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I. Protasov, K. Protasova

CLOSENESS AND LINKNESS IN BALLEANS

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A set X endowed with a coarse structure is called ballean or coarse space. For a ballean $(X,\mathcal{E})$, we say that two subsets A, B of X are close (linked) if there exists an entourage $E\in\mathcal{E}$ such that $A\subseteq E[B]$, $B\subseteq E[A]$ (either A,B are bounded or contain unbounded close subsets). We explore the following general question: which information about a ballean is contained and can be extracted from the relations of closeness and linkness.

1. Introduction. Given a set X, a family $\mathcal{E}$ of subsets of $X\times X$ is called a *coarse structure* on X if

- each $E\in\mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X=\{(x,x)\in X\colon x\in X\}$;
- if E, $E'\in\mathcal{E}$ then $E\circ E'\in\mathcal{E}$ and $E^{-1}\in\mathcal{E}$, where $E\circ E'=\{(x,y)\colon \exists z((x,z)\in E, (z,y)\in E')\}$, $E^{-1}=\{(y,x)\colon (x,y)\in E\}$;
- if $E\in\mathcal{E}$ and $\triangle_X\subseteq E'\subseteq E$ then $E'\in\mathcal{E}$;
- $\bigcup\mathcal{E}=X\times X$.

A subfamily $\mathcal{E}'\subseteq\mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $E\in\mathcal{E}$, there exists $E'\in\mathcal{E}'$ such that $E\subseteq E'$. For $x\in X$, $A\subseteq X$ and $E\in\mathcal{E}$, we denote $E[x]=\{y\in X\colon (x,y)\in E\}$, $E[A]=\bigcup_{a\in A}E[a]$ and say that $E[x]$ and $E[A]$ are *balls of radius E around* x and A.

The pair $(X,\mathcal{E})$ is called a *coarse space* [14] or a ballean [11], [13].

For a ballean $(X,\mathcal{E})$, a subset $B\subseteq X$ is called *bounded* if $B\subseteq E[x]$ for some $E\in\mathcal{E}$ and $x\in X$. A ballean $(X,\mathcal{E})$ is called *bounded (unbounded)* if X is bounded (unbounded). The family $\mathcal{B}_{(X,\mathcal{E})}$ of all bounded subsets of $(X,\mathcal{E})$ is called the *bornology* of $(X,\mathcal{E})$. We denote by $\mathcal{B}^{\sharp}_{(X,\mathcal{E})}$ the set of all ultrafilters φ on X such that $X\setminus B\in\varphi$ for each $B\in\mathcal{B}_{(X,\mathcal{E})}$.

Definition 1. Given a ballean $(X,\mathcal{E})$, we say that subsets A, B of X are

- *close* (write $A\delta B$) if there exists $E\in\mathcal{E}$ such that $A\subseteq E[B]$, $B\subseteq E[A]$;
- *linked* (write $A\lambda B$) if either A,B are bounded or there exist unbounded subsets $A'\subseteq A$, $B'\subseteq B$ such that $A'\delta B'$.

The relations δ and λ, called *closeness* and *linkness*, are special kinds of asymptotic proximities [9]. We note that δ and λ play the central part in ultrafilter description of the Higson's corona $\nu(X,\mathcal{E})$ of $(X,\mathcal{E})$, see [8] and Section 6. The negation of λ (namely,

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asymptotic disjointness) is used for definition of normal balleans, see [7] and Section 4. The relation δ is an equivalence and each δ-class is a connected component in the hyperballean of $(X, \mathcal{E})$, see [5], [12].

In above and other cases, δ and λ were used as a tool for studing balleans, but δ and λ are interesting for their own sake. In this paper, we concentrate around the following general questions:

Which properties of a ballean $(X, \mathcal{E})$ can be recognized by $\delta_{(X,\mathcal{E})}$ or $\lambda_{(X,\mathcal{E})}$?
How can one detect if a ballean $(X, \mathcal{E})$ is uniquely determined by $\lambda_{(X,\mathcal{E})}$ or $\delta_{(X,\mathcal{E})}$?

To make these questions more precise, we need some definitions.

Definition 2. We say that two balleans $(X, \mathcal{E})$, $(X, \mathcal{E}')$ are δ-*equivalent* (λ-*equivalent*) if $\delta_{(X,\mathcal{E})} = \delta_{(X,\mathcal{E}')}$ ($\lambda_{(X,\mathcal{E})} = \lambda_{(X,\mathcal{E}')}$).

