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GROWTH ESTIMATES FOR A DIRICHLET SERIES AND ITS DERIVATIVE

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Let $A\in(-\infty,+\infty]$, Φ be a continuous function on $[a,A)$ such that for every $x\in\mathbb{R}$ we have $x\sigma-\Phi(\sigma)\to-\infty$ as $\sigma\uparrow A$, $\widetilde{\Phi}(x)=\max\{x\sigma-\Phi(\sigma)\colon \sigma\in[a,A)\}$ be the Young-conjugate function of Φ, $\overline{\Phi}(x)=\widetilde{\Phi}(x)/x$ for all sufficiently large x, (λ_n) be a nonnegative sequence increasing to $+\infty$, $F(s)=\sum a_ne^{s\lambda_n}$ be a Dirichlet series absolutely convergent in the half-plane $\operatorname{Re}s<A$, $M(\sigma,F)=\sup\{|F(s)|\colon \operatorname{Re}s=\sigma\}$ and $G(\sigma,F)=\sum|a_n|e^{\sigma\lambda_n}$ for each $\sigma<A$. It is proved that if $\ln G(\sigma,F)\le(1+o(1))\Phi(\sigma)$, $\sigma\uparrow A$, then the inequality

$$\varlimsup_{\sigma\uparrow A}\frac{M(\sigma,F')}{M(\sigma,F)\overline{\Phi}^{\,-1}(\sigma)}\le 1$$

holds, and this inequality is sharp.

1. Introduction. Let Λ be the class of all nonnegative sequences $\lambda=(\lambda_n)_{n=0}^{\infty}$ increasing to $+\infty$, and $A\in(-\infty,+\infty]$. For a sequence $\lambda\in\Lambda$ we put

$$n(t,\lambda)=\sum_{\lambda_n\le t}1,\quad \tau(\lambda)=\varlimsup_{t\to+\infty}\frac{\ln n(t,\lambda)}{t},$$

and denote by $\mathcal{D}_A(\lambda)$ the class of all Dirichlet series of the form

$$F(s)=\sum_{n=0}^{\infty}a_ne^{s\lambda_n}\qquad(1)$$

such that $F(s)\not\equiv0$ and $\sigma_a(F)\ge A$, where $\sigma_a(F)$ is the abscissa of absolute convergence of series (1). Set $\mathcal{D}_A=\bigcup_{\lambda\in\Lambda}\mathcal{D}_A(\lambda)$.

For a Dirichlet series $F\in\mathcal{D}_A$ of the form (1) and every $\sigma<A$ we put

$$M(\sigma,F)=\sup\{|F(s)|\colon \operatorname{Re}s=\sigma\},\quad \mu(\sigma,F)=\max\{|a_n|e^{\sigma\lambda_n}\colon n\ge0\},$$
$$K(\sigma,F)=\frac{M(\sigma,F')}{M(\sigma,F)},\quad G(\sigma,F)=\sum_{n=0}^{\infty}|a_n|e^{\sigma\lambda_n}.$$

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Let $\Phi\colon D_\Phi \to \mathbb{R}$ be a real function. We say that $\Phi \in \Omega_A$ if the domain D_Φ of Φ is an interval of the form $[a, A)$, Φ is continuous on D_Φ, and the following condition

$$\forall x \in \mathbb{R}: \quad \lim_{\sigma\uparrow A}(x\sigma - \Phi(\sigma)) = -\infty \tag{2}$$

holds. It is easy to see that in the case $A < +\infty$ condition (2) is equivalent to the condition $\Phi(\sigma) \to +\infty$, $\sigma \to A - 0$, and in the case $A = +\infty$ this condition is equivalent to the condition $\Phi(\sigma)/\sigma \to +\infty$, $\sigma \to +\infty$. For $\Phi \in \Omega_A$ by $\widetilde{\Phi}$ we denote the Young-conjugate function of Φ, i.e.,

$$\widetilde{\Phi}(x) = \max\{x\sigma - \Phi(\sigma)\colon \sigma \in D_\Phi\}, \quad x \in \mathbb{R}.$$

Note (see Lemma 1 below), that the function $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ is continuous and increasing to A on some interval of the form $(x_0, +\infty)$. Hence the inverse function $\overline{\Phi}^{\,-1}$ is defined on some interval of the form (A_0, A) and $\overline{\Phi}^{\,-1}$ is continuous and increasing to $+\infty$ on (A_0, A).

We say that $\Phi \in \Omega'_A$, if Φ is a function from the class Ω_A continuously differentiable on D_Φ such that Φ' is a positive increasing function on D_Φ.

Let $\Phi \in \Omega'_A$. It is clear that $\Phi'(\sigma) \uparrow +\infty$ as $\sigma \uparrow A$. In addition, Φ' has the inverse function $\varphi\colon [x_0, +\infty) \to D_\Phi$. Set

$$\widehat{\Phi}(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}, \quad \sigma \in D_\Phi.$$

It is easy to prove that $\overline{\Phi}(x) = \widehat{\Phi}(\varphi(x))$ for every $x \in (x_0, +\infty)$. This implies that $\Phi'(\widehat{\Phi}^{-1}(\sigma)) = \overline{\Phi}^{\,-1}(\sigma)$ for all $\sigma \in (A_0, A)$.

