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SECOND HANKEL DETERMINANT FOR A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY SĂLĂGEAN-DIFFERENCE OPERATOR

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In the present investigation, inspired by the work on Yamaguchi type class of analytic functions satisfyingthe analytic criteria $\mathfrak{Re}\{\frac{f(z)}{z}\} > 0$, in the open unit disk $\Delta = \{z \in \mathbb{C}\colon |z| < 1\}$ and making use of Sălăgean-difference operator, which is a special type of Dunkl operator with Dunkl constant ϑ in Δ , we designate definite new classes of analytic functions $\mathcal{R}^{\beta}_{\lambda}(\psi)$ in Δ . For functionsin this new class , significant coefficient estimates $|a_2|$ and $a_3|$ are obtained. Moreover, Fekete-Szegő inequalities and second Hankel determinant for the function belonging to this class are derived. By fixing the parameters a number of special cases are developed are new (or generalization) of the results of earlier researchers in this direction.

1. Introduction and motivation. Let $\mathcal{A}$ be the family of functions f analytic in the open unit disk $\Delta := \{z \in \mathbb{C}\colon |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$ and having Taylor-Maclaurin's series of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta). \tag{1}$$

Denote by $\mathcal{S}$, the class of functions of the form (1) which are univalent in Δ. For given two analytic functions f and g, one will say f is subordinate to g written as $f(z) \prec g(z)$ if there exists an analytic functions $\omega(z)$ satisfying the conditions of Schwarz lemma (i.e. $\omega(0) = 0$ and $|\omega(z)| < 1$) such that $f(z) = g(\omega(z))$.
Denote by $\mathcal{P}$ the class of analytic functions $p(z)$ on Δ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2}$$

satisfying the conditions $p(0) = 1$ and $\operatorname{Re} p(z) > 0\,(z \in \Delta)$. Here $p(z)$ is called the Caratheodory function (see [5]).

Let ψ be an analytic function on Δ such that: 1) $\operatorname{Re}\psi(z) > 0\,(z \in \Delta)$; 2) $\psi(0) = 1$, $\psi'(0) > 0$; 3) ψ maps Δ onto a domain starlike with respect to 1 and symmetric with respect to the real axis. W. Ma and D. Minda (see [19]) introduced the following two classes of analytic functions

$$\mathcal{S}^*(\psi) = \Big\{ f \in \mathcal{A}\colon \frac{zf'(z)}{f(z)} \prec \psi(z)\ (z \in \Delta) \Big\}, \tag{3}$$

$$\mathcal{C}(\psi) = \Big\{ f \in \mathcal{A}\colon 1 + \frac{zf''(z)}{f'(z)} \prec \psi(z)\ (z \in \Delta) \Big\}.$$

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The classes $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$ unified various subclasses of starlike and convex functions in Δ. For example, taking $\psi(z)=\psi_\alpha(z)=\frac{1+(1-2\alpha)z}{1-z}$ $(0\le\alpha<1)$, the class $\mathcal{S}^*(\alpha):=\mathcal{S}^*(\psi)$ is the class of starlike functions of order α and the class $\mathcal{C}(\alpha):=\mathcal{C}(\psi)$ is the class of convex functions of order α. The classes $\mathcal{S}^*:=\mathcal{S}^*(0)$ and $\mathcal{C}:=\mathcal{C}(0)$ are the well-known classes of starlike and convex functions, respectively. Further if we set $\psi(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1, z\in\Delta)$ we get the well know subclasses $\mathcal{S}^*[A,B]$ and $\mathcal{C}[A,B]$. Following the definitions of W. Ma and D. Minda starlike and convex functions, various classes of analytic functions defined by means of subordination are established recently (see [1, 34]). There has been triggering interest to introduce and study new subclasses of analytic functions based on differential and difference operators in geometric function theory.

