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GROWTH ESTIMATES FOR A DIRICHLET SERIES AND ITS DERIVATIVE


Let $A \in (-\infty, +\infty]$, $\Phi$ be a continuous function on $[a; A)$ such that for every $x \in \mathbb{R}$ we have $x - \Phi(x) \to -\infty$ as $x \to A$, $\hat{\Phi}(x) = \max\{x - \Phi(x) : x \in [a, A]\}$ be the Young-conjugate function of $\Phi$, $\Phi(x) = e^{\Phi(x)} = x$ for all sufficiently large $x$, $(\lambda_n)$ be a nonnegative sequence increasing to $+\infty$, $F(s) = \sum a_n e^{\lambda_n s}$ be a Dirichlet series absolutely convergent in the half-plane $\Re s < A$, $M(\sigma, F) = \sup\{|F(s)| : \Re s = \sigma\}$ and $G(\sigma, F) = \sum |a_n| e^{\sigma \lambda_n}$ for each $\sigma < A$.

It is proved that if $\ln G(\sigma, F) \leq (1 + o(1))\Phi(\lambda_A)$, then the inequality

$$\lim_{\sigma \to A} \frac{M(\sigma, F')}{M(\sigma, F)\Phi^{-1}(\sigma)} \leq 1$$

holds, and this inequality is sharp.

1. Introduction. Let $\Lambda$ be the class of all nonnegative sequences $\lambda = (\lambda_n)_{n=0}^{\infty}$ increasing to $+\infty$, and $A \in (-\infty, +\infty]$. For a sequence $\lambda \in \Lambda$ we put

$$n(t, \lambda) = \sum_{\lambda_n \leq t} 1, \quad \tau(\lambda) = \lim_{t \to +\infty} \frac{\ln n(t, \lambda)}{t},$$

and denote by $D_A(\lambda)$ the class of all Dirichlet series of the form

$$F(s) = \sum_{n=0}^{\infty} a_n e^{\lambda_n s} \quad (1)$$

such that $F(s) \not\equiv 0$ and $\sigma_a(F) \geq A$, where $\sigma_a(F)$ is the abscissa of absolute convergence of series (1). Set $D_A = \cup_{\lambda \in \Lambda} D_A(\lambda)$.

For a Dirichlet series $F \in D_A$ of the form (1) and every $\sigma < A$ we put

$$M(\sigma, F) = \sup\{|F(s)| : \Re s = \sigma\}, \quad \mu(\sigma, F) = \max\{|a_n| e^{\sigma \lambda_n} : n \geq 0\},$$

$$K(\sigma, F) = \frac{M(\sigma, F')}{M(\sigma, F)}, \quad G(\sigma, F) = \sum_{n=0}^{\infty} |a_n| e^{\sigma \lambda_n}.$$

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Let \( \Phi: D_\Phi \to \mathbb{R} \) be a real function. We say that \( \Phi \in \Omega_A \) if the domain \( D_\Phi \) of \( \Phi \) is an interval of the form \([a, A]\), \( \Phi \) is continuous on \( D_\Phi \), and the following condition
\[
\forall x \in \mathbb{R}: \quad \lim_{\sigma \uparrow A} (x\sigma - \Phi(\sigma)) = -\infty
\]
holds. It is easy to see that in the case \( A < +\infty \) condition (2) is equivalent to the condition \( \Phi(\sigma) \to +\infty, \sigma \to A - 0 \), and in the case \( A = +\infty \) this condition is equivalent to the condition \( \Phi(\sigma)/\sigma \to +\infty, \sigma \to +\infty \). For \( \Phi \in \Omega_A \) by \( \overline{\Phi} \) we denote the Young-conjugate function of \( \Phi \), i.e.,
\[
\overline{\Phi}(x) = \max\{x\sigma - \Phi(\sigma): \sigma \in D_\Phi\}, \quad x \in \mathbb{R}.
\]
Note (see Lemma 1 below), that the function \( \overline{\Phi}(x) = \Phi(x)/x \) is continuous and increasing to \( A \) on some interval of the form \((x_0, +\infty)\). Hence the inverse function \( \overline{\Phi}^{-1} \) is defined on some interval of the form \((A_0, A)\) and \( \overline{\Phi}^{-1} \) is continuous and increasing to \(+\infty\) on \((A_0, A)\).

We say that \( \Phi \in \Omega_A' \), if \( \Phi \) is a function from the class \( \Omega_A \) continuously differentiable on \( D_\Phi \) such that \( \Phi' \) is a positive increasing function on \( D_\Phi \).

