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EXTREME PROBLEMS IN THE SPACE OF MEROMORPHIC FUNCTIONS OF FINITE ORDER IN THE HALF-PLANE. II

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The extremal problems in the space of meromorphic functions of order $\rho > 0$ in upper half-plane are studed. The method for studying is based on the theory of Fourier coefficients of meromorphic functions. The concept of just meromorphic function of order $\rho > 0$ in upper half-plane is introduced. Using Lemma on the Pólya peaks and the Parseval equality, sharp estimate from below of the upper limits of relations Nevanlinna characteristics of meromorphic functions in the upper half plane are obtained.

1. Introduction. This paper is a direct continuation of [1]. We shall use, without repeating them, the definitions, results and notation of [1]. However, sections, formulas, theorems and other propositions are labeled independently of [1]. We now state the central results of the present paper.

Let $f \in JM$, $\lambda = \lambda_f$ be the corresponding complete measure of f, $\lambda_f := \lambda = \lambda_+ - \lambda_-$ be the Jordan decomposition of λ_f. We set up following notations and terminology

$$m(r,f) := \frac{1}{r} \int_0^\pi \log^+ |f(re^{i\varphi})| \sin \varphi d\varphi, \quad \widehat{N}(r,f) := \int_{r_0}^r \frac{\lambda_-(t)}{t^2} dt,$$

$$N(r,f) := \int_{r/2}^r \frac{\lambda_-(t)}{t^3} dt, \quad T(r,f) := m(r,f) + N(r,f) + m\left(\frac{r}{2}, \frac{1}{f}\right) \text{ if } \rho \le 1,$$

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where r_0 is an arbitrary fixed positive number. We have

$$T(r,f) = T\left(r, \frac{1}{f}\right),$$

(see [1, (3)]). Hence

$$\int\limits_0^\pi |\log|f(re^{i\varphi})|| \sin \varphi d\varphi \le 2rT(r,f). \tag{1}$$

Let us introduce the notions of the order and the indicator of a function. These notions describe growth of a function at infinity.

Definition 1. A function $f \in JM$ is said to be a *function of finite order* if there exists a positive constant $\beta > 1$ such that the inequality $T(r,f) \le r^{\beta-1}$ is valid for all sufficiently large values of r (i.e $r > r_0(\beta)$).

The greatest lower bound ρ of such numbers β is called the *order of the function* $f \in JM$.

The space of such functions is denoted by $JM(\rho)$. By $JA(\rho) \subset JM(\rho)$ we denote the subspace of just analytic functions in $\mathbb{C}_+$. By $m_2(r,f)$ we denote L_2-norm of the function f about the semicircle of $\{re^{i\theta} : 0 \le \theta \le \pi\}$:

$$m_2(r,f) = \left\{ \frac{1}{\pi} \int\limits_0^\pi \left|\log|f(re^{i\theta})|\right|^2 d\theta \right\}^{1/2},$$

The Fourier coefficients of a function $f \in JM$ are defined as

$$c_k(r,f) = \frac{2}{\pi} \int\limits_0^\pi \log|f(re^{i\theta})| \sin k\theta d\theta, \quad k \in \mathbb{N}.$$

Using (1), we obtain

$$|c_k(r,f)| \le \frac{2k}{\pi} \int\limits_0^\pi |\log|f(re^{i\varphi})|| \sin \varphi d\varphi \le \frac{4rk}{\pi} T(r,f) \tag{2}$$

and

$$m(r,f) + m(r,1/f) = \int\limits_0^\pi |\log|f(re^{i\varphi})|| \sin \varphi d\varphi \le 2rT(r,f). \tag{3}$$

Set

$$d\lambda_k(\zeta) = \frac{\sin k\varphi}{\sin \varphi} \tau^{k-1} d\lambda(\zeta) \, (\zeta = \tau e^{i\varphi}), \lambda_m(r) = \lambda_m\left(\overline{C(0,r)}\right),$$

where $\frac{\sin m\varphi}{\sin \varphi}$ is defined for $\varphi = 0, \pi$ by continuity.