To define a new function class we recall the following difference operator. For a function $f\in\mathcal{A}$ and $k\in\mathbb{N}_0=\{0,1,2,3,\cdots\}$ we define a linear operator $\mathfrak{D}^k_\vartheta\colon\mathcal{A}\longrightarrow\mathcal{A}$ as follows:

$$\mathfrak{D}^0_\vartheta f(z)=f(z),$$
$$\mathfrak{D}^k_\vartheta f(z)=\mathfrak{D}^1_\vartheta(\mathfrak{D}^{k-1}_\vartheta f(z))=z+\sum_{n=2}^{\infty}\left[n+\frac{\vartheta}{2}(1+(-1)^{n+1})\right]^k a_nz^n \qquad (z\in\Delta). \tag{4}$$

The operator $\mathfrak{D}^k_\vartheta$ is known as the Sălăgean-difference operator (see [9, 25]). This operator is a modified Dunkel operator of complex variables (see [4, 8]). Dunkel operator describes a major generalization of partial derivatives and realizes the commutative law in $\mathbb{R}^n$. In geometry, it attains the reflexive relation, which is plotting the space into itself as a set of fixed points.When $\vartheta=0$, $\mathfrak{D}^k_\vartheta=\mathfrak{D}^k_0=\mathfrak{D}^k$ is known as the Sălăgean differential operator (see [32]).

Let us give some simple examples of the actions of the introduced operator. For $f(z)=z\cos z=z-\frac{z^3}{2}+\frac{z^5}{4!}-\cdots$ we have $\mathfrak{D}^1_1 f(z)=z-2z^3+\frac{z^5}{4}-\cdots$. For $g(z)=\ln(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots$ we get $\mathfrak{D}^1_1 g(z)=z-z^2+\frac{4}{3}z^3-z^4+\cdots$. For $g(z)=\frac{z}{1-z}=z+z^2+z^3+\cdots$ one has ([9]) $\mathfrak{D}^1_1 g(z)=z+2z^2+4z^3+4z^4+6z^5+6z^6+\cdots$. For $g(z)=\frac{z}{(1-z)^2}=z+2z^2+3z^3+\cdots$ we obtain ([9]) $\mathfrak{D}^1_1 g(z)=z+4z^2+12z^3+16z^4+30z^5+36z^6+\cdots$ and $\dfrac{\mathfrak{D}^k_\vartheta g(z)}{z}=1+2.2^kz+3(3+\vartheta)^kz^2+4.4^kz^3+\cdots$.

In 1966 K. Yamaguchi [37] defined a class satisfying $\operatorname{Re}(f(z)/z)>0$ and obtained some coefficient estimates. Using of Sălăgean-difference operator, we will introduce the following class of functions, defined by subordination.

Definition 1. Let $\psi\colon\Delta\longrightarrow\mathbb{C}$ be an analytic function such that $\operatorname{Re}\psi(z)>0\,(z\in\Delta)$, $\psi(0)=1$, $\psi'(0)>0$ and ψ maps Δ onto a domain starlike with respect to 1 and symmetric with respect to the real axis. A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^k_\vartheta(\psi)$ if

$$\frac{\mathfrak{D}^k_\vartheta f(z)}{z}\prec\psi(z)\quad (z\in\Delta). \tag{5}$$

Remark 1. Taking $\vartheta=0$ and $k=1$ in the above definition, we obtain the class

$$\mathcal{R}^1_0(\psi)=\mathcal{R}(\psi)=\{f\in\mathcal{A}\colon f'(z)\prec\psi(z),\quad z\in\Delta\}.$$

Remark 2. If we set $\psi(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1, z\in\Delta)$ in (5) we get

$$\mathcal{R}^k_\vartheta\left(\frac{1+Az}{1+Bz}\right)=\mathcal{R}^k_\vartheta(A,B)=\left\{f\in\mathcal{A}\colon\left|\frac{\frac{\mathfrak{D}^k_\vartheta f(z)}{z}-1}{A-B\frac{\mathfrak{D}^k_\vartheta f(z)}{z}}\right|<1\right\}.$$

It may be noted that for $\alpha = 0,\ k = 1$ and $\vartheta = 0$, the class $\mathcal{R}_\vartheta^k(\alpha)$ reduces to the class $\mathcal{R}$ of analytic function whose derivative has positive real part in Δ studied in [20].

A lot of works have been done in the direction of finding upper bounds for $a_2,\ a_3$ and $|a_3-\mu a_2^2|$ for the function f in the certain subclasses of $\mathcal{A}$ for some real or complex parameters μ. This work was originated by M. Fekete and G. Szegő [7]. M. Fekete and G. Szegő gave a sharp estimate of non-linear functional $|a_3 - \mu a_2^2|$ for real parameters μ for subclasses of $\mathcal{A}$. This is known as Fekete-Szegő inequality. Several researchers solved the Fekete-Szegő problem for various subclasses of the class of $\mathcal{S}$ (see [6, 13, 14, 27–29, 33]).

