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# MULTIPLICATIVE (GENERALIZED)-DERIVATIONS OF PRIME RINGS THAT ACT AS $n$-(ANTI)HOMOMORPHISMS

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Let $R$ be a prime ring. In this note, we describe the possible forms of multiplicative (generalized)-derivations of $R$ that act as $n$-homomorphism or $n$-antihomomorphism on nonzero ideals of $R$. Consequently, from the given results one can easily deduce the results of Gusić ([7]).

**1. Introduction.** Throughout this paper, $R$ will always denote an associative prime ring with center $Z(R)$ and $C$ the extended centroid of $R$. It is well-known that in this case $C$ is a field. For any $x, y \in R$, the symbol $[x, y]$ denotes the commutator $xy - yx$. Recall, a ring is said to be prime if $xRy = (0)$ (where $x, y \in R$) implies $x = 0$ or $y = 0$. An additive mapping $d\colon R \to R$ is said to be a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In 1991, Brešar [4] introduced the notion of *generalized derivation* as follows: an additive mapping $F\colon R \to R$ is said to be a generalized derivation if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where $d$ is a derivation of $R$. The concept of generalized derivation covers both the notions of derivation and left multiplier (i.e., an additive mapping $T\colon R \to R$ satisfying $T(xy) = T(x)y$ for all $x, y \in R$). Now if we relax the assumption of additivity in the notion of derivation, then it is called *multiplicative derivation*, i.e., a mapping $\delta\colon R \to R$ (not necessarily additive) satisfying $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. Recently, Dhara and Ali [6] extended the notion of multiplicative derivation to multiplicative (generalized)-derivation. Accordingly, a mapping $F\colon R \to R$ (not necessarily additive) is said to be a *multiplicative* (*generalized*) *derivation* of $R$ if $F(xy) = F(x)y + x\delta(y)$ for all $x, y \in R$, where $\delta$ is a multiplicative derivation of $R$. Clearly, every generalized derivation is a multiplicative (generalized)-derivation, however the converse is not generally true (see [6], Example 1.1). Recall that a mapping $f$ of $R$ is said to act as an *homomorphism* (resp. *anti-homomorphism*) on an appropriate subset $K$ of $R$ if $f(xy) = f(x)f(y)$ (resp. $f(xy) = f(y)f(x)$) for all $x, y \in K$. Following Hezajian et al. [8], a mapping $f$ of $R$ is said to act as an *n-homomorphism* (resp. *n-antihomomorphism*) of $R$ if for any $x_i \in R$, where $i = 1, 2, \cdots, n$; $f(\prod_{i=1}^n x_i) = \prod_{i=1}^n f(x_i)$ (resp. $f(\prod_{i=1}^n x_i) = f(x_n)f(x_{n-1}) \cdots f(x_1)$). Initially, the notion of an $n$-homomorphism was introduced and studied for complex algebras by Hejazian et al. [8], where some significant properties of $n$-homomorphisms are discussed on Banach algebras. Moreover, it is not difficult to see that every homomorphism of $R$ is $n$-homomorphism (for $n > 2$), but the converse is not necessarily true (see [8]).

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Till date, there exist many results in the literature showing that the global structure of $R$ is often tightly connected to the behaviour of additive mappings defined on $R$. In 1989, a result due to Bell and Kappe [2] states that if a prime ring $R$ admits a derivation $d$ that acts as homomorphism or anti-homomorphism on a nonzero right ideal $U$ of $R$, then $d = 0$. Later Asma et al. [1] proved that this result also holds on nonzero square-closed Lie ideals of prime rings. Moreover, Rehman [11] established this result for generalized derivations of prime rings. In fact, he proved that if $F$ is a nonzero generalized derivation of a 2-torsion free prime ring $R$ that acts as homomorphism or anti-homomorphism on a nonzero ideal of $R$ and $d \neq 0$, then $R$ is commutative. Recently, Lukashenko [10] provided a new direction to these studies by investigating derivations acting as homomorphisms or anti-homomorphisms in differentially semiprime rings. Now it seems interesting to extend the results of generalized derivations to multiplicative (generalized)-derivations. In this context, Gusić [7] gave the complete form of Rehman's result as follows: *Let $R$ be an associative prime ring, $F$ be a multiplicative (generalized)-derivation of $R$ associated with a multiplicative derivation $\delta$ and $I$ be a nonzero ideal of $R$.*