Note the inequality

$$|\lambda_k(r)| = \left|\iint_{\overline{C(0,r)}} d\lambda_k(\zeta)\right| = \left|\iint_{\overline{C(0,r)}} \frac{\sin k\varphi}{\sin \varphi} \tau^{k-1} d\lambda(\zeta)\right| \le$$

$$\le k \iint_{\overline{C(0,r)}} \tau^{k-1} d|\lambda|(\zeta) \le kr^{k-1}|\lambda|(r). \tag{4}$$

By Carleman's formula in Grishin's notations [1, (1)],

$$c_k(r,f) = \frac{1}{2^k} c_k(2r,f) - \frac{2r^k}{\pi} \int\limits_r^{2r} \frac{\lambda_k(t)}{t^{2k+1}} dt, \quad k \in \mathbb{N}.$$

The following expressions for the Fourier coefficients for $r > r_0$ holds

$$c_k(r, f) = \alpha_k r^k + \frac{2r^k}{\pi} \int\limits_{r_0}^{r} \frac{\lambda_k(t)}{t^{2k+1}} dt, \quad k \in \mathbb{N}, \tag{5}$$

where $\alpha_k = r_0^{-k} c_k(r_0, v)$, and

$$c_k(r, f) = \alpha_k r^k + \frac{r^k}{\pi k r_0^{2k}} \iint\limits_{C_+(0, r_0)} \frac{\sin k\varphi}{\operatorname{Im} \zeta} \tau^k d\lambda(\zeta) +$$

$$+ \frac{r^k}{\pi k} \iint\limits_{r_0 \le |\zeta| \le r} \frac{\sin k\varphi}{\tau^k \operatorname{Im} \zeta} d\lambda(\zeta) - \frac{1}{r^k \pi k} \iint\limits_{\overline{C_+(0, r)}} \frac{\sin k\varphi}{\operatorname{Im} \zeta} \tau^k d\lambda(\zeta), \quad \zeta = \tau e^{i\varphi}, \tag{6}$$

(see [1, formulas (10) and (11)]), where the kernel $\frac{\sin k\varphi}{\tau^k \operatorname{Im} \zeta}$ is extended by continuity to the points on the real axis, $\alpha_k = r_0^{-k} c_k(r_0, v)$.

We will use the following lemma.

Lemma 1. *Let $f \in JM(\rho)$, $\rho > 0$. Then*

$$c_k(r, f) = -\frac{2r^k}{\pi} \int\limits_{r}^{\infty} \frac{\lambda_k(t)}{t^{2k+1}} dt, \quad k \in \mathbb{N}, k > [\rho]. \tag{7}$$

Proof. We divide the equality (5) by r^k, $k > \rho$, and pass to the limit as $r \to \infty$. By inequality (2), the inclusion $f \in JM(\rho)$, we obtain

$$0 = \alpha_k + \frac{2}{\pi} \int\limits_{r_0}^{\infty} \frac{\lambda_k(t)}{t^{2k+1}} dt, \Rightarrow \alpha_k = -\frac{2}{\pi} \int\limits_{r_0}^{\infty} \frac{\lambda_k(t)}{t^{2k+1}} dt, \quad k \in \mathbb{N}, k > \rho.$$

Substituting this value of α_k in (5), we obtain (7). $\qquad \square$

Our main result is the following theorem.

Theorem 1. *Let $f \in JM(\rho)$, $\rho > 0$. Then*

$$\limsup_{r \to \infty} \frac{\widehat{N}(r, f) + \widehat{N}(r, 1/f)}{m_2(r, f)} \ge \frac{|\sin \pi \rho|}{\rho(\rho + 1)} \sqrt{1 \left/ \left(1 - \frac{\sin 2\pi \rho}{2\pi \rho}\right)\right.}, \tag{8}$$

and this inequality is sharp, i. e. for some meromorphic function f, $f \in JM(\rho)$, $\rho > 0$, equality (8) holds.