In 1976, J. W. Noonan and D. K. Thomas [26] defined q^{th} Hankel determinant of function f for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1}\cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2}\cdots & a_{n+q} \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q}\cdots & a_{n+2q-2} \end{vmatrix},$$

where $a_1 = 1$. For brief history of Hankel determinant (see [30]). A good amount of literature exist for finding upper bound on $|H_2(2)|$ for various subclasses of $\mathcal{S}$. A. Janteng, S. A. Halim, M. Darus [10,11] derived the exact bounds for $|H_2(2)|$ for the class of starlike functions ($\mathcal{S}^*$), the class of convex functions ($\mathcal{C}$) and the class of functions whose derivatives have positive real parts ($\mathcal{R}$) in Δ . S. K. Lee, V. Ravichandran, S. Supramaniam [16] investigated $|H_2(2)|$ in the general class $\mathcal{S}^*(\psi)$ of starlike functions with respect to a given function ψ. D. V. Krishna, T. Ramreddy [15] generalized the result from [11] giving the sharp bound of $|H_2(2)|$ in the class of starlike and convex functions of α. P. Zaprawa [36] showed that if $f \in \mathrm{T}$, the class of typically real functions, then $|H_2(2)| \leq 9$. Apart from these, many research all over the globe obtained the upper bounds for various subclasses of univalent analytic functions and their results are available in literature (see [2, 3, 12, 22–24]).

In the present paper, following the technique used by R. J. Libera, E. J. Zlotkiewicz (see [17,18]), Fekete-Szegő inequality for the class $\mathcal{R}_\vartheta^k(\psi)$ is completely settled for real and complex parameters μ. Further, we obtain the upper bounds of $|H_2(2)|$ for the above mentioned class.

2. Preliminaries. We need the following lemmas in order to investigate the main results:

Lemma 1 ([5, 17–19])**.** *If $p \in \mathcal{P}$ is of form (2), then*

$$|c_n| \leq 2 \quad (n \geq 1), \tag{6}$$

$$|c_2 - \nu c_1^2| \leq 2\max\{1, |2\nu - 1|\} \quad (\nu \in \mathbb{C}), \tag{7}$$

$$c_2 = \frac{1}{2}[c_1^2 + (4 - c_1^2)x], \tag{8}$$

$$c_3 = \frac{1}{4}[c_1^3 + 2(4 - c_1^2)c_1 x - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - |x|^2)z], \tag{9}$$

for some complex numbers $x,\ z$ satisfying $|x| \leq 1$ and $|z| \leq 1$. The estimates in (6) and (7) are sharp for the functions $p(z) = \frac{1+z}{1-z}$ and $p(z) = \frac{1+z^2}{1-z^2}$ $(z \in \Delta)$.

Lemma 2 ([19])**.** *Let $p \in \mathcal{P}$ be of form (2). Then*

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \nu \leq 0, \\ 2, & 0 \leq \nu \leq 1, \\ 4\nu - 2, & \nu > 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(z) = \frac{1+z}{1-z}$ or one of its rotation. If $0 < \nu < 1$, then the equality holds if and only if $p(z) = \frac{1+z^2}{1-z^2}$ or one of its rotation. If $\nu = 0$, the equality holds if and only if $p(z) = \left(\frac{1}{2} + \frac{\eta}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\eta}{2}\right)\frac{1-z}{1+z}, \quad (0 \le \eta \le 1)$ or one of its rotation. If $\nu = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that the equality holds in the case of $\nu = 0$.

Although the above upper bound is sharp, when $0 < \nu < 1$, it can be improved as follows

$$|c_2 - \nu c_1^2| + \nu|c_1|^2 \le 2 \left(0 < \nu \le \frac{1}{2}\right), \quad |c_2 - \nu c_1^2| + (1-\nu)|c_1|^2 \le 2 \left(\frac{1}{2} < \nu \le 1\right).$$