Every δ-equivalent balleans are λ-equivalent, but not vice versa, see Section 2.

If $(X, \mathcal{E})$, $(X, \mathcal{E}')$, are λ-equivalent then $\mathcal{B}_{(X,\mathcal{E})} = \mathcal{B}_{(X,\mathcal{E}')}$.

Definition 3. A ballean $(X, \mathcal{E})$ is called δ-*rigid* (λ-*rigid*) if, for every coarse structure $\mathcal{E}'$ on X such that $\delta_{(X,\mathcal{E})} = \delta_{(X,\mathcal{E}')}$ ($\lambda_{(X,\mathcal{E})} = \lambda_{(X,\mathcal{E}')}$), we have $\mathcal{E} = \mathcal{E}'$.

Thus, every δ-rigid (λ-rigid) ballean $(X, \mathcal{E})$ is uniquely determined by the relation $\delta_{(X,\mathcal{E})}$ ($\lambda_{(X,\mathcal{E})}$). Every λ-rigid ballean is δ-rigid.

For balleans $(X, \mathcal{E})$, uniquely determined by $\mathcal{B}_{(\mathcal{X},\mathcal{E})}$ see [4].

Definition 4. We say that a class $\mathcal{K}$ of balleans is

- δ-*stable* (λ-*stable*), for any two δ-equivalent (λ-equivalent) balleans $(X, \mathcal{E})$, $(X, \mathcal{E}')$, if $(X, \mathcal{E}) \in \mathcal{K}$ then $(X, \mathcal{E}') \in \mathcal{K}$;
- δ-*rigid* (λ-*rigid*), for any two δ-equivalent (λ-equivalent) balleans $(X, \mathcal{E})$, $(X, \mathcal{E}') \in \mathcal{K}$, we have $\mathcal{E} = \mathcal{E}'$.

By [9, Theorem 3.4], for every ballean $(X, \mathcal{E})$, there exists the strongest by inclusion coarse structure $\mathcal{E}_\delta$ on X such that $(X, \mathcal{E})$, $(X, \mathcal{E}_\delta)$ are δ-equivalent. Analogously, there exists the strongest by inclusion coarse structure $\mathcal{E}_\lambda$ on X such that $(X, \mathcal{E})$, $(X, \mathcal{E}_\lambda)$ are λ-equivalent.

Definition 5. We say that a ballean $(X, \mathcal{E})$ is δ-*strong* (λ-*strong*) if $\mathcal{E} = \mathcal{E}_\delta$ ($\mathcal{E} = \mathcal{E}_\lambda$).

In Section 2 we prove that every discrete ballean is λ-rigid, every coarsely discrete ballean is δ-rigid but needs not to be λ–rigid.

In Section 3 we show that the class of metrizable balleans is λ-rigid but not λ-stable, the class of submetrizable balleans is λ-stable. Every metrizable ballean $(X, \mathcal{E})$ is λ-strong (so δ-strong) but $(X, \mathcal{E})$ is λ-rigid if and only if $(X, \mathcal{E})$ is discrete. Is every metrizable ballean δ-rigid? This question remains open.

In Section 4 we prove that the class of normal balleans is λ-stable, but not δ-rigid.

In Section 5 we prove that the class of finitary group balleans is λ-rigid.

In Section 6 we show that the parallelity relation on $\mathcal{B}^\sharp_{(X,\mathcal{E})}$ uniquely defines $\delta_{(X,\mathcal{E})}$ and apply this statement to balleans $(X, \mathcal{E})$ with finite $\mathcal{B}^\sharp_{(X,\mathcal{E})}$. We conclude the paper with an ultrafilter characterization of the normality.

2. Discrete balleans. We recall that a ballean $(X, \mathcal{E})$ is *discrete* if, for every $E \in \mathcal{E}$, there exists a bounded subset B such that $E[x] = \{x\}$ for each $x \in X \backslash B$. A subset $Y \subseteq X$ is called *discrete* if the subballean $(Y, \mathcal{E}|_Y)$ is discrete.