For a Dirichlet series $F \in \mathcal{D}_A$ and a function $\Phi \in \Omega_A$ we put

$$T_\Phi(F) = \varlimsup_{\sigma\uparrow A} \frac{\ln M(\sigma, F)}{\Phi(\sigma)}, \quad t_\Phi(F) = \varlimsup_{\sigma\uparrow A} \frac{\ln \mu(\sigma, F)}{\Phi(\sigma)}, \quad \mathcal{T}_\Phi(F) = \varlimsup_{\sigma\uparrow A} \frac{\ln G(\sigma, F)}{\Phi(\sigma)}.$$

Suppose that $\lambda \in \Lambda$, $\Phi \in \Omega_A$, and $F \in \mathcal{D}_A(\lambda)$ is a Dirichlet series such that $T_\Phi(F) = 1$. Then the following theorem, which is proved in [1], gives an estimate of the growth for the quantity $K(\sigma, F)$ as $\sigma \uparrow A$ by some conditions on λ and Φ.

Theorem A. *Let $A \in (-\infty, +\infty]$, $\lambda \in \Lambda$, α be a positive increasing to $+\infty$ function on $[0, +\infty)$ such that $\alpha(t) = o(t)$ as $t \to +\infty$, $F \in \mathcal{D}_A(\lambda)$, $\Phi \in \Omega'_A$, and $\gamma(\sigma) = 2/\alpha(\overline{\Phi}^{\,-1}(\sigma))$ for every $\sigma \in (A_0, A)$. Suppose that $\ln n(t, \lambda) \le t/\alpha(t)$, $t \ge t_0$, and $\sigma + \gamma(\sigma) < A$, $\sigma \in [\sigma_0, A)$. If $T_\Phi(F) = 1$, then*

$$\varlimsup_{\sigma\uparrow A} \frac{K(\sigma, F)}{\overline{\Phi}^{\,-1}(\sigma + \gamma(\sigma))} \le 1. \tag{3}$$

It is shown in [1] that in many cases estimate (3) is sharp. To substantiate the exactness of inequality (3), in [1], in particular, the following theorem was proved.

Theorem B. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega'_A$ be a twice continuously differentiable function on D_Φ, and φ be the inverse function of Φ'. If*

$$\Phi((1 + o(1))\sigma) \sim (1 + o(1))\Phi(\sigma), \quad \sigma \uparrow A,$$

and $t^2\varphi'(t) \uparrow +\infty$ as $t \uparrow +\infty$, then for every sequence $\lambda \in \Lambda$ there exists a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ such that $T_\Phi(F) = t_\Phi(F) = 1$ and

$$\varlimsup_{\sigma\uparrow A} \frac{K(\sigma, F)}{\overline{\Phi}^{\,-1}(\sigma)} = 1. \tag{4}$$

It is easily seen that the conditions of Theorem A imply the equality $\tau(\lambda) = 0$. Therefore, in the case $\tau(\lambda) > 0$ Theorem A does not give any information about the growth of the quantity $K(\sigma, F)$. Moreover, if $A < +\infty$, then even in the case $\tau(\lambda) = 0$ the conclusion of Theorem A is true only under some conditions on Φ.

Let $F \in \mathcal{D}_A$ be a Dirichlet series of the form (1) with nonnegative coefficients a_n. Then $M(\sigma, F) = G(\sigma, F) = F(\sigma)$, $\sigma < A$. Hence, $\mathcal{T}_\Phi(F) = T_\Phi(F)$ and $K(\sigma, F) = (\ln M(\sigma, F))'$, $\sigma < A$. Therefore, as is easy to prove (see Lemma 4 below; see also Lemma 1 in [2]), for such series, without any conditions on the sequence λ of its exponents and on a function $\Phi \in \Omega_A$, the equality $T_\Phi(F) = 1$ (or the identical equality $\mathcal{T}_\Phi(F) = 1$) implies the inequality

$$\varlimsup_{\sigma\uparrow A} \frac{K(\sigma, F)}{\overline{\Phi}^{\,-1}(\sigma)} \le 1. \tag{5}$$

It turns out that inequality (5) follows from the equality $\mathcal{T}_\Phi(F) = 1$ for any other Dirichlet series $F \in \mathcal{D}_A$. The following theorem confirms this fact.

Theorem 1. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $F \in \mathcal{D}_A$. If $\mathcal{T}_\Phi(F) \le 1$, then inequality (5) holds.*

For an arbitrary function $\Phi \in \Omega_A$, inequality (5) is sharp in each of the classes $\mathcal{D}_A(\lambda)$, $\lambda \in \Lambda$. This conclusion can be drawn from the following theorem which is a generalization of Theorem B.