3. Coefficient estimate results.

Theorem 1. *Let $\psi(z) = 1 + \mathrm{A}_1 z + \mathrm{A}_2 z^2 + \cdots$ with $\mathrm{A}_1 > 0$ and $\mathrm{A}_n \in \mathbb{R}$. If the function $f \in \mathcal{A}$ of form (1) belongs to the class $\mathcal{R}_\vartheta^k(\psi)$, then*

$$|a_2| \le \frac{\mathrm{A}_1}{2^k}, \tag{10}$$

$$|a_3| \le \frac{\mathrm{A}_1}{(3+\vartheta)^k} \max\left\{1, \left|\frac{\mathrm{A}_2}{\mathrm{A}_1}\right|\right\}, \tag{11}$$

$$|a_3 - \mu a_2^2| \le \frac{\mathrm{A}_1}{(3+\vartheta)^k} \max\left\{1, \left|\frac{\mathrm{A}_2}{\mathrm{A}_1} - \frac{(3+\vartheta)^k}{2^{2k}}\mu \mathrm{A}_1\right|\right\}, \ \mu \in \mathbb{C}. \tag{12}$$

Proof. Let the function $f \in \mathcal{A}$ be in the class $\mathcal{R}_\vartheta^k(\psi)$. Hence by Definition 1 there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$\frac{\mathfrak{D}_\vartheta^k f(z)}{z} = \psi(w(z)) \quad (z \in \Delta). \tag{13}$$

Define the function $p(z)$ given by

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{14}$$

Clearly $p \in \mathcal{P}$. From (14) it follows that

$$w(z) = \frac{p(z)+1}{p(z)-1} = \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_1 c_2}{2} + \frac{c_1^2}{8}\right)z^3 + \cdots. \tag{15}$$

Now

$$\psi[w(z)] = 1 + \frac{\mathrm{A}_1 c_1}{2}z + \frac{1}{2}\left[\mathrm{A}_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{\mathrm{A}_2 c_1^2}{2}\right]z^2 + $$
$$+ \frac{1}{2}\left[\mathrm{A}_1\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \mathrm{A}_2\left(c_1 c_2 - \frac{c_1^3}{2}\right) + \frac{\mathrm{A}_3 c_1^3}{4}\right]z^3 + \cdots. \tag{16}$$

From (2) we have

$$\frac{\mathfrak{D}_\vartheta^k f(z)}{z} = 1 + 2^k a_2 z + (3+\vartheta)^k a_3 z^2 + 4^k a_4 z^3 + \cdots. \tag{17}$$

Using (16) and (17) in (13) and equating the coefficients of z, z^2 and z^3 on both sides we get

$$a_2 = \frac{A_1 c_1}{2^{k+1}}, \quad a_3 = \frac{1}{2(3+\vartheta)^k}\left[A_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_2 c_1^2}{2}\right],$$
$$a_4 = \frac{1}{2^{2k+1}}\left[A_1\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + A_2\left(c_1c_2 - \frac{c_1^3}{2}\right) + \frac{A_3 c_1^3}{4}\right]. \tag{18}$$

By applying Lemma1, we get $|a_2| \leq \frac{A_1}{2^k}$,

$$|a_3| = \frac{1}{2(3+\vartheta)^k}\left|A_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_2c_1^2}{2}\right| \leq \frac{A_1}{(3+\vartheta)^k}\max\left\{1, \left|\frac{A_2}{A_1}\right|\right\}.$$

Therefore,

$$|a_3 - \mu a_2^2| = \frac{A_1}{2(3+\vartheta)^k}|c_2 - \nu c_1^2|, \tag{19}$$

where $\nu = \frac{1}{2}\left[1 - \frac{A_2}{A_1} + \frac{(3+\vartheta)^k}{2^{2k}}\mu A_1\right]$.

By application of (7) of Lemma 1 we obtain the desire estimate (12). □

Corollary 1. *If $f(z)$ given by (1) belongs to the class $\mathcal{R}_\vartheta^k(A, B)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{A - B}{(3+\vartheta)^k}\max\left\{1, \left|B + \frac{(3+\vartheta)^k}{2^{2k}}\mu(A - B)\right|\right\}.$$

Proof. For the function $\psi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ we have $\psi(z) = 1 + (A - B)z - B(A - B)z^2 + \cdots$. Taking $A_1 = (A - B)$ and $A_2 = -B(A - B)$ in Theorem 1 we get the desire result. □

Setting $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$ and $B = -1$ in Theorem 1 we get the result for the class $\mathcal{R}_\vartheta^k(\alpha)$ as follows:

