-1}P, \tag{3.11}$$

where Q^{-1} is the operator inverse to $Q : \mathbb{C}^{\mathbb{T}} \longrightarrow Q\mathbb{C}^{\mathbb{T}}$ and P is the orthoprojector $\mathbf{H}_\mu \longrightarrow Q\mathbb{C}^{\mathbb{T}}$.

☐ Let $t \in \mathbb{T}$. Then $\Pi(\mu_\xi)\{t\} = \mu_\xi Q\{t\} = \int_{Qt} \xi(\tau)\mu(d\tau)$. On the other hand, $(\Pi\mu)_{\Pi\xi}\{t\} = \sum_{s\in\{t\}} \Pi\xi(s)(\Pi\mu)_s = \Pi\xi(t)\Pi\mu\{t\} = \Pi\xi(t)\mu Qt$. Therefore

$$\forall \xi \in \mathbf{H}_\mu \quad \forall t \in \mathbb{T} \quad \Pi\xi(t) = \frac{1}{\mu Qt}\int_{Qt} \xi(\tau)d\tau, \tag{3.12}$$

which is equivalent to the formula (3.11) ■

3.4.2. Obviously, the equalities

$$\|Q\| = \|\Pi\| = 1, \quad Q\Pi = P, \quad \Pi Q = I_{\mathbb{C}^{\mathbb{T}}}, \quad PQ = Q \tag{3.13}$$

are true for the operators Q and Π acting on functions.

3.4.3. Let $\mu \in \mathcal{M}_+,\ \nu := \Pi\mu$, e.g. $\mu = Q\nu$ for some $\nu \in \mathcal{N}_+$. Then the operators $Q : \mathbb{H}_\nu \longrightarrow \mathbf{H}_\mu,\ \Pi : \mathbf{H}_\mu \longrightarrow \mathbb{H}_\nu$ are mutually adjoint:

$$\Pi = Q^*, \qquad Q = \Pi^*. \tag{3.14}$$

☐ Let $x \in \mathbb{H}_\nu, \xi \in \mathbf{H}_\mu$. Considering (3.13) and isometricity of Q we find $(Qx|\xi) = (PQx|\xi) = (Qx|P\xi) = (x|Q^{-1}P\xi) = (x|\Pi\xi)$ ■

3.4.4. Let $\mu \in \mathcal{M}_+, \quad \nu := \Pi\mu$. Then the embedding $Q : \mathbb{H}_\nu \longrightarrow \mathbf{H}_\mu$ is isometrical.

☐ By the equality $(\Pi\mu)_t = \mu Qt, \quad \forall x \in \mathbb{H}_\nu \quad \|Qx\|^2 = \int_{\mathbf{T}} |Qx(\tau)|^2\nu(d\tau) = \sum_{t\in\mathbf{T}} |x(t)|^2\mu Qt = \sum_{t\in\mathbb{T}} |x(t)|^2\nu_t = \|x\|^2$ ■

3.5. Let $\mu \in \mathcal{M}_+, \quad \mathcal{E} \in \Lambda$, with $\mu\mathcal{E} < \infty$. Then the formula (3.3) defines $\mu_{\mathcal{E}}\xi$ when $\xi \in \mathbf{H}_\mu$, with $\mu_{\mathcal{E}} \in \mathbf{H}_\mu^*$ and $\|\mu\| = \sqrt{\mu\mathcal{E}}$.

3.5.1. Let $\mu \in \mathcal{M}_+, \nu := \Pi\mu$, e.g. $\mu = Q\nu$ for some $\nu \in \mathcal{N}_+$. If $\mu QE < \infty$ then

$$\forall \xi \in \mathbf{H}_\mu \quad \mu_{QE}\xi = \nu_E\Pi\xi. \tag{3.15}$$

☐ This can be proved as follows

$$\mu_{QE}\xi = \mu_\xi QE = (\Pi\mu_\xi)E = (\Pi\mu)_{\Pi\xi}E = \nu_{\Pi\xi}E = \nu_E\Pi\xi \quad ■$$