Let \( \Phi \in \Omega_A' \). It is clear that \( \Phi' (\sigma) \uparrow +\infty \) as \( \sigma \uparrow A \). In addition, \( \Phi' \) has an inverse function \( \varphi: [x_0, +\infty) \to D_\Phi \). Set
\[
\hat{\Phi}(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}, \quad \sigma \in D_\Phi.
\]
It is easy to prove that \( \overline{\Phi}(x) = \hat{\Phi}(\varphi(x)) \) for every \( x \in (x_0, +\infty) \). This implies that \( \Phi'(\overline{\Phi}^{-1}(\sigma)) = \overline{\Phi}^{-1}(\sigma) \) for all \( \sigma \in (A_0, A) \).

For a Dirichlet series \( F \in D_A \) and a function \( \Phi \in \Omega_A \) we put
\[
T_\Phi(F) = \lim_{\sigma \uparrow A} \frac{\ln M(\sigma, F)}{\Phi(\sigma)}, \quad t_\Phi(F) = \lim_{\sigma \uparrow A} \frac{\ln \mu(\sigma, F)}{\Phi(\sigma)}, \quad T_s(F) = \lim_{\sigma \uparrow A} \frac{\ln G(\sigma, F)}{\Phi(\sigma)}.
\]

Suppose that \( \lambda \in \Lambda, \Phi \in \Omega_A, \) and \( F \in D_A(\lambda) \) is a Dirichlet series such that \( T_\Phi(F) = 1 \). Then the following theorem, which is proved in [1], gives an estimate of the growth for the quantity \( K(\sigma, F) \) as \( \sigma \uparrow A \) by some conditions on \( \lambda \) and \( \Phi \).

**Theorem A.** Let \( A \in (-\infty, +\infty], \lambda \in \Lambda, \alpha \) be a positive increasing to \(+\infty\) function on \([0, +\infty)\) such that \( \alpha(t) = o(t) \) as \( t \to +\infty \), \( F \in D_A(\lambda), \Phi \in \Omega_A', \) and \( \gamma(\sigma) = 2/\alpha(\Phi^{-1}(\sigma)) \) for every \( \sigma \in (A_0, A) \). Suppose that \( \ln n(t, \lambda) \leq t/\alpha(t), \ t \geq t_0, \) and \( \sigma + \gamma(\sigma) < A, \sigma \in [\sigma_0, A) \). If \( T_\Phi(F) = 1 \), then
\[
\lim_{\sigma \uparrow A} \frac{K(\sigma, F)}{\Phi^{-1}(\sigma + \gamma(\sigma))} \leq 1.
\]

(3)

It is shown in [1] that in many cases estimate (3) is sharp. To substantiate the exactness of inequality (3), in [1], in particular, the following theorem was proved.

**Theorem B.** Let \( A \in (-\infty, +\infty], \Phi \in \Omega_A' \) be a twice continuously differentiable function on \( D_\Phi \), and \( \varphi \) be the inverse function of \( \Phi' \). If
\[
\Phi((1 + o(1))\sigma) \sim (1 + o(1))\Phi(\sigma), \quad \sigma \uparrow A,
\]
and \( t^2\varphi'(t) \uparrow +\infty \) as \( t \uparrow +\infty \), then for every sequence \( \lambda \in \Lambda \) there exists a Dirichlet series \( F \in D_A(\lambda) \) such that \( T_\Phi(F) = t_\Phi(F) = 1 \) and
\[
\lim_{\sigma \uparrow A} \frac{K(\sigma, F)}{\Phi^{-1}(\sigma)} = 1.
\]

(4)
It is easily seen that the conditions of Theorem A imply the equality \( \tau(\lambda) = 0 \). Therefore, in the case \( \tau(\lambda) > 0 \) Theorem A does not give any information about the growth of the quantity \( K(\sigma, F) \). Moreover, if \( A < +\infty \), then even in the case \( \tau(\lambda) = 0 \) the conclusion of Theorem A is true only by some conditions on \( \Phi \).

Let \( F \in \mathcal{D}_A \) be a Dirichlet series of the form (1) with nonnegative coefficients \( a_n \). Then \( M(\sigma, F) = G(\sigma, F) = F(\sigma), \sigma < A \). Hence, \( T_\Phi(F) = T_\Phi(F) \) and \( K(\sigma, F) = (\ln M(\sigma, F))' \), \( \sigma < A \). Therefore, as is easy to prove (see Lemma 4 below; see also Lemma 1 in [2]), for such series, without any conditions on the sequence \( \lambda \) of its exponents and on a function \( \Phi \in \Omega_A \), the equality \( T_\Phi(F) = 1 \) (or the identical equality \( T_\Phi(F) = 1 \)) implies the inequality

\[
\lim_{\sigma \to A} \frac{K(\sigma, F)}{\Phi^{-1}(\sigma)} \leq 1. \tag{5}
\]

It turns out that inequality (5) follows from the equality \( T_\Phi(F) = 1 \) for any other Dirichlet series \( F \in \mathcal{D}_A \). The following theorem confirms this fact.