*(a) Assume that $F$ acts as homomorphism on $I$. Then $\delta = 0$, and $F = 0$ or $F(x) = x$ for all $x \in R$.*

*(b) Assume that $F$ acts as anti-homomorphism on $I$. Then $\delta = 0$, and $F = 0$ or $F(x) = x$ for all $x \in R$ (in this case $R$ should be commutative).*

In view of our above discussion, we find it reasonable to extend the results of derivations acting as homomorphisms (resp. anti-homomorphisms) to $n$-homomorphisms (resp. $n$-antihomomorphisms) with multiplicative derivations. More specifically, we study multiplicative (generalized)-derivations of prime rings that act as $n$-homomorphism or $n$-antihomomorphism.

**2. The results.** We begin with the following observations in this subject, which we shall use frequently.

**Lemma 1.** *Let $R$ be a prime ring and $I$ be a nonzero ideal of $R$. Then for any $a, b \in R$, $aIb = (0)$ implies $a = 0$ or $b = 0$.*

**Lemma 2.** *Let $R$ be a prime ring and $I$ be a nonzero ideal of $R$. If for any fixed positive integer $n$, $[x^n, y^n] \in Z(R)$ for all $x, y \in I$, then $R$ is commutative.*

*Proof.* By hypothesis, we have $[[x^n, y^n], r] = 0$ for all $x, y \in I$ and $r \in R$. It is well-known that $I$ and $R$ satisfy same polynomial identities. Thus, we have $[[x^n, y^n], r] = 0$ for all $x, y, r \in R$. If possible suppose that $R$ is not commutative. By a famous result of Lanski [9], $R \subseteq M_n(F)$, where $M_n(F)$ be a ring of $n \times n$ matrices, with $n \geq 2$ over a field $F$. Moreover, $R$ and $M_n(F)$ satisfy the same polynomial identities. Choose $x = e_{11}, y = e_{12} + e_{22}$ and $r = e_{21}$, where $e_{ij}$ denotes matrix with 1 at $ij-$entry and 0 elsewhere. In this view, it follows that

$$0 = [[x^n, y^n], r] = e_{11},$$

a contradiction. Hence, $R$ is commutative. □

**Lemma 3.** *Let $R$ be a ring and $\delta$ be a multiplicative derivation of $R$. Then the followings are true:*

*(i) $\delta(0) = 0$.*

*(ii) If $a \in Z(R)$, then $\delta(a) \in Z(R)$.*

*Proof.* (*i*) $\delta(0) = \delta(0.0) = \delta(0).0 + 0.\delta(0) = 0$. (*ii*) Let $a \in Z(R)$ and $\delta$ be a multiplicative derivation of $R$. Then for each $x \in R$, we have

$$\delta(ax) = \delta(a)x + a\delta(x), \quad \delta(ax) = \delta(xa) = \delta(x)a + x\delta(a).$$

Together with above two equations, we get

$$[x, \delta(a)] = 0 \text{ for all } x \in R.$$

Hence $\delta(a) \in R$. □

**Theorem 1.** *Let $R$ be a prime ring, $I$ a nonzero ideal of $R$. Suppose that $F\colon R \to R$ is a multiplicative (generalized)-derivation associated with a multiplicative derivation $\delta$ of $R$ such that $F$ acts as $n$-homomorphism on $I$. Then $\delta = 0$, and $F = 0$ or there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$ and $\lambda^{n-1} = 1$.*