3.5.2. Corollary. Let $\nu \in \mathcal{N}_+, \quad \mu = Q\nu, \quad E \in 2^{\mathbb{T}}$. Then

$$\forall x \in \mathbb{H}_\nu \quad \mu_{QE}Qx = \nu_E x. \tag{3.16}$$

3.6. Let $\nu \in {}^{\mathrm{rst}}\mathcal{N}_+$ and μ is the standardized image of the measure ν. A function $x \in \mathbb{C}^{\mathbb{T}}$ is called ν-standard (we write $x \in {}^{\mathrm{rst}}\mathbb{H}_\nu$) if $x = \Pi\xi$ for some function $\xi \in {}^{\mathrm{st}}\mathbf{H}_\mu$. Then we call ξ *ν-standardized image* of the function x.

3.6.1. The ν-standardized image of the function $x \in {}^{\mathrm{rst}}\mathbb{H}_\nu$ is determined uniquely.

☐ Assume that $\xi \in {}^{\mathrm{st}}\mathbf{H}_\mu$ and $\Pi\xi = 0$. As $\Pi = Q^{-1}P, \quad P\xi = 0$. Assuming that P is a quasi-unity, $\|\xi\| = \|\xi - P\xi\| \approx 0$. As the number $\|\xi\|$ is standard, $\xi = 0$ ■

3.6.2. Each ν-standard function is ν-integrable.

$\square$ Let $\nu \in {}^{\rm rst}\mathcal{N}_+, x \in {}^{\rm rst}\mathbb{H}_\nu$. Denote by μ and ξ respectively the standardized images of ν and x so that $\nu = \Pi\mu, \quad x = \Pi\xi$. Let $E \in 2^{\mathbb{T}}$ and $\mathcal{E} := QE$. By (3.15), $\nu_E x = \mu_{QE}\xi = \int_{\mathcal{E}} \xi(\tau)\mu(d\tau)$. If $\nu E \approx 0$, then $\mu\mathcal{E} \approx 0$ and by absolute continuity property of Lebesgue integral, $\nu_E x \approx 0$. Moreover, if $\nu E \ll \infty$, then $\mu\mathcal{E} \ll \infty$ and $|\int_{\mathcal{E}} \xi(\tau)\mu(d\tau)| \ll \infty$, because ξ is standard. Therefore $|\nu_E x| \ll \infty$ $\blacksquare$

3.6.3. Definition (cf. with [1],[2]). Let $\nu \in {}^{\rm nst}\mathcal{N}_+, \quad \mu := {}^\circ\nu, \ \xi \in \mathbf{H}_\mu, \quad E \in 2^{\mathbb{T}}$. A function $x \in \mathbb{C}^{\mathbb{T}}$ is called *a lifting* of the function ξ on E if x is both ν-integrable and ${}^\bullet\nu$-integrable, and ν-quasi-everywhere $x(t) \approx \Pi\xi(t)$ on E.

3.6.4. Theorem. Let $\nu \in^{\rm nst} \mathcal{N}_+, \quad \mu = {}^\circ\nu, \quad \xi \in {}^{\rm st}\mathbf{H}_\mu, \quad E \in 2^{\mathbb{T}}$ and x is a lifting of x on E. If $\nu E \ll \infty$, then $\int_{QE} \xi(\tau)\mu(d\tau) \approx \sum_{t\in E} x(t)\nu_t$.

$\square$ Let $\nu E \ll \infty$ and ${}^\bullet\nu := \Pi\mu$. According to (3.15), $\mu_{QE}\xi = ({}^\bullet\nu)_E\Pi\xi$. By 3.6.2, the function $\Pi\xi$ is ${}^\bullet\nu$-integrable. Then the function $x - \Pi\xi$ is ${}^\bullet\nu$-integrable, too. Let $E_0 \subseteq E, \quad ({}^\bullet\nu)E_0 \approx 0$ and $\forall t \in E\backslash E_0 \quad x(t) \approx \Pi\xi(t)$. Then $({}^\bullet\nu)_E|x - \Pi\xi| = ({}^\bullet\nu)_{E_0}|x - \Pi\xi| + ({}^\bullet\nu)_{E\backslash E_0}|x - \Pi\xi| \approx 0$. So $({}^\bullet\nu)_E x \approx ({}^\bullet\nu)_E\Pi\xi$. Thus $\mu_{QE}\xi \approx ({}^\bullet\nu)_E x$ and it remains to apply (2.15) $\blacksquare$