Theorem 1. *For a ballean $(X,\mathcal{E})$, the following statements are equivalent:*

1. *$(X,\mathcal{E})$ is discrete;*
2. *if $Y\subseteq X$, $Z\subseteq X$ and $Y\delta Z$ then $Y\setminus Z$ and $Z\setminus Y$ are bounded;*
3. *if $Y\subseteq X$, $Z\subseteq X$ and $Y\lambda Z$ then either Y,Z are bounded or $Y\cap Z$ is unbounded.*

Proof. The implications $(1)\Longrightarrow(2)\Longrightarrow(3)$ are evident.

We assume that (3) holds but $(X,\mathcal{E})$ is not discrete. Then there exists $E\in\mathcal{E}$ such that the set $\{x\in X\colon |E[x]|>1\}$ is unbounded. We choose an unbounded subset Y of X such that $|E[y]|>1$ for each $y\in Y$ and $E[y]\cap E[y']=\varnothing$ for all distinct $y,y'\in Y$. For each $y\in Y$, we pick $z_y\in E[y]\setminus\{y\}$ and denote $Z=\{z_y\colon y\in Y\}$. Clearly, $Y\lambda Z$ but $Y\cap Z=\varnothing$ contradicting (3). □

Corollary 1. *Every discrete ballean is λ-rigid and hence δ-rigid.*

To continue, we need the following construction from [6].

Let $(X,\mathcal{E})$ be an unbounded ballean and φ be a filter on X such that $X\setminus B\in\varphi$ for each $B\in\mathcal{B}_{(X,\mathcal{E})}$. We denote by $\mathcal{E}_\varphi$ a coarse structure on X with the base $\{H_{(E,\phi)}\colon E\in\mathcal{E},\,\Phi\in\varphi\}$, where

$$H_{(E,\phi)}[x]=\begin{cases}\{x\} & \text{if } x\in\Phi,\\ E[x]\setminus\Phi & \text{if } x\in X\setminus\Phi.\end{cases}$$

Clearly, $\mathcal{E}_\varphi\subseteq\mathcal{E}$ and $\mathcal{B}_{(X,\mathcal{E})}=\mathcal{B}_{(X,\mathcal{E}_\varphi)}$.

Theorem 2. *Let $(X,\mathcal{E})$ be a ballean, $E_0\in\mathcal{E}$. Assume that there exists an unbounded subset Y of X such that $|E_0[y]|>1$, $y\in Y$ and every unbounded subset of Y can be partitioned in two unbounded subsets. Then there exists an ultrafilter φ on X such that $\mathcal{E}_\varphi\subsetneq\mathcal{E}$ and $(X,\mathcal{E})$, $(X,\mathcal{E}_\varphi)$ are λ-equivalent, so $(X,\mathcal{E})$ is not λ-rigid.*

Proof. We take an arbitrary ultrafilter $\varphi\in\mathcal{B}^\sharp_{(X,\mathcal{E})}$ such that $Y\in\varphi$. The definition of the coarse structure $\mathcal{E}_\varphi$ ensures that $\mathcal{E}_\varphi\subsetneq\mathcal{E}$.

We take arbitrary subsets A,B of X linked in $(X,\mathcal{E})$ and show that A,B are linked in $(X,\mathcal{E}_\varphi)$. If A,B are bounded in $(X,\mathcal{E})$ then A,B are bounded in $(X,\mathcal{E}_\varphi)$, so we suppose that A,B are unbounded in $(X,\mathcal{E})$. We choose unbounded subsets $A'\subseteq A$ and $B'\subseteq B$ which are close in $(X,\mathcal{E})$. If $A'\cap Y$ and $B'\cap Y$ are bounded then A',B' are close in $(X,\mathcal{E}_\varphi)$ by the construction of $\mathcal{E}_\varphi$. We suppose that $A'\cap Y$ is unbounded and partition $A'\cap Y$ in two unbounded subsets C,D. Since φ is an ultrafilter, either $C\notin\varphi$ or $D\notin\varphi$. Hence, either C or D is linked to B' in $(X,\mathcal{E}_\varphi)$. □

Let $(X,\mathcal{E})$ be a ballean. We recall that a subset Y of X is *large* if there exists $E\in\mathcal{E}$ such that $X=E[Y]$.

Clearly, Y is large if and only if $Y\delta X$. We note that every unbounded subset of X is linked with X.