Corollary 2. *Let $f(z) \in \mathcal{R}_\vartheta^k(\alpha)$. Then*

$$|a_3 - \mu a_2^2| \leq \frac{2(1-\alpha)}{(3+\vartheta)^k}\max\left\{1, \left|1 - \frac{(3+\vartheta)^k}{2^{2k-1}}\mu(1-\alpha)\right|\right\}.$$

Remark 3. Fixing $\vartheta = 0$ in Theorem 1, we get the upper bound for the function belonging to the subclass of $\mathcal{A}$ associated with the Sălăgean differential operator given by

$$|a_3 - \mu a_2^2| \leq \frac{A_1}{3^k}\max\left\{1, \left|\frac{A_2}{A_1} - \frac{3^k}{2^{2k}}\mu A_1\right|\right\}.$$

The bounds of $|H_2(1)|$ for $\mu = 1$ follows from the Theorem 1 as follows:

Corollary 3. *If the function f given by (1) is in the class $\mathcal{R}_\vartheta^k(\psi)$, then*

$$|H_2(1)| = |a_3 - a_2^2| \leq \frac{A_1}{(3+\vartheta)^k}\max\left\{1, \left|\frac{A_2}{A_1} - \frac{(3+\vartheta)^k}{2^{2k}}A_1\right|\right\}.$$

Theorem 2. *Let $\mu \in \mathbb{R}$. If the function f given by (1) belongs to the class $\mathcal{R}_\vartheta^k(\psi)$, then*

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{\mathrm{A}_1}{2(3+\vartheta)^k}\left(\frac{2\mathrm{A}_2}{\mathrm{A}_1}-\frac{(3+\vartheta)^k}{2^{2k-1}}\mu\mathrm{A}_1\right), & \mu\le\delta_1,\\ \frac{\mathrm{A}_1}{(3+\vartheta)^k}, & \delta_1\le\mu\le\delta_2,\\ \frac{\mathrm{A}_1}{2(3+\vartheta)^k}\left(\frac{(3+\vartheta)^k}{2^{2k-1}}\mu\mathrm{A}_1-\frac{2\mathrm{A}_2}{\mathrm{A}_1}\right), & \mu\ge\delta_2.\end{cases}$$

Furthermore, for $\delta_1<\mu\le\delta_1+k$,

$$|a_3-\mu a_2^2|+(\mu-\delta_1)|a_2^2|\le\frac{\mathrm{A}_1}{(3+\vartheta)^k},$$

and for $\delta_1+k<\mu<\delta_1+2k$,

$$|a_3-\mu a_2^2|+(\delta_1+2k-\mu)|a_2|^2\le\frac{\mathrm{A}_1}{(3+\vartheta)^k},$$

where $\delta_1=\frac{2^{2k}}{(3+\vartheta)^k}\left(\frac{\mathrm{A}_2-\mathrm{A}_1}{\mathrm{A}_1^2}\right)$, $\delta_2=\frac{2^{2k}}{(3+\vartheta)^k}\frac{\mathrm{A}_1+\mathrm{A}_2}{\mathrm{A}_1^2}$, $k=\frac{2^{2k}}{(3+\vartheta)^k\mathrm{A}_1}$. *The obtained inequalies is sharp.*

Proof. From (19), we have $|a_3-\mu a_2^2|=\frac{\mathrm{A}_1}{2(3+\vartheta)^k}|c_2-\nu c_1^2|$. The result follows by application of Lemma 1 in (19). This completes the proof of Theorem 2. To show that the bounds are sharp, we define the functions the functions F_η and G_η $(0\le\eta\le1)$, respectively, with $F_\eta(0)=0=F'_\eta(0)-1$ and $G_\eta(0)=0=G'_\eta(0)-1$ by

$$\frac{\mathfrak{D}_\vartheta^k F_\eta(z)}{z}=\psi\left(\frac{z(z+\eta)}{1+\eta z}\right)\quad\text{and}\quad\frac{\mathfrak{D}_\vartheta^k G_\eta(z)}{z}=\psi\left(-\frac{z(z+\eta)}{1+\eta z}\right),\text{ respectively.}$$

Clearly the functions $K_{\psi_n}=\psi(z^{n-1}), F_\eta, G_\eta\in\mathcal{R}_\vartheta^k(\psi)$. Also we write $K_\psi:=K_{\psi_2}$.