Remark 1. In contrast to Theorem 1 and Corollary from [1] which are true for $\rho > 1$, this Theorem holds for all $\rho > 0$.

2. Proof of Theorem 1. Let us prove Theorem 1. Let $f \in JM(\rho)$, $\rho > 0$. Let ρ be a non-integer. In the case of integer ρ, the theorem is obvious.

We define a measure $\tilde{\lambda}$ by the equality $\tilde{\lambda} = |\lambda_f|$. Without loss of generality, we can suppose that the measure $\tilde{\lambda}$ does not load some neighborhood of zero. Denote by $q = [\rho]$, $\beta = \{\beta_k\}$,

$$
\beta_k = \begin{cases} |\alpha_k(f)| + \left| \dfrac{1}{\pi k r_0^{2k}} \iint\limits_{C_+(0,r_0)} \dfrac{\sin k\varphi}{\operatorname{Im}\zeta} \tau^k d\lambda_f(\zeta) \right|, & 1 < k < q+1, \\ -S_+(+\infty; k, \tilde{\lambda}), & k \geq q+1, \end{cases}
$$

where

$$
S_+(+\infty; k, \tilde{\lambda}) := \lim_{r \to \infty} S_+(r; k, \tilde{\lambda}).
$$

The following estimate holds

$$
|c_k(r; \tilde{\lambda}, \beta)| \leq \frac{(k+1)r^k}{\pi} \int\limits_r^\infty \frac{\tilde{\lambda}(t)}{t^{k+2}} dt - \frac{k-1}{r^k \pi} \int\limits_0^r t^{k-2} \tilde{\lambda}(t) dt \leq \frac{A(k+1)r^{\rho+\varepsilon}}{\pi(k-\rho-\varepsilon)}, \quad k \geq q+1, \quad (9)
$$

(see [1, (20)]), for some $A > 0$, $\varepsilon > 0$, $\rho + \varepsilon < q + 1$.

A similar inequality holds for $1 \leq k < q+1$. Thus the pair $(\tilde{\lambda}, \beta)$ is ρ-admissible. By Lemma 6 from [1], there exists the function $F \in JA(\rho)$ such that $c_k(r, F) = c_k(r; \tilde{\lambda}, \beta)$ for all $r > 0$ and for all $k \in \mathbb{N}$.

Set

$$
\widehat{N}_1(r) = \widehat{N}_1(r, F) := \int\limits_1^r \frac{\tilde{\lambda}(t)}{t^2} dt.
$$

The order of the function $\widehat{N}(r)$ is less than or equal to ρ. In fact, it is equal ρ, because otherwise it follows from Theorem 3 [10] that this order of $T(r, f)$ is an integer, but we excluded this case.

Integrating by parts in (9), we obtain

$$
|c_k(r, F)| \leq \frac{(k+1)r^k}{\pi} \int\limits_0^r \frac{d\widehat{N}_1(t)}{t^k} - \frac{k-1}{r^k \pi} \int\limits_0^r t^k d\widehat{N}_1(t) =
$$

$$
= \frac{k}{\pi} \left\{ (k-1) \int\limits_0^r \left(\frac{t}{r}\right)^k \frac{\widehat{N}_1(t)}{t} dt + (k+1) \int\limits_r^\infty \left(\frac{r}{t}\right)^k \frac{\widehat{N}_1(t)}{t} dt \right\} - \frac{2k}{\pi} r \widehat{N}_1(r) \qquad (10)
$$

for $k \geq q + 1$.

By (6), we have

$$
|c_k(r, F)| \leq r^k \left(|\alpha_k| + \frac{2}{\pi r_0^{2k}} \int\limits_0^{r_0} t^{k-1} d\tilde{\lambda}(t) \right) + \frac{1}{\pi} \int\limits_{r_0}^r \left[\left(\frac{r}{t}\right)^k - \left(\frac{t}{r}\right)^k \right] \frac{d\tilde{\lambda}(t)}{t},
$$

for $1 \leq k \leq q$.