*Proof.* By hypothesis, we have

$$F\left(\prod_{i=1}^{n} x_i\right) = \prod_{i=1}^{n} F(x_i) \tag{1}$$

for all $x_i \in I$. On the other hand, we find

$$F\left(\prod_{i=1}^{n} x_i\right) = F\left(\prod_{i=1}^{n-1} x_i\right)x_n + \prod_{i=1}^{n-1} x_i\delta(x_n) \tag{2}$$

for all $x_i \in I$. Combining (1) and (2), we obtain

$$\prod_{i=1}^{n} F(x_i) = F\left(\prod_{i=1}^{n-1} x_i\right)x_n + \prod_{i=1}^{n-1} x_i\delta(x_n) \tag{3}$$

for all $x_i \in I$. Replace $x_n$ by $x_n r$ in (3), where $r \in R$, we get

$$\prod_{i=1}^{n-1} F(x_i)x_n\delta(r) = \prod_{i=1}^{n} x_i\delta(r).$$

That is

$$\left(\prod_{i=1}^{n-1} F(x_i) - \prod_{i=1}^{n-1} x_i\right)x_n\delta(r) = 0.$$

In view of Lemma 1, we find that either $\prod_{i=1}^{n-1} F(x_i) = \prod_{i=1}^{n-1} x_i$ or $\delta = 0$. Let us consider

$$\prod_{i=1}^{n-1} F(x_i) = \prod_{i=1}^{n-1} x_i \tag{4}$$

for all $x_i \in I$. Replace $x_{n-1}$ by $x_{n-1}r$ in (4), we find

$$\prod_{i=1}^{n-1} F(x_i)r + \prod_{i=1}^{n-2} F(x_i)x_{n-1}\delta(r) = \prod_{i=1}^{n-1} x_i r \tag{5}$$

for all $x_i \in I$ and $r \in R$. Right multiply (4) by $r$ and subtract from (5), we get

$$\prod_{i=1}^{n-2} F(x_i)x_{n-1}\delta(r) = 0$$

for all $x_i \in I$ and $r \in R$. Again by invoking Lemma 1, we find that either $\prod_{i=1}^{n-2} F(x_i) = 0$ or $\delta = 0$. But $\delta \neq 0$, so we have $\prod_{i=1}^{n-2} F(x_i) = 0$ for all $x_i \in I$. Substitute $x_{n-2}r$ in place of $x_{n-2}$ in above expression, where $r \in R$, we find that $\prod_{i=1}^{n-3} F(x_i)I\delta(r) = (0)$. By Lemma 1, it follows that either $\prod_{i=1}^{n-3} F(x_i) = 0$ for all $x_i \in I$ or $\delta = 0$. But $\delta \neq 0$, thus we have $\prod_{i=1}^{n-3} F(x_i) = 0$ for all $x_i \in I$. Continuing in this way, we arrive at $F(x) = 0$ for all $x \in I$. Replace $x$ by $xr$, where $r \in R$, we get $x\delta(r) = 0$ for all $x \in I$ and $r \in R$. It implies that $\delta = 0$, which is a contradiction.

Let us now consider the latter case $\delta = 0$, we find that

$$F\left(\prod_{i=1}^{n} x_i\right) = F(x_i)\prod_{i=2}^{n} x_i \tag{6}$$

for all $x_i \in I$. Combining (1) and (6), we obtain

$$F(x_1)\left(\prod_{i=2}^{n} F(x_i) - \prod_{i=2}^{n} x_i\right) = 0$$

for all $x_i \in I$. Replace $x_1$ by $x_1 r$, where $r \in R$, we may infer that

$$F(x_1)R\left(\prod_{i=2}^{n} F(x_i) - \prod_{i=2}^{n} x_i\right) = (0)$$

for all $x_i \in I$. Since $R$ is prime, we find that either $F(x) = 0$ for all $x \in I$ or $\prod_{i=2}^{n} F(x_i) = \prod_{i=2}^{n} x_i$ for all $x_i \in I$. It is straightforward to see that the former case implies $F = 0$. On the other side, we have