Theorem 2. *Let $A \in (-\infty, +\infty]$ and $\Phi \in \Omega_A$. For every sequence $\lambda \in \Lambda$ there exists a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ such that $\mathcal{T}_\Phi(F) = t_\Phi(F) = 1$ and equality (4) holds.*

In order to prove Theorems 1 and 2, we will need some auxiliary results, which are given in the next section.

2. Auxiliary results. The following lemma is well known (see, for example, [3, § 3.2], [4]).

Lemma 1. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $\varphi(x) = \max\{\sigma \in D_\Phi \colon x\sigma - \Phi(\sigma) = \widetilde{\Phi}(x)\}$, $x \in \mathbb{R}$. Then the following statements are true:*

(i) the function φ is nondecreasing on $\mathbb{R}$;

(ii) the function φ is continuous from the right on $\mathbb{R}$;

(iii) $\varphi(x) \to A$, $x \to +\infty$;

(iv) the right-hand derivative of $\widetilde{\Phi}(x)$ is equal to $\varphi(x)$ at every point $x \in \mathbb{R}$;

(v) if $x_0 = \inf\{x > 0 \colon \Phi(\varphi(x)) > 0\}$, then the function $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ increase to A on $(x_0, +\infty)$;

(vi) the function $\alpha(x) = \Phi(\varphi(x))$ is nondecreasing on $[0, +\infty)$.

In the following lemmas φ and x_0 are defined by Φ in the same way as in Lemma 1.

Lemma 2 ([5])**.** *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, $\sigma_0 = \overline{\Phi}(x_0 + 0)$, and $\sigma \in (\sigma_0, A)$. Then the minimum value of the function*

$$h(y) = \frac{\Phi(y)}{y - \sigma}, \quad y \in (\sigma, A),$$

is $\overline{\Phi}^{\,-1}(\sigma)$ and this value is attained at the point $y = \varphi(\overline{\Phi}^{\,-1}(\sigma))$.

Lemma 3 ([5]). *Let* $\delta \in (0,1)$, $A \in (-\infty,+\infty]$, $\Phi \in \Omega_A$, $\sigma_0 = \overline{\Phi}(x_0+0)$, *and* $y(\sigma) = \varphi(\overline{\Phi}^{-1}(\sigma))$ *for all* $\sigma \in (\sigma_0, A)$. *Then*

$$\overline{\Phi}^{-1}\left(\sigma + \frac{\delta\Phi(y(\sigma))}{\overline{\Phi}^{-1}(\sigma)}\right) \le \frac{\overline{\Phi}^{-1}(\sigma)}{1-\delta}, \quad \sigma \in (\sigma_0, A).$$

Lemma 4. *Let* $A \in (-\infty,+\infty]$, $\Phi \in \Omega_A$, $\sigma_0 = \overline{\Phi}(x_0+0)$, $b \in [\sigma_0, A)$, Ψ *be a convex function on* (b,A) *such that* $\Psi(y) \le \Phi(y)$ *for all* $y \in (b,A)$, *and*

$$E = \{\sigma \in (b,A)\colon\ \Psi(y) - \Psi(\sigma) \le \Phi(y) \text{ for all } y \in (\sigma, A)\}.$$

Then $\Psi'_+(\sigma) \le \overline{\Phi}^{-1}(\sigma)$ *for every* $\sigma \in E$.

Proof. Suppose that $\sigma \in E$. Then $\sigma \in (\sigma_0, A)$ and therefore, setting $y = \varphi(\overline{\Phi}^{-1}(\sigma))$ and using the convexity of the function Ψ and Lemma 2, we obtain

$$\Psi'_+(\sigma) \le \frac{\Psi(y) - \Psi(\sigma)}{y - \sigma} \le \frac{\Phi(y)}{y-\sigma} = \overline{\Phi}^{-1}(\sigma).$$

□

Remark 1. It is easy to see that if functions Φ and Ψ satisfy the conditions of Lemma 4, then there exists a number $c \in [b, A)$ such that $(c, A) \subset E$. In addition, the set E contains every point $\sigma \in (b,A)$ such that $\Psi(\sigma) \ge 0$.

Lemma 5. *Let* $A \in (-\infty,+\infty]$, $\Phi \in \Omega_A$, $\sigma_0 = \overline{\Phi}(x_0+0)$, $y(\sigma) = \varphi(\overline{\Phi}^{-1}(\sigma))$ *for each* $\sigma \in (\sigma_0, A)$, $b \in [\sigma_0, A)$, $F \in \mathcal{D}_A$ *be a Dirichlet series of the form* (1), *and* $q > 0$. *If* $\ln G(y,F) \le q\Phi(y)$ *for all* $y \in (b,A)$, *then for every* $\sigma \in (b,A)$ *and arbitrary* $p \ge q$ *we have*

$$\sum_{\lambda_n > p\overline{\Phi}^{-1}(\sigma)} |a_n| e^{\sigma\lambda_n} < \frac{1}{e^{(p-q)\Phi(y(\sigma))}}. \tag{6}$$

Proof. We first prove inequality (6) in the case $p = q$, i.e. we show that

$$\sum_{\lambda_n > q\overline{\Phi}^{-1}(\sigma)} |a_n| e^{\sigma\lambda_n} < 1 \tag{7}$$

for every $\sigma \in (b, A)$.