Therefore, by double integration by parts, we obtain the inequality

$$
|c_k(r, F)| \leq r^k \gamma_k + \frac{2k}{\pi} r \widehat{N}_1(r) + \frac{k(k-1)}{\pi} \int\limits_{r_0}^r \left[\left(\frac{r}{t}\right)^k - \left(\frac{t}{r}\right)^k \right] \frac{\widehat{N}_1(t)}{t} dt, \quad 1 \leq k \leq q, \quad (11)
$$

where

$$\gamma_k = |\alpha_k| + \frac{2}{\pi r_0^{2k}} \int\limits_0^{r_0} t^{k-1} d\tilde{\lambda}(t) + \frac{\tilde{\lambda}(r_0)}{\pi r_0^{k+1}}.$$

Besides, $|c_k(r, f)| \leq |c_k(r, F)|$, $1 \leq k \leq q$, and $|c_k(r, f)| \leq -c_k(r, F)$, $k \geq q + 1$. Hence

$$m_2(r, f) \leq m_2(r, F) \tag{12}$$

Let $\varepsilon > 0$ be a fixed number. Applying the lemma of Pólya peaks [11] for functions $\widehat{N}_1(r)$, $r^{\rho-\varepsilon}$, $r^{\rho+\varepsilon}$, we find the sequence (r_n), $\lim\limits_{n\to\infty} r_n = \infty$, ([8, p. 62]) such that

$$\widehat{N}_1(t) \leq \left(\frac{t}{r_n}\right)^{\rho-\varepsilon} \widehat{N}_1(r_n),\ 0 < t \leq r_n, \quad \widehat{N}_1(t) \leq \left(\frac{t}{r_n}\right)^{\rho+\varepsilon} \widehat{N}_1(r_n),\ t > r_n. \tag{13}$$

Using inequalities (10), (11), (13), we obtain

$$\begin{aligned}
|c_k(r_n, F)| &\leq r_n^k \gamma_k + \frac{2k}{\pi} \widehat{N}_1(r_n) \left(\frac{k^2 + \rho - \varepsilon}{(\rho - \varepsilon)^2 - k^2} + 1\right), \quad 1 \leq k \leq [\rho], \\
|c_k(r_n, F)| &\leq \frac{2k}{\pi} \widehat{N}_1(r_n) \left(\frac{k^2 + \rho - \varepsilon k}{(k - \varepsilon)^2 - \rho^2} - 1\right), \quad k > [\rho].
\end{aligned} \tag{14}$$

Inequality (13) implies, in particular, that $r_n^{[\rho]} = o(\widehat{N}_1(r_n))$ as $n \to \infty$, since for t' such that $r_0 < t'$ and $\widehat{N}_1(t') \geq (t')^{\rho-\varepsilon}$ for all $r_n \geq t'$ the inequality

$$r_n^{q+\varepsilon} < r_n^{\rho-\varepsilon} \leq r_n^{\rho-\varepsilon} \frac{\widehat{N}_1(t')}{(t')^{\rho-\varepsilon}} \leq \widehat{N}_1(r_n)$$

holds if $2\varepsilon < \rho - q$.