A ballean $(X,\mathcal{E})$ is called *coarsely discrete* if X contains a large discrete subset.

Theorem 3. *Every coarsely discrete ballean $(X,\mathcal{E})$ is δ-rigid.*

Proof. Assuming that a coarse structure $\mathcal{E}'$ on X is δ-equivalent to $\mathcal{E}$, we shall prove that $\mathcal{E}'=\mathcal{E}$.

Let Y be a large discrete subset of $(X,\mathcal{E})$. Then there exists $E\in\mathcal{E}$ such that $E[Y]=X$. Since $Y\delta X$, there exists $E'\in\mathcal{E}'$ such that $E'[Y]=X$. The δ-equivalence of the coarse structures $\mathcal{E}$, $\mathcal{E}'$ implies the δ-equivalence of the induced coarse structures $\mathcal{E}|_Y$ and $\mathcal{E}'_Y$. Now Corollary 1 implies that $\mathcal{E}|_Y=\mathcal{E}'|_Y$. Taking into account that Y is large in both balleans $(X,\mathcal{E})$ and $(X,\mathcal{E}')$, we conclude that $\mathcal{E}=\mathcal{E}'$. □

Example 1. We define a coarse structure $\mathcal{E}$ on the set $\mathbb{N}$ of natural numbers such that the ballean $(\mathbb{N},\mathcal{E})$ is δ-rigid but not λ-rigid. We denote

$$A=\{(2n+1,2n+2),(2n+2,2n+1),(n,n)\colon n\in\omega\}$$

and put $\mathcal{E}=\{(F\times F)\cup A\colon F\subset\mathbb{N},\ |F|<\omega\}$. Since the set $2\mathbb{N}$ is large and discrete in $(\mathbb{N},\mathcal{E})$, by Theorem 3, $(\mathbb{N},\mathcal{E})$ is δ-rigid. By Theorem 2 with $Y=2\mathbb{N}$, $(\mathbb{N},\mathcal{E})$ is not λ-rigid.

3. Ordinal and submetrizable balleans. We recall that a ballean $(X,\mathcal{E})$ is *ordinal* if $\mathcal{E}$ has a base linearly ordered by inclusion. In this case $\mathcal{E}$ has a base well-ordered by inclusion.

A ballean $X,\mathcal{E}$ is called *metrizable* if there exists a metric d on X such that $\mathcal{E}$ has the base $\{\{(x,y)\colon d(x,y)<r\}\colon r\in\mathbb{R}^+\}$. A ballean $(X,\mathcal{E})$ is metrizable if and only if $\mathcal{E}$ has a countable base [12, Theorem 2.1.1]. Hence, every metrizable ballean is ordinal.

We denote by $\mathcal{M}$ and $\mathcal{L}$ the cases of metrizable and ordinal balleans.

Theorem 4. *The following statements hold:*

1. *every ordinal ballean is λ-strong;*
2. *$\mathcal{M}$ and $\mathcal{L}$ are λ-rigid;*
3. *an ordinal ballean $(X,\mathcal{E})$ is λ-rigid if and only if $(X,\mathcal{E})$ is discrete;*
4. *$\mathcal{M}$ and $\mathcal{L}$ are not λ-stable.*

Proof. (1) let $(X,\mathcal{E})$ be an unbounded ordinal ballean, $\{L_\lambda\colon\lambda<\kappa\}$ be a base of $\mathcal{E}$ well-ordered by inclusion. We take a ballean $(X,\mathcal{E}')$ λ-equivalent to $(X,\mathcal{E})$ and show that $\mathcal{E}'\subseteq\mathcal{E}$. Assume the contrary and choose $E'\in\mathcal{E}'\setminus\mathcal{E}$. Then we can choose a κ-sequence $(x_\alpha)_{\alpha<\kappa}$ in X such that

- $L_\alpha[x_\alpha]\cap L_\beta[x_\beta]=\varnothing$, $\alpha<\beta<\kappa$;
- $E'[x_\alpha]\cap E'[x_\beta]=\varnothing$, $\alpha<\beta<\kappa$;
- $E'[x_\alpha]\setminus L_\alpha[x_\alpha]\neq\varnothing$, $\alpha<\kappa$.