We fix an arbitrary $\sigma \in (b,A)$ and consider the function

$$H(y) = \sum_{\lambda_n > q\overline{\Phi}^{-1}(\sigma)} |a_n| s^{y\lambda_n}, \quad y < A.$$

Note that inequality (7) can be rewritten as $H(\sigma) < 1$.

Suppose on the contrary that $H(\sigma) \ge 1$. For all $y < A$ we get

$$H'(y) = \sum_{\lambda_n > q\overline{\Phi}^{-1}(\sigma)} \lambda_n |a_n| s^{y\lambda_n} > \sum_{\lambda_n > q\overline{\Phi}^{-1}(\sigma)} q\overline{\Phi}^{-1}(\sigma)|a_n| s^{y\lambda_n} = q\overline{\Phi}^{-1}(\sigma) H(y). \tag{8}$$

On the other hand, setting $\Psi(y) = (\ln H(y))/q$, $y < A$, we see that the function Ψ is convex on the interval $(-\infty, A)$ and $\Psi(y) \le (\ln G(y,F))/q \le \Phi(y)$ for all $y \in (b, A)$. Thus by Lemma 4 (see Remark 1) we have

$$\frac{H'(\sigma)}{qH(\sigma)} = \Psi'(\sigma) \le \overline{\Phi}^{\,-1}(\sigma),$$

which contradicts (8) with $y = \sigma$. Therefore, inequality (7) is proved.

We now prove inequality (6) for $p > q$. Put

$$\delta = \frac{p-q}{p}, \quad \varepsilon = \frac{\delta\Phi(y(\sigma))}{\overline{\Phi}^{\,-1}(\sigma)}.$$

Then by Lemma 3 we have $q\overline{\Phi}^{\,-1}(\sigma+\varepsilon) \le p\overline{\Phi}^{\,-1}(\sigma)$. Using inequality (7) with $\sigma+\varepsilon$ instead of σ, we get

$$\sum_{\lambda_n > p\overline{\Phi}^{\,-1}(\sigma)} |a_n| e^{\sigma\lambda_n} = \sum_{\lambda_n > p\overline{\Phi}^{\,-1}(\sigma)} \frac{1}{e^{\varepsilon\lambda_n}} |a_n| e^{(\sigma+\varepsilon)\lambda_n} \le \frac{1}{e^{\varepsilon p \overline{\Phi}^{\,-1}(\sigma)}} \sum_{\lambda_n > p\overline{\Phi}^{\,-1}(\sigma)} |a_n| e^{(\sigma+\varepsilon)\lambda_n} \le$$
$$\le \frac{1}{e^{\varepsilon p \overline{\Phi}^{\,-1}(\sigma)}} \sum_{\lambda_n > q\overline{\Phi}^{\,-1}(\sigma+\varepsilon)} |a_n| e^{(\sigma+\varepsilon)\lambda_n} < \frac{1}{e^{\varepsilon p \overline{\Phi}^{\,-1}(\sigma)}} = \frac{1}{e^{(p-q)\Phi(y(\sigma))}}.$$

□

The following lemma was proved by I. V. Ostrovskii (see [1]).

Lemma 6. *Suppose that* $0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_N$. *Then for each exponential polynomial*

$$P(s) = \sum_{n=0}^{N} a_n e^{s\lambda_n}$$

and every $\sigma \in \mathbb{R}$ *the inequality* $M(\sigma, P') \le \lambda_N M(\sigma, P)$ *holds.*

Let $\lambda \in \Lambda$. Consider a Dirichlet series F of the form (1) and put

$$\beta(F) = \varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}.$$

It is well known (for instance, see [6, p. 114–115]) that

$$\sigma_a(F) \le \beta(F) \le \sigma_a(F) + \tau(\lambda)$$

and these inequalities are sharp (moreover, it was shown in [7] that for any $A, B \in [-\infty, +\infty]$ such that $A \le B \le A + \tau(\lambda)$ there exists a Dirichlet series F of the form (1) such that $\sigma_a(F) = A$ and $\beta(F) = B$).