$$\prod_{i=2}^{n} F(x_i) = \prod_{i=2}^{n} x_i \tag{7}$$

for all $x_i \in I$. Take $rx_2$ instead of $x_2$ in (7), where $r \in R$, we get

$$F(r)x_2\prod_{i=3}^{n} F(x_i) = rx_2\prod_{i=3}^{n} x_i. \tag{8}$$

Left multiply (7) by $r$ and then subtract from (8), we obtain

$$(F(r)x_2 - rF(x_2))\prod_{i=3}^{n} F(x_i) = 0$$

for all $x_i \in I$ and $r \in R$. Substitute $x_2 s$ in place of $x_2$ in above equation, where $s \in R$, we obtain

$$(F(r)x_2 - rF(x_2))R\prod_{i=3}^{n} F(x_i) = (0)$$

for all $x_i \in I$ and $r \in R$. It implies that either $F(r)x - rF(x) = 0$ for all $x \in I$ and $r \in R$ or $\prod_{i=3}^{n} F(x_i) = 0$ for all $x_i \in I$. One may observe that in both of these cases we get the situation $F(r)x - rF(x) = 0$ for all $x \in I$ and $r \in R$. Replace $x$ by $sx$, we get $(F(r)s - rF(s))x = 0$ for all $x \in I$ and $r, s \in R$. By Lemma 1, we get $F(r)s = rF(s)$ for all $r, s \in R$. Replace $r$ by $rp$, we get $F(r)p1_R(s) = 1_R(r)pF(s)$ for all $r, s, p \in R$, where $1_R$ is the identity mapping of $R$. With the aid of a result of Brešar [[3], Lemma], it follows that there exists some $\lambda \in C$ such that $F = \lambda 1_R$ and hence $F(x) = \lambda x$ for all $x \in R$. In view of our hypothesis, we have $\lambda \prod_{i=1}^{n} x_i = \prod_{i=1}^{n} \lambda x_i$. It forces that $\lambda^{n-1} = 1$. It completes the proof. □

**Corollary 1** ([7], Theorem 1$(a)$). *Let $R$ be an associative prime ring, $I$ a nonzero ideal of $R$. Suppose that $F\colon R \to R$ is a multiplicative (generalized)-derivation associated with a multiplicative derivation $\delta$ of $R$ such that $F$ acts a homomorphism on $I$. Then $\delta = 0$, and $F = 0$ or $F(x) = x$ for all $x \in R$.*

In spirit of a result of Gusić ([7], Theorem 1$(b)$), it is natural to investigate multiplicative (generalized)-derivations that act as $n$-antihomomorphisms. However, we could not get this result in its complete form, but we obtain the following:

**Theorem 2.** *Let $R$ be a prime ring, $I$ a nonzero ideal of $R$. Suppose that $F\colon R \to R$ is a multiplicative (generalized)-derivation associated with a multiplicative derivation $\delta$ of $R$ such that $F$ acts as $n$-antihomomorphism on $I$. If $F = \delta$, then $\delta(x)^{n-1} \in Z(R)$ for all $x \in I$. Moreover, if $\delta$ is additive, then either $\delta = 0$ or $R$ is commutative or $R$ is an order in a $4-$dimensional simple algebra.*

*Proof.* By hypothesis, we have

$$F\left(\prod_{i=1}^{n}\right) = F(x_n)F(x_{n-1})\cdots F(x_2)F(x_1) \tag{9}$$

for all $x_i \in I$. On the other hand, we may infer that

$$F\left(\prod_{i=1}^{n}\right) = F(x_1)\prod_{i=2}^{n} x_i + \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1} x_j\delta(x_i)\prod_{k=i+1}^{n} x_k\right) \tag{10}$$