For each $\alpha<\kappa$, we pick $y_\alpha\in E'[x_\alpha]\setminus L_\alpha[x_\alpha]$ and put $X'=\{x_\alpha\colon\alpha<\kappa\}$, $Y=\{y_\alpha\colon\alpha<\kappa\}$. Then X',Y are close in $(X,\mathcal{E}')$. For each $E\in\mathcal{E}$, there exists $\beta<\kappa$ such that $E[x_\alpha]\subseteq L_\alpha[x_\alpha]$, $\alpha<\beta$. Hence, $E[X']\cap Y$ is bounded and X',Y are not linked in $\mathcal{E}$ contradicting λ-equivalence of $(X,\mathcal{E})$ and $(X,\mathcal{E}')$.

(2) follows from (1).

(3) We note that every unbounded subset of an ordinal ballean can be partitioned in two unbounded subsets and apply Theorem 2.

(4) follows from Theorem 2. □

By Theorem 3, the statement (3) of Theorem 4 does not hold for δ in place of λ.

Question 1. *Is every metrizable ballean δ-rigid? Equivalently, is $\mathcal{M}$ δ-stable?*

We recall that an unbounded ballean $(X, \mathcal{E})$ is *submetrizable* if there exists a coarse structure $\mathcal{E}'$ on X such that $\mathcal{E} \subseteq \mathcal{E}'$ and $(X, \mathcal{E}')$ is unbounded and metrizable.

For a ballean $(X, \mathcal{E})$, a function $f\colon X \longrightarrow \mathbb{R}$ is called *macro-uniform* if, for every $E \in \mathcal{E}$, there exists $r_E \in \mathbb{R}^+$ such that $\operatorname{diam} f(E[x]) \le r_E$ for each $x \in X$. We denote $\mathsf{mu}(X, \mathcal{E})$ the family of all macro-uniform functions on X.

By [13, Theorem 2.2.3], an unbounded ballean $(X, \mathcal{E})$ is submetrizable if and only if there exists an unbounded function $f \in \mathsf{mu}(X, \mathcal{E})$.

Theorem 5. *If balleans $(X, \mathcal{E})$, $(X, \mathcal{E}')$ are λ-equivalent, then $\mathsf{mu}(X, \mathcal{E}) = \mathsf{mu}(X, \mathcal{E}')$.*

Proof. We assume that there exists $f \in \mathsf{mu}(X, \mathcal{E}') \setminus \mathsf{mu}(X, \mathcal{E})$ and choose $E \in \mathcal{E}$ and a sequence $(y_n)_{n\in\omega}$ in X such that $\operatorname{diam} f(E[y_n]) > 2n$. For each $n \in \omega$, we pick $z_n \in E[y_n]$ such that $|f(y_n)| - f(z_n)| > n$. Passing to subsequences, we may suppose that $|f(y_n)| - f(z_m)| > n$ for all $m \ge n$. We denote $Y = \{y_n\colon n \in \omega\}$, $Z = \{z_n\colon n \in \omega\}$ and note that $Y \delta_{(X,\mathcal{E})} Z$. Since $f \in \mathsf{mu}(X, \mathcal{E}')$, for every $E' \in \mathcal{E}'$, there exists $k \in \mathbb{N}$ such that $\operatorname{diam} f(\mathcal{E}'[y_n]) < k$ for all $n \in \omega$. Then $\mathcal{E}'[Y] \cap Z$ is finite, so Y, Z are not linked in $(X, \mathcal{E}')$. □

Corollary 2. *The class of submetrizable balleans is λ-stable.*

A ballean $(X, \mathcal{E})$ is called mu-*bounded* if $(X, \mathcal{E})$ is not submetrizable. For mu-bounded balleans, see [2]. By Corollary 2, the class of mu-bounded balleans is λ-stable.

4. Normal balleans. For a ballean $(X, \mathcal{E})$, two subsets A, B of X are called *asymptotically disjoint* if for every $E \in \mathcal{E}$ the intersection $E[A] \cap E[B]$ is bounded. We note that unbounded subsets A, B are asymptotically disjoint if and only if A, B are not linked.

A subset U of X is called an *asymptotic neighbourhood* of a subset A if for every $E \in \mathcal{E}$ the set $E[A] \setminus U$ is bounded.

A ballean $(X, \mathcal{E})$ is called *normal* [7] if any two asymptotically disjoint subsets have disjoint asymptotic neighbourhoods.