From this remark, arbitrariness ε and (14), it follows that

$$\limsup\limits_{n\to\infty} \frac{|c_k(r_n, F)|}{\widehat{N}_1(r_n)} \leq \frac{2k}{\pi} \frac{\rho^2 + \rho}{|\rho^2 - k^2|}, \quad k = 1, 2, \dots.$$

By the Parseval's equality

$$(m_2(r, F))^2 = \frac{1}{2} \sum\limits_{k=-\infty}^{\infty} |c_k(r, F)|^2,$$

we obtain the inequality

$$\liminf\limits_{n\to\infty} \frac{m_2(r, F)}{\widehat{N}_1(r)} \leq \left\{\sum\limits_{k=-\infty}^{\infty} \frac{2k^2}{\pi^2} \frac{(\rho^2 + \rho)^2}{|\rho^2 - k^2|^2}\right\}^{1/2}.$$

The sum on the right-hand side of this inequality can be easily found using residues [12]. It is equal

$$\sum\limits_{k=-\infty}^{\infty} \frac{2k^2}{\pi^2} \frac{(\rho^2 + \rho)^2}{|\rho^2 - k^2|^2} = \left(\frac{\rho(\rho + 1)}{|\sin \pi\rho|}\right)^2 \left(1 - \frac{\sin 2\pi\rho}{2\pi\rho}\right). \tag{15}$$

Since $\widehat{N}_1(r) = \widehat{N}(r, f) + \widehat{N}(r, 1/f)$, by (15) we get the statement of Theorem 1.

We show that the estimate (8) is sharp.

Example. The function

$$f(z) = \exp\left\{\frac{1}{\pi i} \int\limits_{t \geq 1} \frac{(tz+1)^{q+1} t^\rho}{(t-z)(t^2+1)^{q+1}} dt\right\}, \quad \rho > 0, \ q = [\rho],$$

is the analytic function in $\mathbb{C}_+$ with the complete measure $\lambda_f(t) = t^{\rho+1} = \lambda_+(t) = \tilde{\lambda}(t)$ if $t \in \mathbb{R}$, $t \geq 1$, and $d\lambda_f(z) \equiv 0$ if $z \in \mathbb{C}_+ \cup (-\infty, 1)$,

$$N_1(r) = N(r, 1/f) = \int\limits_1^r t^{\rho-2} dt = \frac{r^{\rho-1}}{\rho-1} - \frac{1}{\rho-1}, \ N(r, f) \equiv 0, \ \lambda_k(t) = \frac{k(\rho+1)}{k+\rho} t^{k+\rho}.$$

Now estimate the function

$$u(z) = \text{Re}\left\{\frac{1}{\pi i} \int\limits_{t \geq 1} \frac{(tz+1)^{q+1} t^\rho}{(t-z)(t^2+1)^{q+1}} dt\right\}.$$

We have

$$u(z) = \frac{1}{\pi i} \int\limits_{t \geq 1} = \frac{1}{\pi i} \int\limits_{1 \leq t \leq \frac{1}{2}|z|} + \frac{1}{\pi i} \int\limits_{\frac{1}{2}|z| < t \leq 2|z|} + \frac{1}{\pi i} \int\limits_{t > 2|z|} = I_1 + I_2 + I_3.$$

Further

$$|I_1| \leq \frac{1}{\pi} \int\limits_{1 \leq t \leq \frac{1}{2}|z|} \frac{(t|z|+1)^{q+1} t^\rho}{\left(\frac{1}{2}|z|\right)(t^2+1)^{q+1}} dt \leq \frac{1}{\pi} \int\limits_{t \geq 1} \frac{dt}{t^{1+\varepsilon}} \times \max_{1 \leq t \leq \frac{1}{2}|z|} \frac{t^{\rho+1+\varepsilon}(t|z|+1)^{q+1}}{\left(\frac{1}{2}|z|\right)(t^2+1)^{q+1}} \leq C_{q,\varepsilon}^{(1)} |z|^{\rho+\varepsilon},$$

$$|I_3| \leq \frac{1}{\pi} \int\limits_{2|z|}^{\infty} \frac{(t|z|+1)^{q+1} t^\rho}{\left(\frac{1}{2}t\right)(t^2+1)^{q+1}} dt \leq \frac{1}{\pi} \int\limits_{t \geq 1} \frac{dt}{t^{1+\varepsilon}} \times \max_{t \geq 2|z|} \frac{t^{\rho+1+\varepsilon}(t|z|+1)^{q+1}}{\left(\frac{1}{2}t\right)(t^2+1)^{q+1}} \leq C_{q,\varepsilon}^{(2)} |z|^{\rho+\varepsilon},$$