Note also that for a Dirichlet series F the interval $(-\infty, \beta(F))$ is the domain of existence of the maximum term $\mu(\sigma, F)$. If $F(s) \not\equiv 0$, then this interval is also the domain of existence of the central index

$$\nu(\sigma, F) = \max\{n \ge 0\colon |a_n| e^{\sigma\lambda_n} = \mu(\sigma, F)\}.$$

Lemma 7 ([8]). *Let $\lambda\in\Lambda$, $A\in(-\infty,+\infty]$. If for a Dirichlet series of the form* (1) *there exists an increasing sequence $(n_k)_{k=0}^{\infty}$ of nonnegative integers such that $a_n=0$ for all $n<n_0$, $a_{n_k}\neq 0$ for every $k\ge 0$, and*

$$\varkappa_k:=\frac{\ln|a_{n_k}|-\ln|a_{n_{k+1}}|}{\lambda_{n_{k+1}}-\lambda_{n_k}}\uparrow A,\quad k\uparrow\infty,\quad |a_n|\le|a_{n_k}|e^{\varkappa_k(\lambda_{n_k}-\lambda_n)},\quad n\in(n_k,n_{k+1}),\ k\ge 0,$$

then $\beta(F)=A$ and, moreover, $\nu(\sigma,F)=n_0$ for every $\sigma<\varkappa_0$ and $\nu(\sigma,F)=n_{k+1}$ for all $\sigma\in[\varkappa_k,\varkappa_{k+1})$ and $k\ge 0$.

Lemma 8 ([4]). *Let $\lambda\in\Lambda$, $A\in(-\infty,+\infty]$, and $\Phi\in\Omega_A$. If the condition*

$$\forall t>0:\quad \ln n=o(\Phi(\varphi(\lambda_n/t))),\quad n\to\infty, \tag{9}$$

holds, then each Dirichlet series F of the form (1) *such that $\beta(F)=A$ belongs to the class $\mathcal{D}_A(\lambda)$ and for this series we have $T_\Phi(F)=t_\Phi(F)$.*

3. Proof of Theorems.

Proof of Theorem 1. Suppose that $A\in(-\infty,+\infty]$, $\Phi\in\Omega_A$, $F\in\mathcal{D}_A$ is a Dirichlet series of the form (1) such that $\mathcal{T}_\Phi(F)\le 1$, and prove that inequality (5) holds.

For all $s\in\mathbb{C}$ with $\operatorname{Re}z<A$ and each $N\in\mathbb{R}$ we put

$$P_N(s)=\sum_{\lambda_n\le N}a_ne^{s\lambda_n},\quad R_N(s)=\sum_{\lambda_n>N}a_ne^{s\lambda_n}.$$

Then $F(s)=P_N(s)+R_N(s)$ and therefore

$$M(\sigma,F)-M(\sigma,R_N)\le M(\sigma,P_N)\le M(\sigma,F)+M(\sigma,R_N),\quad \sigma<A. \tag{10}$$

As above, let $x_0=\inf\{x>0\colon \Phi(\varphi(x))>0\}$, $\sigma_0=\overline{\Phi}(x_0+0)$, and $y(\sigma)=\varphi(\overline{\Phi}^{\,-1}(\sigma))$ for all $\sigma\in(\sigma_0,A)$.

We fix an arbitrary $\eta>1$ and choose numbers p and q such that $1<q<p<\eta$. Since $\mathcal{T}_\Phi(F)\le 1$, we have $\ln G(y,F)\le q\Phi(y)$, $y\in(b,A)$, for some $b\in[\sigma_0,A)$. Setting $N(\sigma)=\eta\overline{\Phi}^{\,-1}(\sigma)$, by Lemma 5 we obtain

$$M(\sigma,R_{N(\sigma)})\le\sum_{\lambda_n>\eta\overline{\Phi}^{\,-1}(\sigma)}|a_n|e^{\sigma\lambda_n}<\frac{1}{e^{(\eta-q)\Phi(y(\sigma))}},\quad \sigma\in(b,A).$$

Therefore, $M(\sigma,R_{N(\sigma)})=o(1)$, $\sigma\uparrow A$. Then it follows from (10) that

$$M(\sigma,P_{N(\sigma)})=M(\sigma,F)+o(1),\quad \sigma\uparrow A. \tag{11}$$

Let $\varepsilon(\sigma)=1/N(\sigma)$, $y\in(b,A)$. By Lemma 3 we have

$$\overline{\Phi}^{\,-1}(\sigma+\varepsilon(\sigma))\sim\overline{\Phi}^{\,-1}(\sigma),\ \sigma\uparrow A.$$

Hence for some $b_0\in(b,A)$ we obtain

$$\eta\overline{\Phi}^{\,-1}(\sigma)\ge p\overline{\Phi}^{\,-1}(\sigma+\varepsilon(\sigma)),\quad \sigma\in(b_0,A).$$

Taking into account that for every fixed $\varepsilon > 0$ and an arbitrary $x \geq 0$ the inequality