If $\mu<\delta_1$ or $\mu>\delta_2$, then the equality holds if and only if f is K_ψ or one of its rotations. When $\delta_1<\mu<\delta_2$, then the equality holds if and only if f is K_{ψ_3} or one of its rotations. If $\mu=\delta_1$ then the equality holds if and only if f is F_η or one of its rotations. If $\mu=\delta_2$ then the equality holds if and only if f is G_η or one of its rotations. □

In the following theorem, we obtain the upper bound of the second Hankel determinant $|H_2(2)|$ for $f\in\mathcal{R}_\vartheta^k(\psi)$.

Theorem 3. *Let* $k\in\mathbb{N}_0$, $\vartheta\in\mathbb{R}$. *Suppose that* $f\in\mathcal{R}_\vartheta^k(\psi)$.
(i) If A_1, A_2 *and* A_3 *satisfy the inequalities*

$$2((3+\vartheta)^k-2^{3k})|\mathrm{A}_2|+(3+\vartheta)^k-2^{k+1})\mathrm{A}_1\le0, |(3+\vartheta)^{2k}\mathrm{A}_1\mathrm{A}_3-2^{3k}\mathrm{A}_2^2|-2^{3k}\mathrm{A}_1^2\le0,$$

then $|a_2a_4-a_3^2|\le\frac{\mathrm{A}_1^2}{(3+\vartheta)^{2k}}$.
(ii) If A_1, A_2 *and* A_3 *satisfy the conditions*

$$2[(3+\vartheta)^{2k}-2^{3k}]|\mathrm{A}_2|+((3+\vartheta)^{2k}-2^{3k+1})\mathrm{A}_1\ge0,$$
$$2|(3+\vartheta)^{2k}\mathrm{A}_1\mathrm{A}_3-2^{3k}\mathrm{A}_2^2|-2[(3+\vartheta)^{2k}-2^{3k}]|\mathrm{A}_2|\mathrm{A}_1-(3+\vartheta)^{2k}\mathrm{A}_1^2\ge0,$$

or the conditions

$$2[(3+\vartheta)^{2k}-2^{3k}]|\mathrm{A}_2|+((3+\vartheta)^{2k}-2^{3k+1})\mathrm{A}_1\le0,$$
$$|(3+\vartheta)^{2k}\mathrm{A}_1\mathrm{A}_3-2^{3k}\mathrm{A}_2^2|-2^{3k}\mathrm{A}_1^2\ge0,$$

then $|a_2a_4-a_3^2|\le\frac{1}{2^{3k}(3+k)^{2k}}|(3+\vartheta)^{2k}\mathrm{A}_1\mathrm{A}_3-2^{3k}\mathrm{A}_2^2|$.
(iii) If A_1, A_2 *and* A_3 *satisfy the conditions*

$$2[(3+\vartheta)^{2k}-2^{3k}]|\mathrm{A}_2|+((3+\vartheta)^{2k}-2^{3k+1})\mathrm{A}_1>0,$$
$$2|(3+\vartheta)^{2k}\mathrm{A}_1\mathrm{A}_3-2^{3k}\mathrm{A}_2^2|-2[(3+\vartheta)^{2k}-2^{3k}]|\mathrm{A}_2|\mathrm{A}_1-(3+\vartheta)^{2k}\mathrm{A}_1^2\le0$$

then

$$|a_2a_4-a_3^2|\leq \frac{\mathrm{A}_1^2}{2^{3k+2}(3+\vartheta)^{2k}}\times\begin{cases} R, & Q\leq 0,\ P\leq -\frac{Q}{4},\\ 16P+4Q+R, & Q\geq 0,\ P\geq -\frac{Q}{8};\quad Q\leq 0,\quad P\geq -\frac{Q}{4},\\ \frac{4PR-Q^2}{4P} & Q>0,\ P\leq -\frac{Q}{8},\end{cases}$$

where

$$P=|(3+\vartheta)^{2k}\mathrm{A}_3-2^{3k}\frac{\mathrm{A}_2^2}{\mathrm{A}_1}|c^4-2|(3+\vartheta)^{2k}-2^{3k}||\mathrm{A}_2|-[(3+\vartheta)^{2k}-2^{3k}]\mathrm{A}_1,$$
$$Q=4\{2[(3+\vartheta)^{2k}-2^{3k}]|\mathrm{A}_2|+[(3+\vartheta)^{2k}-(2^{3k+1})\mathrm{A}_1]\},\ R=162^{3k}\mathrm{A}_1.$$