**Theorem 1.** Let \( A \in (-\infty, +\infty], \Phi \in \Omega_A, \) and \( F \in \mathcal{D}_A \). If \( T_\Phi(F) \leq 1 \), then inequality (5) holds.

For an arbitrary function \( \Phi \in \Omega_A \), inequality (5) is sharp in each of the classes \( \mathcal{D}_A(\lambda), \lambda \in \Lambda \). This conclusion can be drawn from the following theorem which is a generalization of Theorem B.

**Theorem 2.** Let \( A \in (-\infty, +\infty] \) and \( \Phi \in \Omega_A \). For every sequence \( \lambda \in \Lambda \) there exists a Dirichlet series \( F \in \mathcal{D}_A(\lambda) \) such that \( T_\Phi(F) = t_\Phi(F) = 1 \) and equality (4) holds.

In order to prove Theorems 1 and 2, we will need some auxiliary results, which are given in the next section.

2. **Auxiliary results** The following lemma is well known (see, for example, [3, § 3.2], [4]).

**Lemma 1.** Let \( A \in (-\infty, +\infty], \Phi \in \Omega_A, \) and \( \varphi(x) = \max\{\sigma \in \mathcal{D}_\Phi: x\sigma - \Phi(\sigma) = \Phi(x)\} \), \( x \in \mathbb{R} \). Then the following statements are true:

(i) the function \( \varphi \) is nondecreasing on \( \mathbb{R} \);
(ii) the function \( \varphi \) is continuous from the right on \( \mathbb{R} \);
(iii) \( \varphi(x) \to A, x \to +\infty \);
(iv) the right-hand derivative of \( \Phi(x) \) is equal to \( \varphi(x) \) at every point \( x \in \mathbb{R} \);
(v) if \( x_0 = \inf\{x > 0: \Phi(\varphi(x)) > 0\} \), then the function \( \Phi(x) = \Phi(x)/x \) increase to \( A \) on \( (x_0, +\infty) \);
(vi) the function \( \alpha(x) = \Phi(\varphi(x)) \) is nondecreasing on \([0, +\infty)\).

In the following lemmas \( \varphi \) and \( x_0 \) are defined by \( \Phi \) in the same way as in Lemma 1.

**Lemma 2** ([5]). Let \( A \in (-\infty, +\infty], \Phi \in \Omega_A, \sigma_0 = \Phi(x_0 + 0), \) and \( \sigma \in (\sigma_0, A) \). Then the minimum value of the function

\[
h(y) = \frac{\Phi(y)}{y - \sigma}, \quad y \in (\sigma, A),
\]

is \( \Phi^{-1}(\sigma) \) and this value is attained at the point \( y = \varphi(\Phi^{-1}(\sigma)) \).
Lemma 3 ([5]). Let $δ ∈ (0, 1)$, $A ∈ (−∞, +∞]$, $Φ ∈ Ω_A$, $σ_0 = \bar{Φ}(x_0 + 0)$, and $y(σ) = \varphi(\Phi^{-1}(σ))$ for all $σ ∈ (σ_0, A)$. Then

$$\Phi^{-1}\left(σ + \frac{δΦ(y(σ))}{Φ^{-1}(σ)}\right) ≤ \frac{Φ^{-1}(σ)}{1 - δ}, \quad σ ∈ (σ_0, A).$$

Lemma 4. Let $A ∈ (−∞, +∞]$, $Φ ∈ Ω_A$, $σ_0 = \bar{Φ}(x_0 + 0)$, $b ∈ [σ_0, A]$, $Ψ$ be a convex function on $(b, A)$ such that $Ψ(y) ≤ Φ(y)$ for all $y ∈ (b, A)$, and

$$E = \{σ ∈ (b, A): \; Ψ(y) - Ψ(σ) ≤ Φ(y) \; \text{for all} \; y ∈ (σ, A)\}.$$

Then $Ψ'_+(σ) ≤ \Phi^{-1}(σ)$ for every $σ ∈ E$.