$$\frac{x}{e^{\varepsilon x}} \leq \frac{1}{\varepsilon e}$$

holds and again using Lemma 5 for all $\sigma \in (b_0, A)$ we have

$$M(\sigma, R'_{N(\sigma)}) \leq \sum_{\lambda_n > \eta\overline{\Phi}^{-1}(\sigma)} \lambda_n |a_n| s^{y\lambda_n} = \sum_{\lambda_n > \eta\overline{\Phi}^{-1}(\sigma)} \frac{\lambda_n}{e^{\varepsilon(\sigma)\lambda_n}} |a_n| s^{(\sigma+\varepsilon(\sigma))\lambda_n} \leq$$
$$\leq \frac{1}{\varepsilon(\sigma)e} \sum_{\lambda_n > \eta\overline{\Phi}^{-1}(\sigma)} |a_n| s^{(\sigma+\varepsilon(\sigma))\lambda_n} \leq \frac{1}{\varepsilon(\sigma)e} \sum_{\lambda_n > p\overline{\Phi}^{-1}(\sigma+\varepsilon(\sigma))} |a_n| s^{(\sigma+\varepsilon(\sigma))\lambda_n} \leq$$
$$\leq \frac{1}{\varepsilon(\sigma)e} \frac{1}{e^{(p-q)\Phi(y(\sigma+\varepsilon(\sigma)))}} = \frac{\eta\overline{\Phi}^{-1}(\sigma)}{e^{(p-q)\Phi(y(\sigma+\varepsilon(\sigma)))+1}}.$$

Therefore,

$$M(\sigma, R'_{N(\sigma)}) = o(\overline{\Phi}^{-1}(\sigma)), \quad \sigma \uparrow A. \tag{12}$$

Further, using Lemma 6 and relations (11) and (12), we obtain

$$M(\sigma, F') \leq M(\sigma, P'_{N(\sigma)}) + M(\sigma, R'_{N(\sigma)}) \leq N(\sigma)M(\sigma, P_{N(\sigma)}) + M(\sigma, R'_{N(\sigma)}) =$$
$$= \eta\overline{\Phi}^{-1}(\sigma)M(\sigma, F) + o(\overline{\Phi}^{-1}(\sigma)), \quad \sigma \uparrow A,$$

so that

$$\varlimsup_{\sigma\uparrow A} \frac{K(\sigma, F)}{\overline{\Phi}^{-1}(\sigma)} \leq \eta.$$

Since $\eta > 1$ is arbitrary, we have (5). □

Proof of Theorem 2. Suppose that $\Phi \in \Omega_A$ and $\lambda \in \Lambda$, and prove that there exists a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ such that $\mathcal{T}_\Phi(F) = t_\Phi(F) = 1$ and equality (4) holds.

As above, we put $x_0 = \inf\{x > 0 \colon \Phi(\varphi(x)) > 0\}$, $\sigma_0 = \overline{\Phi}(x_0 + 0)$, and $y(\sigma) = \varphi(\overline{\Phi}^{-1}(\sigma))$ for all $\sigma \in (\sigma_0, A)$. From condition (2) and Lemmas 1 and 3 it follows that there exists a subsequence $\lambda^* = (\lambda_{n_k})$ of the sequence λ such that for it and for the sequences $(\varkappa_k)$ and (δ_k), where

$$\varkappa_k = \overline{\Phi}(\lambda_{n_{k+1}}), \quad \delta_k = \frac{1}{\sqrt{\Phi(\varphi(\lambda_{n_{k+1}}))}} = \frac{1}{\sqrt{\Phi(y(\varkappa_k))}}$$

for all integers $k \geq 0$, we have $n_0 = 0$, $\Phi(\varphi(\lambda_{n_1})) > 1$, and also

$$\forall t > 0 : \quad \ln^2 k = o(\Phi(\varphi(\lambda_{n_k}/t))), \quad n \to \infty; \tag{13}$$
$$(k+1)\lambda_{n_k}\sigma - \Phi(\sigma) \leq (k+1)\lambda_{n_k}\varkappa_0, \quad \sigma \in [\varkappa_k, A),\ k \geq 0; \tag{14}$$
$$\tau_k := \varkappa_k + \frac{\delta_k\Phi(y(\varkappa_k))}{\overline{\Phi}^{-1}(\varkappa_k)} = \varkappa_k + \frac{1}{\delta_k\lambda_{n_{k+1}}} < \varkappa_{k+1}, \quad k \geq 0; \tag{15}$$
$$2\lambda_{n_k} \leq \lambda_{n_{k+1}}, \quad k \geq 0. \tag{16}$$

Note that (δ_k) is a nonincreasing sequence of points with $(0, 1)$ tending to 0. Therefore, using (15) and Lemma 3, we obtain

$$\overline{\Phi}^{-1}(\tau_k) \sim \overline{\Phi}^{-1}(\varkappa_k), \quad k \to \infty. \tag{17}$$

In addition, according to (16) and (15),

$$(\lambda_{n_{k+1}}-\lambda_{n_k})(\tau_k-\varkappa_k)\ge \frac12\lambda_{n_{k+1}}(\tau_k-\varkappa_k)=\frac{1}{2\delta_k}=\frac12\sqrt{\Phi(\varphi(\lambda_{n_{k+1}}))},$$

and hence, using (13), we see that

$$\frac{k+1}{e^{(\lambda_{n_{k+1}}-\lambda_{n_k})(\tau_k-\varkappa_k)}}\to 0,\quad k\to\infty. \tag{18}$$

Put $a_0=1$,

$$a_{n_{k+1}}=\prod_{j=0}^{k}e^{\varkappa_j(\lambda_{n_j}-\lambda_{n_{j+1}})},\quad k\ge 0,$$

and $a_n=0$ if $n\in(n_k,n_{k+1})$ for some $k\ge 0$. By Lemma 7 for Dirichlet series (1) with such coefficients a_n we have $\beta(F)=A$ and, moreover, $\nu(\sigma,F)=n_0$ for every $\sigma<\varkappa_0$ and $\nu(\sigma,F)=n_{k+1}$ for all $\sigma\in[\varkappa_k,\varkappa_{k+1})$ and $k\ge 0$.