A function $f\colon X \longrightarrow \mathbb{R}$ is called *slowly oscillating* if for any $E \in \mathcal{E}$ and $\varepsilon > 0$ there exists a bounded subset B of X such that $\operatorname{diam} f(E[x]) < \varepsilon$ for each $x \in X$. By [6, Theorem 2.2], a ballean $(X, \mathcal{E})$ is normal if and only if, for any two disjoint and asymptotically disjoint subsets A, B of X, there exists a slowly oscillating function $f\colon X \longrightarrow [0, 1]$ such that $f|_A = 0$, $f|_B = 1$.

We denote by $\mathsf{so}(X, \mathcal{E})$ and $\mathsf{sob}(X, \mathcal{E})$ the families of all and all bounded slowly oscillating functions on X.

Theorem 6. *If balleans $(X, \mathcal{E})$, $(X, \mathcal{E}')$ are λ-equivalent then $\mathsf{sob}(X, \mathcal{E}) = \mathsf{sob}(X, \mathcal{E}')$.*

Proof. We suppose the contrary and let $f \in \mathsf{sob}(X, \mathcal{E}') \setminus \mathsf{sob}(X, \mathcal{E})$. We denote $\mathcal{B} = \mathcal{B}_{(X,\mathcal{E})} = \mathcal{B}_{(X,\mathcal{E}')}$ and take $\varepsilon > 0$ and $E \in \mathcal{E}$ such that, for every $B \in \mathcal{B}$, there exists $x_B \in X \setminus B$ such that $\operatorname{diam} f(E[x_B]) > \varepsilon$. We pick $y_B \in E[x_B]$ such that $|f(x_B) - f(y_B)| > \varepsilon$. Then we enumerate $\mathcal{B} = \{B_\alpha\colon \alpha < \kappa\}$, denote $Z = \{x_B\colon B \in \mathcal{B}\}$ and define a mapping $h\colon Z \longrightarrow X$ as follows.

We put $h(x_{B_0}) = y_{B_0}$ and let h is already defined for each x_{B_α}, $\alpha < \beta$. If h is not defined at x_{B_β} then we put $f(x_{B_\beta}) = y_{B_\beta}$.

We take an ultrafilter $p \in \mathcal{B}^\sharp$ such that $Z \in p$, denote $q = h^\beta(p)$ and observe that $|f^\beta(p) - f^\beta(q)| \ge \varepsilon$. We choose $P \in p$, $Q \in p$ such that $P \subseteq Z$, $Q \subseteq Z$, $|f(x) - f^\beta(p)| < \frac{1}{2}\varepsilon$, $x \in P$ and $|f(x) - f^\beta(q)| < \frac{1}{2}\varepsilon$, $x \in Q$. By the construction of h, we have $P \delta_{(X,\mathcal{E})} Q$, so $P \lambda_{(X,\mathcal{E})} Q$. By the assumption, $P \lambda_{(X,\mathcal{E}')} Q$. Hence, there exist $E' \in \mathcal{E}'$ and an unbounded

$P' \subseteq P$ such that $E'[x] \cap Q \neq \varnothing$ for each $x \in P'$. Since f is slowly oscillating in $(X, \mathcal{E}')$, there exists $B \in \mathcal{B}$ such that $\operatorname{diam} f(E'[x]) < \frac{1}{2}\varepsilon$ for each $x \in X \setminus B$. Then $f(E'[x]) \cap Q = \varnothing$ for each $x \in P' \setminus B$ and we get a contradiction with the choice of P'. □

Corollary 3. *The class of normal balleans is λ-stable.*

If $(X, \mathcal{E})$, $(X, \mathcal{E}')$ are metrizable and $\mathsf{sob}(X, \mathcal{E}) = \mathsf{sob}(X, \mathcal{E}')$, then $\lambda_{(X,\mathcal{E})} = \lambda_{(X,\mathcal{E}')}$ and, by Theorem 4(2), $\mathcal{E} = \mathcal{E}'$.

We show that the class of normal balleans is not δ-rigid.