Proof. Suppose that $σ ∈ E$. Then $σ ∈ (σ_0, A)$ and therefore, setting $y = \varphi(Φ^{-1}(σ))$ and using the convexity of the function $Ψ$ and Lemma 2, we obtain

$$Ψ'_+(σ) ≤ \frac{Ψ(y) - Ψ(σ)}{y - σ} ≤ \frac{Φ(y)}{y - σ} = Φ^{-1}(σ).$$

Remark 1. It is easy to see that if functions $Φ$ and $Ψ$ satisfy the conditions of Lemma 4, then there exists a number $c ∈ [b, A]$ such that $(c, A) ⊂ E$. In addition, the set $E$ contains every point $σ ∈ (b, A)$ such that $Ψ(σ) ≥ 0$.

Lemma 5. Let $A ∈ (−∞, +∞]$, $Φ ∈ Ω_A$, $σ_0 = \bar{Φ}(x_0 + 0)$, $y(σ) = \varphi(Φ^{-1}(σ))$ for each $σ ∈ (σ_0, A)$, $b ∈ [σ_0, A]$, $F ∈ D_A$ be a Dirichlet series of the form (1), and $q > 0$. If $ln G(y, F) ≤ qΦ(y)$ for all $y ∈ (b, A)$, then for every $σ ∈ (b, A)$ and arbitrary $p ≥ q$ we have

$$\sum_{λ_n > pΦ^{-1}(σ)} |a_n|e^{σλ_n} < \frac{1}{e^{(p-q)Φ(y(σ))}}.$$ (6)

Proof. We first prove inequality (6) in the case $p = q$, i.e. we show that

$$\sum_{λ_n > qΦ^{-1}(σ)} |a_n|e^{σλ_n} < 1$$ (7)

for every $σ ∈ (b, A)$.

We fix an arbitrary $σ ∈ (b, A)$ and consider the function

$$H(y) = \sum_{λ_n > qΦ^{-1}(σ)} |a_n|s^{yλ_n}, \quad y < A.$$

Note that inequality (7) can be rewritten as $H(σ) < 1$.

Suppose on the contrary that $H(σ) ≥ 1$. For all $y < A$ we get

$$H'(y) = \sum_{λ_n > qΦ^{-1}(σ)} λ_n |a_n|s^{yλ_n} ≥ \sum_{λ_n > qΦ^{-1}(σ)} qΦ^{-1}(σ)|a_n|s^{yλ_n} = qΦ^{-1}(σ)H(y).$$ (8)
On the other hand, setting $\Psi(y) = (\ln H(y))/q$, $y < A$, we see that the function $\Psi$ is convex on the interval $(-\infty, A)$ and $\Psi(y) \leq (\ln G(y, F))/q \leq \Phi(y)$ for all $y \in (b, A)$. Thus by Lemma 4 (see Remark 1) we have

$$\frac{H'(\sigma)}{qH(\sigma)} = \Psi'(\sigma) \leq \Phi^{-1}(\sigma),$$

which contradicts (8) with $y = \sigma$. Therefore, inequality (7) is proved.

We now prove inequality (6) for $p > q$. Put

$$\delta = p - q, \quad \varepsilon = \frac{\delta \Phi(y(\sigma))}{\Phi^{-1}(\sigma)}.$$

Then by Lemma 3 we have $q\Phi^{-1}(\sigma + \varepsilon) \leq p\Phi^{-1}(\sigma)$. Using inequality (7) with $\sigma + \varepsilon$ instead of $\sigma$, we get

$$\sum_{\lambda_n > p\Phi^{-1}(\sigma)} |a_n|e^{\lambda_n} = \sum_{\lambda_n > p\Phi^{-1}(\sigma)} \frac{1}{e^{p\Phi^{-1}(\sigma)}} |a_n|e^{(\sigma + \varepsilon)\lambda_n} \leq \frac{1}{e^{p\Phi^{-1}(\sigma)}} \sum_{\lambda_n > p\Phi^{-1}(\sigma)} |a_n|e^{(\sigma + \varepsilon)\lambda_n} \leq \frac{1}{e^{p\Phi^{-1}(\sigma)}} \sum_{\lambda_n > q\Phi^{-1}(\sigma + \varepsilon)} |a_n|e^{(\sigma + \varepsilon)\lambda_n} < \frac{1}{e^{p\Phi^{-1}(\sigma)}} = \frac{1}{e^{(p-q)\Phi(y(\sigma))}}.$$

\[\Box\]

The following lemma was proved by I. V. Ostrovskii (see [1]).

**Lemma 6.** Suppose that $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_N$. Then for each exponential polynomial

$$P(s) = \sum_{n=0}^{N} a_n e^{\lambda_n s}$$

and every $\sigma \in \mathbb{R}$ the inequality $M(\sigma, P') \leq \lambda_N M(\sigma, P)$ holds.