for all $x_i \in I$. Combining (9) and (10), we find that

$$F(x_n)\cdots F(x_1) = F(x_1)\prod_{i=2}^{n} x_i + \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1} x_j\delta(x_i)\prod_{k=i+1}^{n} x_k\right) \tag{11}$$

for all $x_i \in I$. Replace $x_1$ by $x_1x_n$ in (11), we obtain

$$\begin{aligned} F(x_n)\cdots F(x_2)F(x_1)x_n + F(x_n)\cdots F(x_2)x_1\delta(x_n) = F(x_1)x_n\prod_{i=2}^{n} x_i + \\ + x_1\delta(x_n)\prod_{i=2}^{n} x_i + x_1x_n\delta(x_2)\prod_{i=3}^{n} x_i + x_1x_n\sum_{i=3}^{n}\left(\prod_{j=2}^{i-1} x_j\delta(x_i)\prod_{k=i+1}^{n} x_k\right) \end{aligned} \tag{12}$$

for all $x_i \in I$. Using (9) in (12), we get

$$F\left(\prod_{i=1}^{n} x_i\right)x_n + F(x_n)\cdots F(x_2)x_1\delta(x_n) = F(x_1)x_n\prod_{i=2}^{n} x_i + x_1\delta(x_n)\prod_{i=2}^{n} x_i = \\ +x_1x_n\delta(x_2)\prod_{i=3}^{n} x_i + x_1x_n\sum_{i=3}^{n}\left(\prod_{j=2}^{i-1} x_j\delta(x_i)\prod_{k=i+1}^{n} x_k\right)$$

for all $x_i \in I$. It implies that

$$\left(F(x_1)\prod_{i=2}^{n} x_i + \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1} x_j\delta(x_i)\prod_{k=i+1}^{n} x_k\right)\right)x_n + F(x_n)\cdots F(x_2)x_1\delta(x_n) = \\ = F(x_1)x_n\prod_{i=2}^{n} x_i + x_1\delta(x_n)\prod_{i=2}^{n} x_i + x_1x_n\delta(x_2)\prod_{i=3}^{n} x_i + x_1x_n\sum_{i=3}^{n}\left(\prod_{j=2}^{i-1} x_j\delta(x_i)\prod_{k=i+1}^{n} x_k\right)$$

for all $x_i \in I$. In particular, for $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = y$, we find

$$F(x)y^n + x\left(\sum_{i=0}^{n-2} y^i\delta(y)y^{n-1-i}\right) + F(y)^{n-1}x\delta(y) = F(x)y^n + x\delta(y)y^{n-1} + \\ +xy\delta(y)y^{n-2} + x\left(\sum_{i=2}^{n-1} y^i\delta(y)y^{n-1-i}\right)$$

for all $x, y \in I$. It yields that

$$F(y)^{n-1}x\delta(y) = xy^{n-1}\delta(y) \tag{13}$$

for all $x, y \in I$. Replace $x$ by $rx$, where $r \in R$ in (13), we get

$$F(y)^{n-1}rx\delta(y) = rxy^{n-1}\delta(y). \tag{14}$$

Left multiply (13) by $r$ and combine with (14), we obtain $[F(y)^{n-1}, r]x\delta(y) = 0$ for all $x, y \in I$ and $r \in R$.

In particular, we take $F = \delta$. Thus we have $[\delta(y)^{n-1}, r]x\delta(y) = 0$ for all $x, y \in I$ and $r \in R$. Since $R$ is a prime ring, it follows that for each $y \in I$, either $[\delta(y)^{n-1}, r] = 0$ for all $r \in R$ or $\delta(y) = 0$. In each case we have $[\delta(y)^{n-1}, r] = 0$ for all $y \in I$ and $r \in R$, i.e., $\delta(y)^{n-1} \in Z(R)$ for all $y \in I$. If $\delta$ is additive, we are done by ([5], Theorem B). □