3.7. *ν-nearstandardness.* Let $\nu \in {}^{\rm nst}\mathcal{N}_+$ and $\quad \mu := {}^\circ\nu$. A function $x \in \mathbb{C}^{\mathbb{T}}$ is called ν-nearstandard (we write $x \in {}^{\rm nst}\mathbb{H}_\nu$) if $\|Qx - \xi\| \approx 0$ for some function $\xi \in {}^{\rm st}\mathbf{H}_\mu$. In this case ξ is called *ν-shadow* of the function x and we write ${}^\circ x := \xi$; function ${}^\bullet x := \Pi\xi = \Pi({}^\circ x)$ is called *ν-shadow* of the function x *on* $\mathbb{T}$.

3.7.1. As $(\forall \xi \in {}^{\rm st}\mathbf{H}_\mu)$ $(\|\xi\| \approx 0 \Rightarrow \xi = 0)$, the shadows ${}^\circ x$ and ${}^\bullet x$ are uniquely determined.

3.7.2. Let $\nu \in {}^{\rm nst}\mathcal{N}_+, \mu := {}^\circ\nu, x \in \mathbb{C}^{\mathbb{T}}$. Then $x \in {}^{\rm nst}\mathbb{H}_\nu$ if and only if $(\exists \xi \in {}^{\rm st}\mathbf{H}_\mu)$ $(\|x - \Pi\xi\| \approx 0)$. If $\xi \in {}^{\rm st}\mathbf{H}_\mu$ and $\|x - \Pi\xi\| \approx 0$, then $\xi = {}^\circ x$.

$\square$ As the embedding $Q : \mathbb{H}_\nu \longrightarrow \mathbf{H}_\mu$ is isometrical (s. 3.4.4) and the orthoprojector $P : \mathbf{H}_\mu \longrightarrow Q\mathbb{C}^{\mathbb{T}}$ is a quasi-unity, then

$$\forall x \in \mathbb{H}_\nu \ \forall \xi \in {}^{\rm st}\mathbf{H}_\mu \ \|x - \Pi\xi\| \approx \|Qx - \xi\| \quad \blacksquare$$

3.7.3. Corollary. Let $x \in {}^{\rm nst}\mathbb{H}_\nu$. Then $\|x - {}^\bullet x\| \approx 0$.

3.7.4. Let $\nu \in {}^{\rm nst}\mathcal{N}_+$ and $\mu := {}^\circ\nu$. Each function $\xi \in {}^{\rm st}\mathbf{H}_\mu$ is a shadow for some function $x \in \mathbb{H}_\nu$, for example of the function $x := \Pi\xi$. Moreover, if $\xi \in {}^{\rm nst}\mathbf{H}_\mu$ then $\Pi\xi \in {}^{\rm nst}\mathbb{H}_\nu$ and ${}^\circ\xi = {}^\circ(\Pi\xi)$.

$\square$ As P is a quasi-unity, then $\|Q\Pi\xi - \xi\| = \|P\xi - \xi\| \approx \|P({}^\circ\xi) - {}^\circ\xi\| \approx 0 \quad \blacksquare$

3.7.5. Theorem. Let $\nu \in {}^{\rm nst}\mathcal{N}_+, \ \mu := {}^\circ\nu, \ x \in {}^{\rm nst}\mathbb{H}_\nu$. Suppose that the function x is ν-integrable and ${}^\bullet\nu$-integrable. Then x is a lifting for its shadow ${}^\circ x$ and for all $E \in 2^{\mathbb{T}}$ if $\nu E \ll \infty$ then

$$\int_{QE} ({}^\circ x)(\tau)\mu(d\tau) \approx \sum_{t\in E} x(t)\nu_t.$$

$\square$ Let $\xi := {}^\circ x$. Then, by (3.31), $\|x - \Pi\xi\| \approx 0$. According to 1.5.5 $x(t) \approx \Pi\xi(t)$ quasi everywhere on $\mathbb{T}$. Consequently, x is a lifting of ξ and we can apply the theorem 3.6.4 $\blacksquare$

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