Let $\lambda \in \mathbb{R}$. Consider a Dirichlet series $F$ of the form (1) and put

$$\beta(F) = \lim_{n \to \infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}.$$

It is well known (for instance, see [6, p. 114–115]) that

$$\sigma_a(F) \leq \beta(F) \leq \sigma_a(F) + \tau(\lambda)$$

and these inequalities are sharp (moreover, it was shown in [7] that for any $A, B \in [-\infty, +\infty]$ such that $A \leq B \leq A + \tau(\lambda)$ there exists a Dirichlet series $F$ of the form (1) such that $\sigma_a(F) = A$ and $\beta(F) = B$).

Note also that for a Dirichlet series $F$ the interval $(-\infty, \beta(F))$ is the domain of existence of the maximum term $\mu(\sigma, F)$. If $F(s) \neq 0$, then this interval is also the domain of existence of the central index

$$\nu(\sigma, F) = \max\{n \geq 0 : |a_n|e^{\sigma \lambda_n} = \mu(\sigma, F)\}.$$
Lemma 7 ([8]). Let \( \lambda \in \Lambda, A \in (-\infty, +\infty] \). If for a Dirichlet series of the form (1) there exists an increasing sequence \((a_k)_{k=0}^\infty\) of nonnegative integers such that \( a_n = 0 \) for all \( n < n_0 \), \( a_n \neq 0 \) for every \( k \geq 0 \), and
\[
\kappa_k := \frac{\ln |a_{n_k}| - \ln |a_{n_{k+1}}|}{\lambda_{n_{k+1}} - \lambda_{n_k}} \uparrow A, \quad k \uparrow \infty, \quad |a_n| \leq |a_{n_k}|e^{\sigma_{n_k} - \lambda_{n_k}}, \quad n \in (n_k, n_{k+1}), \quad k \geq 0,
\]
then \( \beta(F) = A \) and, moreover, \( \nu(\sigma, F) = n_0 \) for every \( \sigma < \kappa_0 \) and \( \nu(\sigma, F) = n_{k+1} \) for all \( \sigma \in (\kappa_k, \kappa_{k+1}) \) and \( k \geq 0 \).

Lemma 8 ([4]). Let \( \lambda \in \Lambda, A \in (-\infty, +\infty], \) and \( \Phi \in \Omega_A \). If the condition
\[
\forall t > 0: \quad \ln n = o(\Phi(\varphi(\lambda_n/t))), \quad n \to \infty,
\]
holds, then each Dirichlet series \( F \) of the form (1) such that \( \beta(F) = A \) belongs to the class \( \mathcal{D}_A(\lambda) \) and for this series we have \( T_\Phi(F) = t_\Phi(F) \).

3. Proof of Theorems.

Proof of Theorem 1. Suppose that \( A \in (-\infty, +\infty], \Phi \in \Omega_A, F \in \mathcal{D}_A \) is a Dirichlet series of the form (1) such that \( T_\Phi(F) \leq 1 \), and prove that inequality (5) holds.

For all \( s \in \mathbb{C} \) with \( \Re z < A \) and each \( N \in \mathbb{R} \) we put
\[
P_N(s) = \sum_{\lambda_n \leq N} a_n e^{\lambda_n}, \quad R_N(s) = \sum_{\lambda_n > N} a_n e^{\lambda_n}.
\]
Then \( F(s) = P_N(s) + R_N(s) \) and therefore
\[
M(\sigma, F) - M(\sigma, R_N) \leq M(\sigma, P_N) \leq M(\sigma, F) + M(\sigma, R_N), \quad \sigma < A.
\]

As above, let \( x_0 = \inf\{x > 0: \Phi(\varphi(x)) > 0\} \), \( \sigma_0 = \Phi(x_0 + 0) \), and \( y_\sigma = \varphi(\Phi^{-1}(\sigma)) \) for all \( \sigma \in (\sigma_0, A) \).

We fix an arbitrary \( \eta > 1 \) and choose numbers \( p \) and \( q \) such that \( 1 < q < p < \eta \). Since \( T_\Phi(F) \leq 1 \), we have \( \ln G(y, F) \leq q\Phi(y), \quad y \in (b, A) \), for some \( b \in [\sigma_0, A) \). Setting \( N(\sigma) = \eta \Phi^{-1}(\sigma) \), by Lemma 5 we obtain
\[
M(\sigma, R_{N(\sigma)}) \leq \sum_{\lambda_n > \eta \Phi^{-1}(\sigma)} |a_n|e^{\sigma \lambda_n} < \frac{1}{e^{(\eta - q)\Phi(y(\sigma))}}, \quad \sigma \in (b, A).
\]