Note that series (1) can be represented as

$$F(s)=\sum_{m=0}^{\infty}a_{n_m}e^{s\lambda_{n_m}}.$$

Since $\beta(F)=A$ and condition (13) holds, $F\in\mathcal{D}_A(\lambda^*)$ and $\mathcal{T}_\Phi(F)=t_\Phi(F)$ by Lemma 8. Then also $F\in\mathcal{D}_A(\lambda)$.

Let $\sigma\in[\varkappa_k,\varkappa_{k+1})$ and $k\ge 0$. Then

$$\varkappa_k=\overline{\Phi}(\lambda_{n_{k+1}})=\max\left\{y-\frac{\Phi(y)}{\lambda_{n_{k+1}}}: y\in D_\Phi\right\}\ge\sigma-\frac{\Phi(\sigma)}{\lambda_{n_{k+1}}}. \tag{19}$$

From (19) and (14) we obtain, respectively, the following inequalities

$$\lambda_{n_{k+1}}(\sigma-\varkappa_k)\le\Phi(\sigma),\quad \lambda_{n_k}(\varkappa_k-\varkappa_0)\le\lambda_{n_k}(\sigma-\varkappa_0)\le\frac{\Phi(\sigma)}{k+1}.$$

Using these inequalities, we have

$$\ln\mu(\sigma,F)=\int_{\varkappa_0}^{\sigma}\lambda_{\nu(t,F)}dt=\int_{\varkappa_0}^{\varkappa_k}\lambda_{\nu(t,F)}dt+\int_{\varkappa_k}^{\sigma}\lambda_{\nu(t,F)}dt\le$$
$$\le\lambda_{n_k}(\varkappa_k-\varkappa_0)+\lambda_{n_{k+1}}(\sigma-\varkappa_k)\le\frac{\Phi(\sigma)}{k+1}+\Phi(\sigma)=\frac{k+2}{k+1}\Phi(\sigma).$$

Thus we see that $t_\Phi(F)\le 1$. Then also $\mathcal{T}_\Phi(F)\le 1$, and therefore by Theorem 1 for the constructed series inequality (5) holds.

Next, for an arbitrary $\sigma<A$ and each integer $p\ge 0$ we set

$$Q_p(\sigma)=\sum_{m\le p}\lambda_{n_m}a_{n_m}e^{\sigma\lambda_{n_m}},\quad R_p(\sigma)=\sum_{m>p}\lambda_{n_m}a_{n_m}e^{\sigma\lambda_{n_m}},$$
$$S_p(\sigma)=\sum_{m\le p}a_{n_m}e^{\sigma\lambda_{n_m}},\quad T_p(\sigma)=\sum_{m>p}a_{n_m}e^{\sigma\lambda_{n_m}}.$$

Since $Q_p(\sigma)T_p(\sigma) \le S_p(\sigma)\lambda_{n_p}T_p(\sigma) \le S_p(\sigma)R_p(\sigma)$, we obtain

$$K(\sigma, F) = \frac{F'(\sigma)}{F(\sigma)} = \frac{Q_p(\sigma) + R_p(\sigma)}{S_p(\sigma) + T_p(\sigma)} \ge \frac{Q_p(\sigma)}{S_p(\sigma)}. \tag{20}$$

Let $k \ge 0$ be an arbitrary integer. According to (15) we have $\tau_k \in (\varkappa_k, \varkappa_{k+1})$, and therefore $\mu(\tau_k, F) = a_{n_{k+1}} e^{\tau_k \lambda_{n_{k+1}}}$. If $m \le k$, then

$$a_{n_m} e^{\tau_k \lambda_{n_m}} = a_{n_m} e^{\varkappa_k \lambda_{n_m}} e^{(\tau_k - \varkappa_k)\lambda_{n_m}} \le \mu(\varkappa_k, F) e^{(\tau_k - \varkappa_k)\lambda_{n_k}} = a_{n_{k+1}} e^{\varkappa_k \lambda_{n_{k+1}}} e^{(\tau_k - \varkappa_k)\lambda_{n_k}} =$$
$$= \frac{a_{n_{k+1}} e^{\tau_k \lambda_{n_{k+1}}}}{e^{(\lambda_{n_{k+1}} - \lambda_{n_k})(\tau_k - \varkappa_k)}} = \frac{\mu(\tau_k, F)}{e^{(\lambda_{n_{k+1}} - \lambda_{n_k})(\tau_k - \varkappa_k)}},$$

and so, using (18), we get

$$S_k(\tau_k) = \sum_{m \le k} a_{n_m} e^{\tau_k \lambda_{n_m}} \le \frac{(k+1)\mu(\tau_k, F)}{e^{(\lambda_{n_{k+1}} - \lambda_{n_k})(\tau_k - \varkappa_k)}} = o(\mu(\tau_k, F)), \quad k \to \infty. \tag{21}$$