Proof. Substituting the values of $a_2,\ a_3$ and a_4 from (18) in $(a_2a_4-a_3^2)$ we have

$$a_2a_4-a_3^2=M[\alpha_1c_1^4+\alpha_2c_1^2c_2+\alpha_3c_1c_3+\alpha_4c_2^2],\tag{20}$$

where $M=\frac{\mathrm{A}_1}{2^{3k+4}(3+\vartheta)^{2k}}$, $\alpha_1=((3+\vartheta)^{2k}-2^{3k})(\mathrm{A}_1-2\mathrm{A}_2)+(3+\vartheta)^{2k}\mathrm{A}_3-\frac{\mathrm{A}_2^2}{\mathrm{A}_1}2^{3k}$, $\alpha_2=-4((3+\vartheta)^{2k}-2^{3k})(\mathrm{A}_1-\mathrm{A}_2), \alpha_3=4(3+\vartheta)^{2k}\mathrm{A}_1,\ \alpha_4=-2^{3k+2}\mathrm{A}_1$. Since the functions $p(z)$ and $p(e^{i\theta}z)$ $(\theta\in\mathbb{R})$ are in the class $\mathcal{P}$, without loss of generality, we can assume that $c_1=c\in[0,2]$. Substituting the values of c_2 and c_3 from (8) and (9) in (20), it follows that

$$|a_2a_4-a_3^2|=M\Big|(4\alpha_1+2\alpha_2+\alpha_3+\alpha_4)\frac{c^4}{4}+(\alpha_2+\alpha_3+\alpha_4)\frac{c^2x(4-c^2)}{2}+$$
$$+\left(\alpha_4(4-c^2)-\alpha_3c^2\right)\frac{x^2(4-c^2)}{4}+\frac{\alpha_3c}{2}(4-c^2)(1-|x|^2)z\Big|.\tag{21}$$

Now,

$$4\alpha_1+2\alpha_2+\alpha_3+\alpha_4=4\left[(3+\vartheta)^{2k}\mathrm{A}_3-2^{3k}\frac{\mathrm{A}_2^2}{\mathrm{A}_1}\right],\quad \alpha_2+\alpha_3+\alpha_4=4[(3+\vartheta)^{2k}-2^{3k}]\mathrm{A}_2,$$
$$\alpha_4(4-c^2)-\alpha_3c^2=-2^{3k+2}\mathrm{A}_1(4-c^2)-4(3+\vartheta)^{2k}\mathrm{A}_1c^2,$$
$$\frac{\alpha_3c}{2}(4-c^2)=2(3+\vartheta)^{2k}\mathrm{A}_1c(4-c^2).$$

Substituting these values in (21) and applying triangle inequality $|x|=\rho$ we obtain

$$|a_2a_4-a_3^2|\leq MH(c,\rho),\tag{22}$$

where

$$H(c,\rho)=\Big|(3+\vartheta)^{2k}\mathrm{A}_3-2^{3k}\frac{\mathrm{A}_2^2}{\mathrm{A}_1}\Big|c^4+2|(3+\vartheta)^{2k}-2^{3k}||\mathrm{A}_2|c^2\rho(4-c^2)+$$
$$+2(3+\vartheta)^{2k}\mathrm{A}_1(4-c_1^2)+\rho^2(4-c^2)\mathrm{A}_1(2-c)[2^{3k+1}-((3+\vartheta)^{2k}-2^{3k})c],$$
$$\frac{\partial H}{\partial\rho}=2|(3+\vartheta)^{2k}-2^{3k}||\mathrm{A}_2|c^2(4-c^2)+2\rho(4-c^2)\mathrm{A}_1(2-c)[2^{3k+1}-((3+\vartheta)^{2k}-2^{3k})]>0.$$