Therefore, \( M(\sigma, R_{N(\sigma)}) = o(1), \quad \sigma \uparrow A \). Then it follows from (10) that
\[
M(\sigma, P_{N(\sigma)}) = M(\sigma, F) + o(1), \quad \sigma \uparrow A.
\]

Let \( \varepsilon(\sigma) = 1/N(\sigma), \quad y \in (b, A) \). By Lemma 3 we have
\[
\Phi^{-1}(\sigma + \varepsilon(\sigma)) \sim \Phi^{-1}(\sigma), \quad \sigma \uparrow A.
\]

Hence for some \( b_0 \in (b, A) \) we obtain
\[
\eta \Phi^{-1}(\sigma) \geq p \Phi^{-1}(\sigma + \varepsilon(\sigma)), \quad \sigma \in (b_0, A).
\]
Taking into account that for every fixed \( \varepsilon > 0 \) and an arbitrary \( x \geq 0 \) the inequality
\[
\frac{x}{e^{\varepsilon x}} \leq \frac{1}{\varepsilon e}
\]
holds and again using Lemma 5 for all \( \sigma \in (b_0, A) \) we have
\[
M(\sigma, R_{N(\sigma)}') \leq \sum_{\lambda_n > \eta \Phi^{-1}(\sigma)} \lambda_n |a_n| s^{y_{\lambda_n}} = \sum_{\lambda_n > \eta \Phi^{-1}(\sigma)} \frac{\lambda_n}{\varepsilon(\sigma)\lambda_n} |a_n| s^{(\sigma + \varepsilon(\sigma))\lambda_n} \leq \leq \frac{1}{\varepsilon(\sigma)e} \sum_{\lambda_n > \eta \Phi^{-1}(\sigma)} |a_n| s^{(\sigma + \varepsilon(\sigma))\lambda_n} \leq \frac{1}{\varepsilon(\sigma)e} \sum_{\lambda_n > \eta \Phi^{-1}(\sigma)} |a_n| s^{(\sigma + \varepsilon(\sigma))\lambda_n} \leq \frac{1}{\varepsilon(\sigma)e} \frac{1}{e^{(p-q)\Phi(y(\sigma + \varepsilon(\sigma)))}} = \frac{\eta \Phi^{-1}(\sigma)}{e^{(p-q)\Phi(y(\sigma + \varepsilon(\sigma)))}+1}.
\]
Therefore,
\[
M(\sigma, R_{N(\sigma)}') = o(\Phi^{-1}(\sigma)), \quad \sigma \uparrow A.
\] (12)
Further, using Lemma 6 and relations (11) and (12), we obtain
\[
M(\sigma, F') \leq M(\sigma, P_{N(\sigma)}') + M(\sigma, R_{N(\sigma)}') \leq N(\sigma)M(\sigma, P_{N(\sigma)}) + M(\sigma, R_{N(\sigma)}') = \eta \Phi^{-1}(\sigma)M(\sigma, F) + o(\Phi^{-1}(\sigma)), \quad \sigma \uparrow A,
\]
so that
\[
\lim_{\sigma \uparrow A} \frac{K(\sigma, F)}{\Phi^{-1}(\sigma)} \leq \eta.
\]
Since \( \eta > 1 \) is arbitrary, we have (5).

\( \square \)

**Proof of Theorem 2.** Suppose that \( \Phi \in \Omega_A \) and \( \lambda \in \Lambda \), and prove that there exists a Dirichlet series \( F \in \mathcal{D}_A(\lambda) \) such that \( T_\Phi(F) = t_\Phi(F) \) = 1 and equality (4) holds.