**Corollary 2** ([7], Theorem 1($b$)). *Let $R$ be an associative prime ring, $I$ a nonzero ideal of $R$. Suppose that $F\colon R \to R$ is a multiplicative (generalized)-derivation associated with a multiplicative derivation $\delta$ of $R$ such that $F$ acts a homomorphism on $I$. Then $\delta = 0$, and $F = 0$ or $F(x) = x$ for all $x \in R$.*

*Proof.* For $n = 2$, in view of equation (13) and (14), we have $[F(y), t]x\delta(y) = 0$ for all $x, y, t \in I$. This same expression appeared in the beginning of the proof of Theorem 1(b) in [7], hence the conclusion follows in the same way. □

**Definition 1.** Let $F\colon R \to R$ be a function. Then $F$ is called *right multiplicative (generalized) derivation* of $R$ if it satisfies

$$F(xy) = F(x)y + x\delta(y)$$

for all $x, y \in R$ and $\delta$ is any mapping of $R$. And $F$ is called *left multiplicative (generalized)* derivation of $R$ if it satisfies

$$F(xy) = \delta(x)y + xF(y)$$

for all $x, y \in R$ and $\delta$ is any mapping of $R$. Then it is not difficult to see that the associated mapping $\delta$ of right and left multiplicative (generalized)-derivation $F$ is a multiplicative derivation. Now, $F$ is said to be *two-sided multiplicative (generalized) derivation* of $R$ if it satisfies

$$F(xy) = F(x)y + x\delta(y) = \delta(x)y + xF(y)$$

for all $x, y \in R$, where $\delta$ is a multiplicative derivation of $R$.

**Theorem 3.** *Let $R$ be a prime ring, $I$ a nonzero ideal of $R$. Suppose that $F\colon R \to R$ is a two-sided multiplicative (generalized)-derivation associated with a multiplicative derivation $\delta$ of $R$ such that $F$ acts as $n$-antihomomorphism on $I$. Then $\delta = 0$, and $F = 0$ or there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$ and $\lambda^{n-1} = 1$ (in this case $R$ should be commutative).*

*Proof.* From equation (13), we have $F(y)^{n-1}x\delta(y) = xy^{n-1}\delta(y)$ for all $x, y \in I$. Take $F(z)x$ in place of $x$ in this equation, we get

$$F(y)^{n-1}F(z)x\delta(y) = F(z)xy^{n-1}\delta(y), \quad F(zy^{n-1})x\delta(y) = F(z)xy^{n-1}\delta(y),$$
$$F(z)y^{n-1}x\delta(y) + z\delta(y^{n-1})x\delta(y) = F(z)xy^{n-1}\delta(y)$$

for all $x, y, z \in I$. It implies that

$$F(z)[y^{n-1}, x]\delta(y) + z\delta(y^{n-1})x\delta(y) = 0 \tag{15}$$

for all $x, y, z \in I$. Replace $z$ by $rz$ in (15), where $r \in R$, we get

$$\delta(r)z[y^{n-1}, x]\delta(y) + rF(z)[y^{n-1}, x]\delta(y) + rz\delta(y^{n-1})x\delta(y) = 0.$$

Using (15), we find $\delta(r)z[y^{n-1}, x]\delta(y) = 0$ for all $x, y, z \in I$ and $r \in R$. In view of Lemma 1, it implies that either $\delta = 0$ or $[y^{n-1}, x]\delta(y) = 0$ for all $x, y \in I$. Assume that $[y^{n-1}, x]\delta(y) = 0$ for all $x, y \in I$. It implies that for each $y \in I$, either $y^{n-1} \in Z(R)$ or $\delta(y) = 0$. Together these both cases (using Lemma 3) imply that $\delta(y^{n-1}) \in Z(R)$ for all $y \in I$.