Example 2. We denote by S_ω the group of all permutations of ω, by $\mathcal{E}$ the coarse structure on ω with the base

$$\{\{(x, y) \colon y \in \{x\} \cup Fx\} \colon F \in [S_\omega]^{<\omega}\}$$

and take the strongest coarse structure $\mathcal{E}'$ on ω such that $\mathcal{B}_{(\omega,\mathcal{E}')} = [\omega]^{<\omega}$, see Example 3 in [3]. Then $\mathcal{E} \subsetneq \mathcal{E}'$ and every infinite subset of ω is large in $(\omega, \mathcal{E})$ and $(\omega, \mathcal{E}')$. Hence, $\delta_{(\omega,\mathcal{E})} = \delta_{(\omega,\mathcal{E}')}$ and $(\omega, \mathcal{E})$, $(\omega, \mathcal{E}')$ are normal because any two infinite subsets of X are not asymptotically disjoint in $(\omega, \mathcal{E})$ and $(\omega, \mathcal{E}')$.

Question 2. *Is $\mathsf{so}(X, \mathcal{E}) = \mathsf{so}(X, \mathcal{E}')$ for any λ-equivalent balleans $(X, \mathcal{E})$, $(X, \mathcal{E}')$?*

Following [3], we say that a ballean $(X, \mathcal{E})$ has *bounded growth* if, there exists a mapping $f \colon X \longrightarrow \mathcal{B}_{(X,\mathcal{E})}$ such that for every $E \in \mathcal{E}$, the set $\{x \in X \colon E[x] \nsubseteq f(x)\}$ is bounded.

Question 3. *Is the class of balleans of bounded growth λ-stable?*

5. Finitary balleans. For a group G, we denote by $\mathcal{F}_G$ the coarse structure on G with the base

$$\big\{\{(x, y) \in G \times G \colon y \in \{x\} \cup Fx\} \colon F \in [G]^{<\omega}\big\}$$

and say that $(G, \mathcal{F}_G)$ is the *finitary group ballean* of G. The following theorem show that the class of finitary group balleans is λ-rigid.

Theorem 7. *Let G, H be group on a set X such that $\lambda_{(X,\mathcal{F}_G)} = \lambda_{(X,\mathcal{F}_H)}$. Then $\mathcal{F}_G = \mathcal{F}_H$.*

Proof. If X is countable then the statement follows from Theorem 4(2) because $\mathcal{F}_G, \mathcal{F}_H$ have countable bases, so we suppose that X is uncountable.

We denote by $\cdot$, $*$ the group operations in G, H and, on the contrary, assume that $\mathcal{F}_G \setminus \mathcal{F}_H \neq \varnothing$. Then there exists $F \in [X]^{<\omega}$ such that, for any $H \in [X]^{<\omega}$, there exists $x \in X$ such that $F \cdot x \setminus H * x \neq \varnothing$.

We use the following observation:

($\star$) for every countable subset Y of X, there exists a countable subset Z such that $Y \subset Z$ and, for every $A \in [Y]^{<\omega}$, there exists $z \in Z$ such that $F \cdot z \setminus A * z \neq \varnothing$.

We fix an arbitrary countable subset Y_0 of X containing F and apply ($\star$) to get a countable subgroup Z_0 of G, $Y_0 \subset Z_0$. Then we denote by Y_1 the subgroup of H generated by Z_0. Analogously, for Y_1 we use ($\star$) to get the subgroup Z_1 of G, $Y_1 \subset Z_1$, and denote by Y_2 the subgroup of H generated by Z_1. After ω steps, we put $S = \bigcup_{n\in\omega} Y_n$ and note that S is a subgroup of G and S is a subgroup of H.

By the construction of S, for every $A \in [S]^{<\omega}$, there exists $x \in S$ such that $F \cdot x \setminus A * x \neq \varnothing$. Hence, the restriction of $\mathcal{F}_G$ and $\mathcal{F}_H$ to S are distinct. Since S is countable, we get a contradiction to Theorem 4(2). □

A ballean $(X,\mathcal{E})$ is called *finitary* if for every $E \in \mathcal{E}$ there exists $n \in \omega$ such that $|E[x]| < n$ for each $x \in X$. By [9], for every finitary ballean $(X,\mathcal{E})$, there exists a group S of permutations of X such that $\mathcal{E}$ has the base

$$\big\{\{(x,y)\colon y \in \{x\} \cup Fx\}\colon F \in [S]^{<\omega}\big\}.$$

Example 2 shows that the class of finitary balleans is not δ-stable.