This shows the function $H(c,\rho)$ is an increasing function of ρ on the closed interval [0, 1]. Hence

$$\max_{0\leq\rho\leq 1}H(c,\rho)=H(c,1)=Pt^2+Qt+R,\tag{23}$$

where

$$P = \left|(3+\vartheta)^{2k}\mathrm{A}_3 - 2^{3k}\frac{\mathrm{A}_2^2}{\mathrm{A}_1}\right|c^4 - 2|(3+\vartheta)^{2k} - 2^{3k}||\mathrm{A}_2| - [(3+\vartheta)^{2k} - 2^{3k}]\mathrm{A}_1,$$
$$Q = 4\{2[(3+\vartheta)^{2k} - 2^{3k}]|\mathrm{A}_2| + [(3+\vartheta)^{2k} - (2^{3k+1})\mathrm{A}_1]\}, \quad R = 162^{3k}\mathrm{A}_1, \quad t = c^2.$$

Therefore, in view of (22) and (23) we have

$$|a_2a_4 - a_3^2| \leq \frac{\mathrm{A}_1}{2^{3k+4}(3+\vartheta)^{2k}} \max_{0\leq c\leq 2} H(c,1) = \frac{\mathrm{A}_1}{2^{3k+4}(3+\vartheta)^{2k}} \max_{0\leq t\leq 4}(Pt^2 + Qt + R). \tag{24}$$

Making use of standard result proven for f as defined in (3), that the optimal value of quadratic expression with standard computations as shown in Lee et al., [16](see pp-6, given by equation(16))we have:

$$\max_{0\leq t\leq 4}(Pt^2 + Qt + R) = \begin{cases} R, & Q \leq 0,\ P \leq -\frac{Q}{4}, \\ 16P + 4Q + R, & Q \geq 0,\ P \geq -\frac{Q}{8}; \quad Q \leq 0, \quad P \geq -\frac{Q}{4}, \\ \frac{4PR - Q^2}{4P} & Q > 0,\ P \leq -\frac{Q}{8}. \end{cases} \tag{25}$$

in (24) we obtain the desire estimates. □

Fixing $\vartheta = 0$ and $k = 1$ in Theorem 3, we obtain the upper bound of the second Hankel determinant for the class $\mathcal{R}(\psi)$ as follows:

Corollary 4. *Suppose that* $f \in \mathcal{R}(\psi)$. *(i) If* A_1, A_2 *and* A_3 *satisfy the conditions* $|\mathrm{A}_2| \leq \frac{7\mathrm{A}_1}{2}$ *and* $|9\mathrm{A}_1\mathrm{A}_3 - 8\mathrm{A}_2^2| - 8\mathrm{A}_1^2 \leq 0$, *then* $|a_2a_4 - a_3^2| \leq \frac{\mathrm{A}_1^2}{9}$. *(ii) If* A_1, A_2 *and* A_3 *satisfy the conditions* $|\mathrm{A}_2| \geq \frac{7\mathrm{A}_1}{2}$ *and* $2|9\mathrm{A}_1\mathrm{A}_3 - 8\mathrm{A}_2^2| - 2|\mathrm{A}_2|\mathrm{A}_1 - 9\mathrm{A}_1^2 \geq 0$ *or the conditions* $|\mathrm{A}_2| \leq \frac{7\mathrm{A}_1}{2}$ *and* $|9\mathrm{A}_1\mathrm{A}_3 - 8\mathrm{A}_2^2| - 8\mathrm{A}_1^2 \geq 0$, *then* $|a_2a_4 - a_3^2| \leq \frac{1}{72}|9\mathrm{A}_1\mathrm{A}_3 - 8\mathrm{A}_2^2|$. *(iii) If* A_1, A_2 *and* A_3 *satisfy the conditions* $|\mathrm{A}_2| \geq \frac{7\mathrm{A}_1}{2}$ *and* $2|9\mathrm{A}_1\mathrm{A}_3 - 8\mathrm{A}_2^2| - 2|\mathrm{A}_2|\mathrm{A}_1 - 9\mathrm{A}_1^2 \leq 0$, *then*

$$|a_2a_4 - a_3^2| \leq \frac{\mathrm{A}_1^2}{288}\frac{32|9\mathrm{A}_1\mathrm{A}_3 - 8\mathrm{A}_2^2| - 36|\mathrm{A}_2|\mathrm{A}_1 - 81\mathrm{A}_1^2 - 4\mathrm{A}_2^2}{|9\mathrm{A}_1\mathrm{A}_3 - 8\mathrm{A}_2^2| - 2|\mathrm{A}_2|\mathrm{A}_1 - \mathrm{A}_1^2}.$$

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