We now consider

$$F(xy^{n-1}) = F(x)y^{n-1} + x\delta(y^{n-1}), \quad F(xy^{n-1}) = F(y)^{n-1}F(x)$$

for all $x, y \in I$. Thus we have

$$F(y)^{n-1}F(x) = F(x)y^{n-1} + x\delta(y^{n-1}) = F(x)y^{n-1} + \delta(y^{n-1})x. \tag{16}$$

Take $xz$ in place of $x$ in (16), we find

$$F(y)^{n-1}F(x)z + F(y)^{n-1}x\delta(z) = F(x)zy^{n-1} + x\delta(z)y^{n-1} + \delta(y^{n-1})xz \tag{17}$$

for all $x, y, z \in I$. Using (16), it implies that

$$F(y)^{n-1}x\delta(z) = F(x)[z, y^{n-1}] + x\delta(z)y^{n-1} \tag{18}$$

for all $x, y, z \in I$. Replace $x$ by $rx$ in (18), where $r \in R$, we get

$$F(y)^{n-1}rx\delta(z) = rF(x)[z, y^{n-1}] + \delta(r)x[z, y^{n-1}] + rx\delta(z)y^{n-1}.$$

Using (18), we have

$$[F(y)^{n-1}, r]x\delta(z) = \delta(r)x[z, y^{n-1}] \tag{19}$$

for all $x, y, z \in I$ and $r \in R$. Replace $z$ by $zw^{n-1}$ in (19), we get

$$[F(y)^{n-1}, r]x\delta(z)w^{n-1} + [F(y)^{n-1}, r]xz\delta(w^{n-1}) = \delta(r)x[z, y^{n-1}]w^{n-1} + \delta(r)xz[y^{n-1}, w^{n-1}]$$

for all $x, y, z, w \in I$ and $r \in R$. Equation (19) reduces it to

$$\delta(w^{n-1})[F(y)^{n-1}, r]xz = \delta(r)xz[y^{n-1}, w^{n-1}] \tag{20}$$

for all $x, y, z, w \in I$ and $r \in R$. Take $zs$ in place of $z$ in (20), where $s \in R$, we find

$$\delta(w^{n-1})[F(y)^{n-1}, r]xzs = \delta(r)xzs[y^{n-1}, w^{n-1}]$$

for all $x, y, z, w \in I$ and $r, s \in R$. Using (20) in the above expression, we obtain $\delta(r)xz[[w^{n-1}, y^{n-1}], s] = 0$ for all $x, y, z, w \in I$ and $r, s \in R$. It forces that either $\delta = 0$ or $[w^{n-1}, y^{n-1}] \in Z(R)$ for all $y, w \in I$. But $\delta \neq 0$, thus we have $[w^{n-1}, y^{n-1}] \in Z(R)$ for all $y, w \in I$. In view of Lemma 2, $R$ is commutative. Therefore, $F$ is just an $n$-homomorphism of $R$ and hence by Theorem 1, we get $\delta = 0$, a contradiction.

On the other hand, we assume that $\delta = 0$. Relation (10) implies that

$$F(x_1x_2\cdots x_n) = F(x_1)x_2\cdots x_n$$

for all $x_i \in I$. Using this relation, we obtain

$$\begin{gathered} F(x_1)x_2x_3\cdots x_{n-1}x_nx_{n+1} = F(x_1x_2\cdots x_{n-1}x_n)x_{n+1} = \\ = F(x_n)F(x_{n-1})\cdots F(x_2)F(x_1)x_{n+1} = F(x_n)F(x_{n-1})\cdots F(x_2)F(x_1x_{n+1}) = \\ = F(x_1x_{n+1}x_2\cdots x_n) = F(x_1)x_{n+1}x_2\cdots x_n \end{gathered}$$

for all $x_i \in I$. It gives

$$F(x_1)[x_2\cdots x_n, x_{n+1}] = 0$$

for all $x_i \in I$. Thus we have either $F(x) = 0$ for all $x \in I$ or $[x_2\cdots x_n, x_{n+1}] = 0$ for all $x_i \in I$. The first case implies $F = 0$. In the latter case we find that $R$ is commutative and hence $F$ acts as $n$-homomorphism on $I$. □

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