$$I_2 = \frac{1}{\pi i} \int\limits_{\frac{1}{2}|z| < t \leq 2|z|} \frac{(tz+1)^{q+1} - (t^2+1)^{q+1}}{(t^2+1)^{q+1}(t-z)} t^\rho dt + \frac{1}{\pi i} \int\limits_{\frac{1}{2}|z| < t \leq 2|z|} \frac{t^\rho dt}{t-z} = I_2^{(1)} + I_2^{(2)}.$$

As can be easily seen,

$$\left|\frac{(tz+1)^{q+1} - (t^2+1)^{q+1}}{(t-z)}\right| \leq C_q |z|^{2q+1}$$

for $\frac{1}{2}|z| < t \leq 2|z|$ (since the expression in the left-hand side is a polynomial in both z and t).

Therefore

$$|I_2^{(1)}| \leq \frac{C_q}{\pi} \int\limits_{\frac{1}{2}|z| < t \leq 2|z|} \frac{|z|^{2q+1} t^\rho}{(t^2+1)^{q+1}} dt \leq C_{q,\varepsilon}^{(3)} |z|^{\rho+\varepsilon} \int\limits_1^{\infty} \frac{dt}{t^{1+\varepsilon}}.$$

Finally, by the Poisson formula for the half-plane, we obtain

$$\text{Re } I_2^{(2)} = \frac{1}{\pi} \int\limits_{\frac{1}{2}|z| < t \leq 2|z|} t^\rho \text{Im} \frac{1}{t-z} dt \leq 2^\rho |z|^\rho.$$

Thus,

$$\log |f(z)| < C_{q,\varepsilon}|z|^{\rho+\varepsilon}.$$

Therefore

$$m(r, f) < C_{q,\varepsilon}\pi r^{\rho+\varepsilon-1}.$$

Since $N(r, f) \equiv 0$ we have

$$T(r, f) = m(r, f) < C_{q,\varepsilon}\pi r^{\rho+\varepsilon-1}.$$

Hence $f \in [\rho, \infty]^+$.

By (7), we obtain

$$c_k(r, f) = -\frac{2r^k k(\rho+1)}{\pi(k+\rho)} \int\limits_r^\infty t^{\rho-k-1} dt = \frac{2r^\rho k(\rho+1)}{\pi(k^2 - \rho^2)}, \quad k > [\rho].$$

Then

$$\frac{c_k(r, f)}{\widehat{N}(r, 1/f)} = \frac{2r^\rho k(\rho^2 - 1)}{\pi(k^2 - \rho^2)} + o(1), \quad r \to \infty, \ k > [\rho]. \tag{16}$$

By (6), we obtain further

$$c_k(r, f) = r^k \tilde{\gamma} + \frac{\rho+1}{\pi} \int\limits_{r_0}^r \left[\left(\frac{r}{t}\right)^k - \left(\frac{t}{r}\right)^k \right] t^{\rho-1} dt =$$

$$= r^k \tilde{\gamma} + \frac{2r^\rho k(\rho^2 - 1)}{\pi(k^2 - \rho^2)} + o(1), \quad r \to \infty, \ 1 \le k \le [\rho],$$

where

$$\tilde{\gamma} = \alpha_k + \frac{2}{\pi r_0^{2k}} \int\limits_0^{r_0} t^{k-1} d\tilde{\lambda}(t).$$

From this and (16), we obtain

$$\lim_{r\to\infty} \frac{c_k(r, f)}{\widehat{N}(r, 1/f)} = \frac{2k(\rho^2 - 1)}{\pi(k^2 - \rho^2)}, \quad k \in \mathbb{N}.$$

Thus, for $f(z)$ the estimate (8) is exact.

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