Using (20) with $\sigma = \tau_k$ and $p = k + 1$, (21), and (17), we have

$$K(\tau_k, F) \ge \frac{Q_{k+1}(\tau_k)}{S_{k+1}(\tau_k)} \ge \frac{\lambda_{n_{k+1}} \mu(\tau_k, F)}{S_k(\tau_k) + \mu(\tau_k, F)} = (1 + o(1))\lambda_{n_{k+1}} = (1 + o(1))\overline{\Phi}^{\,-1}(\varkappa_k) =$$
$$= (1 + o(1))\overline{\Phi}^{\,-1}(\tau_k)$$

as $k \to \infty$. Consequently,

$$\varlimsup_{\sigma \uparrow A} \frac{K(\sigma, F)}{\overline{\Phi}^{\,-1}(\sigma)} \ge \varlimsup_{k \to \infty} \frac{K(\tau_k, F)}{\overline{\Phi}^{\,-1}(\tau_k)} \ge 1.$$

This and (5) imply (4).

Finally, we prove that $\mathcal{T}_\Phi(F) = 1$. Suppose, on the contrary, that $\mathcal{T}_\Phi(F) < 1$. We fix some $q \in (0, 1)$ such that $\mathcal{T}_\Phi(F) \le q$ and put $\Psi(\sigma) = q\Phi(\sigma)$, $\sigma \in D_\Phi$. Then $\mathcal{T}_\Psi(F) \le 1$ and is easy to see $\overline{\Psi}^{\,-1}(\sigma) = q\overline{\Phi}^{\,-1}(\sigma)$ for all $\sigma \in (\sigma_0, A)$. Applying Theorem 1 to Ψ instead of Φ, we obtain

$$\varlimsup_{\sigma \uparrow A} \frac{K(\sigma, F)}{\overline{\Phi}^{\,-1}(\sigma)} = q \varlimsup_{\sigma \uparrow A} \frac{K(\sigma, F)}{\overline{\Psi}^{\,-1}(\sigma)} \le q.$$

This contradicts (4). □

Remark 2. In view of the above results, it is natural to ask whether we can replace the condition $\mathcal{T}_\Phi(F) \le 1$ in Theorem 1 by the condition $T_\Phi(F) \le 1$. Nothing as strong as this is known. It is clear that such replacement is possible, for example, under conditions that ensure the equality $\mathcal{T}_\Phi(F) = T_\Phi(F)$, in particular, provided that (9) holds. Note that the equality $\mathcal{T}_\Phi(F) = T_\Phi(F)$ may not be satisfied in the general case (see, for example, [9, 10]).

REFERENCES

1. M.N. Sheremeta, S.I. Fedynyak, *On the derivative of a Dirichlet series*, Sib. Mat. Zh., **39** (1998), №1, 206–223. (in Russian) English translation in: Siberian Math. J., **39** (1998), №1, 181–197.

2. S.I. Fedynyak, *On maximum modulus points and zero set for an entire function*, Mat. Stud., **30** (2008), №2, 169–172.
3. M.A. Evgrafov, Asymptotic estimates and entire functions, Moscow: Nauka, 1979. (in Russian)
4. T.Ya. Hlova, P.V. Filevych, *Generalized types of the growth of Dirichlet series*, Carpathian Math. Publ., **7** (2015), №2, 172–187.
5. S.I. Fedynyak, P.V. Filevych, *Distance between a maximum modulus point and zero set of an analytic function*, Mat. Stud., **52** (2019), №1, 10–23.
6. A.F. Leont'ev, Series of exponents, Moscow: Nauka, 1976. (in Russian)
7. P.V. Filevych, *On relations between the abscissa of convergence and the abscissa of absolute convergence of random Dirichlet series*, Mat. Stud., **20** (2003), №1, 33–39.
8. P.V. Filevich, *On Valiron's theorem on the relations between the maximum modulus and the maximal term of an entire Dirichlet series*, Izv. Vyssh. Uchebn. Zaved. Mat., (2004), №4, 66–72. (in Russian) English translation in: Russian Math., **48** (2004), №4, 63–69.
9. P.V. Filevich, *On influence of the arguments of coefficients of a power series expansion of an entire function on the growth of the maximum of its modulus*, Sib. Mat. Zh., **44** (2003), №3, 674–685. (in Russian) English translation in: Siberian Math. J., **44** (2003), №3, 529–538.
10. T.Ya. Hlova, P.V. Filevych, *The growth of analytic functions in the terms of generalized types*, J. Lviv Politech. Nat. Univ., Physical and mathematical sciences, (2014), №804, 75–83. (in Ukrainian)

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