As above, we put \( x_0 = \inf\{x > 0: \Phi(\varphi(x)) > 0\}, \) \( \sigma_0 = \Phi(x_0 + 0), \) and \( y(\sigma) = \varphi(\Phi^{-1}(\sigma)) \) for all \( \sigma \in (\sigma_0, A) \). From condition (2) and Lemmas 1 and 3 it follows that there exists a subsequence \( \lambda^* = (\lambda_{n_k}) \) of the sequence \( \lambda \) such that for it and for the sequences \( (x_{k}) \) and \( (\delta_{k}) \), where
\[
x_{k} = \Phi(\lambda_{n_{k+1}}), \quad \delta_{k} = \frac{1}{\sqrt{\Phi(\varphi(\lambda_{n_{k+1}}))}} = \frac{1}{\sqrt{\Phi(y(x_{k}))}}
\]
for all integers \( k \geq 0 \), we have \( n_0 = 0, \) \( \Phi(\varphi(\lambda_{n_1})) > 1, \) and also
\[
\forall t > 0: \quad \ln^2 k = o(\Phi(\varphi(\lambda_{n_k}/t))), \quad n \to \infty; \quad \text{(13)}
\]
\[
(k + 1)\lambda_{n_k} - \Phi(\sigma) \leq (k + 1)\lambda_{n_k} x_0, \quad \sigma \in [x_k, A], \quad k \geq 0; \quad \text{(14)}
\]
\[
\sqrt{\Phi(y(x_{k}))} \leq \Phi^{-1}(x_{k}) = \frac{1}{\delta_{k}} + \frac{1}{\delta_{k} \lambda_{n_{k+1}}} < \lambda_{n_{k+1}}, \quad k \geq 0; \quad \text{(15)}
\]
\[
2\lambda_{n_k} \leq \lambda_{n_{k+1}}, \quad k \geq 0. \quad \text{(16)}
\]
Note that \( (\delta_k) \) is a nonincreasing sequence of points with \( (0, 1) \) tending to 0. Therefore, using (15) and Lemma 3, we obtain
\[
\Phi^{-1}(x_{k}) \sim \Phi^{-1}(x_{k}), \quad k \to \infty. \quad \text{(17)}
\]
In addition, according to (16) and (15),

\[
(\lambda_{n_k+1} - \lambda_{n_k}) (\tau_k - \xi_k) \geq \frac{1}{2} \lambda_{n_k+1} (\tau_k - \xi_k) = \frac{1}{2 \delta_k} = \frac{1}{2} \sqrt{\Phi(\varphi(\lambda_{n_k+1}))},
\]

and hence, using (13), we see that

\[
\frac{k + 1}{e^{(\lambda_{n_k+1} - \lambda_{n_k}) (\tau_k - \xi_k)}} \to 0, \quad k \to \infty. \tag{18}
\]

Put \(a_0 = 1\),

\[
a_{n_k+1} = \prod_{j=0}^{k} e^\xi (\lambda_{n_j} - \lambda_{n_j+1}), \quad k \geq 0,
\]

and \(a_n = 0\) if \(n \in (n_k, n_{k+1})\) for some \(k \geq 0\). By Lemma 7 for Dirichlet series (1) with such coefficients \(a_n\) we have \(\beta(F) = A\) and, moreover, \(\nu(\sigma, F) = n_0\) for every \(\sigma < \xi_0\) and \(\nu(\sigma, F) = n_{k+1}\) for all \(\sigma \in [\xi_k, \xi_{k+1}]\) and \(k \geq 0\).

Note that series (1) can be represented as

\[
F(s) = \sum_{m=0}^{\infty} a_{nm} e^{s\lambda_{nm}}.
\]

Since \(\beta(F) = A\) and condition (13) holds, \(F \in \mathcal{D}_A(\lambda^*)\) and \(T_\Phi(F) = t_\Phi(F)\) by Lemma 8. Then also \(F \in \mathcal{D}_A(\lambda)\).

Let \(\sigma \in [\xi_k, \xi_{k+1}]\) and \(k \geq 0\). Then

\[
\xi_k = \Phi(\lambda_{n_k+1}) = \max \left\{ y - \Phi(y) : y \in D_\Phi \right\} \geq \sigma - \frac{\Phi(\sigma)}{\lambda_{n_k+1}}. \tag{19}
\]

From (19) and (14) we obtain, respectively, the following inequalities

\[
\lambda_{n_k+1}(\sigma - \xi_k) \leq \Phi(\sigma), \quad \lambda_{n_k}(\xi_k - \xi_0) \leq \lambda_{n_k}(\sigma - \xi_0) \leq \frac{\Phi(\sigma)}{k + 1}.
\]

Using these inequalities, we have

\[
\ln \mu(\sigma, F) = \int_{\xi_0}^{\sigma} \lambda_{\nu(t, F)} dt = \int_{\xi_0}^{\xi_k} \lambda_{\nu(t, F)} dt + \int_{\xi_k}^{\sigma} \lambda_{\nu(t, F)} dt \leq \lambda_{n_k}(\xi_k - \xi_0) + \lambda_{n_k+1}(\sigma - \xi_0) \leq \frac{\Phi(\sigma)}{k + 1} + \Phi(\sigma) = \frac{k + 2}{k + 1} \Phi(\sigma).
\]

Thus we see that \(t_\Phi(F) \leq 1\). Then also \(T_\Phi(F) \leq 1\), and therefore by Theorem 1 for the constructed series inequality (5) holds.

Next, for an arbitrary \(\sigma < A\) and each integer \(p \geq 0\) we set

\[
Q_p(\sigma) = \sum_{m \leq p} \lambda_{nm} a_{nm} e^{s\lambda_{nm}}, \quad R_p(\sigma) = \sum_{m > p} \lambda_{nm} a_{nm} e^{s\lambda_{nm}}, \\
S_p(\sigma) = \sum_{m \leq p} a_{nm} e^{s\lambda_{nm}}, \quad T_p(\sigma) = \sum_{m > p} a_{nm} e^{s\lambda_{nm}}.
\]
Since $Q_p(\sigma)T_p(\sigma) \leq S_p(\sigma)\lambda_nT_p(\sigma) \leq S_p(\sigma)R_p(\sigma)$, we obtain

$$K(\sigma, F) = \frac{F'(\sigma)}{F(\sigma)} = \frac{Q_p(\sigma) + R_p(\sigma)}{S_p(\sigma) + T_p(\sigma)} \geq \frac{Q_p(\sigma)}{S_p(\sigma)}$$  \hfill (20)

Let $k \geq 0$ be an arbitrary integer. According to (15) we have $\tau_k \in (\sigma_k, \sigma_{k+1})$, and therefore $\mu(\tau_k, F) = a_{nk+1}e^{r_k\lambda_{nk+1}}$. If $m \leq k$, then

$$a_m e^{r_k\lambda_m} = a_m e^{r_k\lambda_m} e^{(\tau_k - \sigma_k) \lambda_m} \leq \mu(\tau_k, F) e^{(\tau_k - \sigma_k) \lambda_0} = a_{nk+1} e^{r_k\lambda_{nk+1} e^{(\tau_k - \sigma_k) \lambda_0}} =$$

$$= \frac{a_{nk+1} e^{r_k\lambda_{nk+1}}}{e^{(\lambda_{nk+1} - \lambda_0)(\tau_k - \sigma_k)}} = \frac{\mu(\tau_k, F)}{e^{(\lambda_{nk+1} - \lambda_0)(\tau_k - \sigma_k)}},$$

and so, using (18), we get

$$S_k(\tau_k) = \sum_{m \leq k} a_m e^{r_k\lambda_m} \leq \frac{(k + 1) \mu(\tau_k, F)}{e^{(\lambda_{nk+1} - \lambda_0)(\tau_k - \sigma_k)}} = o(\mu(\tau_k, F)), \quad k \to \infty.$$  \hfill (21)

Using (20) with $\sigma = \tau_k$ and $p = k + 1$, (21), and (17), we have

$$K(\tau_k, F) \geq \frac{Q_{k+1}(\tau_k)}{S_{k+1}(\tau_k)} \geq \frac{\lambda_{nk+1} \mu(\tau_k, F)}{S_k(\tau_k) + \mu(\tau_k, F)} = (1 + o(1))\lambda_{nk+1} = (1 + o(1))\Phi^{-1}(\sigma_k) =$$

$$= (1 + o(1))\Phi^{-1}(\tau_k)$$

as $k \to \infty$. Consequently,

$$\lim_{\sigma \uparrow A} K(\sigma, F) \geq \frac{\Phi^{-1}(\tau_k)}{\Phi^{-1}(\tau_k)} \geq 1.$$  \hfill (22)

This and (5) imply (4).

Finally, we prove that $T_\Phi(F) = 1$. Suppose, on the contrary, that $T_\Phi(F) < 1$. We fix some $q \in (0, 1)$ such that $T_\Phi(F) \leq q$ and put $\Psi(\sigma) = q\Phi(\sigma), \sigma \in D_\Phi$. Then $T_\Phi(F) \leq 1$ and is easy to see $\Psi^{-1}(\sigma) = q\Phi^{-1}(\sigma)$ for all $\sigma \in (\sigma_0, A)$. Applying Theorem 1 to $\Psi$ instead of $\Phi$, we obtain

$$\lim_{\sigma \uparrow A} K(\sigma, F) = q \lim_{\sigma \uparrow A} K(\sigma, F) \leq q.$$  \hfill (23)

This contradicts (4).  \hfill $\square$

**Remark 2.** In view of the above results, it is natural to ask whether we can replace the condition $T_\Phi(F) \leq 1$ in Theorem 1 by the condition $T_\Phi(F) \leq 1$. Nothing as strong as this is known. It is clear that such replacement is possible, for example, under conditions that ensure the equality $T_\Phi(F) = T_\Phi(F)$, in particular, provided that (9) holds. Note that the equality $T_\Phi(F) = T_\Phi(F)$ may not be satisfied in the general case (see, for example, [9, 10]).

**REFERENCES**

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