Question 4. *Is the class of finitary balleans δ-rigid?*

The product of two λ-rigid balleans needs not to be λ-rigid: take two metrizable unbounded discrete balleans and apply Corollary 1 and Theorem 2.

Question 5. *Is the product of two δ-rigid balleans δ-rigid?*

A ballean $(X,\mathcal{E})$ is called *cellular* if $\mathcal{E}$ has a base consisting of equivalence relations. Equivalently [4, Theorem 3.1.3], $(X,\mathcal{E})$ is cellular if $\operatorname{asdim}(X,\mathcal{E}) = 0$.

Question 6. *Is the class of cellular balleans λ-stable?*

Remark 1. As was recently shown by Banakh [1], the answers to Questions 4 and 6 are negative under CH (more precisely, under $\mathfrak{b} = \mathfrak{c}$, and $\Delta^{\circ}_{\omega} = \mathfrak{c}$, respectively).

5. Ultrafilters. Let $(X,\mathcal{E})$ be a ballean. We endow X with the discrete topology and consider the *Stone–Čech compactification* βX of X. We take the points of βX to be the ultrafilters on X. Then $\mathcal{B}^{\sharp}_{(X,\mathcal{E})}$ is a closed subset of βX.

Given any $r, q \in \mathcal{B}^{\sharp}_{(X,\mathcal{E})}$, we say that r, q are *parallel* (and write $r||_{(X,\mathcal{E})}q$) if there exists $E \in \mathcal{E}$ such that $E[P] \in q$ for each $P \in p$. By [7, Lemma 4.1], $||_{(X,\mathcal{E})}$ is an equivalence on $\mathcal{B}^{\sharp}_{(X,\mathcal{E})}$. We denote by $\sim_{(X,\mathcal{E})}$ the minimal (by inclusion) closed (in $\mathcal{B}^{\sharp}_{(X,\mathcal{E})} \times \mathcal{B}^{\sharp}_{(X,\mathcal{E})}$) equivalence such that $||_{(X,\mathcal{E})} \subseteq \sim_{(X,\mathcal{E})}$. The quotient $\nu(X,\mathcal{E})$ is called the *Higson corona* of $(X,\mathcal{E})$.

By [8, Proposition 1], $r \sim_{(X,\mathcal{E})} q$ if and only if $h^{\beta}(p) = p^{\beta}(q)$ for every slowly oscillating function $h\colon X \longrightarrow [0,1]$.

Theorem 8. *Let $\mathcal{E}, \mathcal{E}'$ be coarse structures on a set X such that $\mathcal{B}^{\sharp}_{(X,\mathcal{E})} = \mathcal{B}^{\sharp}_{(X,\mathcal{E}')}$ and $||_{(X,\mathcal{E})} = ||_{(X,\mathcal{E}')}$. Then $(X,\mathcal{E}), (X,\mathcal{E}')$ are δ-equivalent. If $\mathcal{B}^{\sharp}_{(X,\mathcal{E})}$ is finite, then $\mathcal{E} = \mathcal{E}'$.*

Proof. We suppose that there exist $P \subseteq X$, $Q \subseteq X$ such that $P\delta_{(X,\mathcal{E})}Q$ and $P \setminus E'(Q) \neq \varnothing$ for each $E' \in \mathcal{E}'$.

Since $P\delta_{(X,\mathcal{E})}Q$, there exists $E \in \mathcal{E}$ such that $P \subseteq E[Q]$ and $E = E^{-1}$. For each $x \in P$, we pick $f(x) \in Q$ such that $(x, f(x)) \in E$. If $p \in \mathcal{B}^{\sharp}_{(X,\mathcal{E})}$ and $P \in p$ then $p||_{(X,\mathcal{E})}f^{\beta}(p)$ and $Q \in f^{\beta}(p)$. Thus, each ultrafilter $p \in \mathcal{B}^{\sharp}_{(X,\mathcal{E})}$ such that $P \in p$ is parallel in $\mathcal{B}^{\sharp}_{(X,\mathcal{E})}$ to some ultrafilter q such that $Q \in q$.

We take an arbitrary ultrafilter r such that $P \setminus E'[Q] \in r$ for each $E' \in \mathcal{E}$. Then $P \in r$ and r is not parallel in $\mathcal{B}^{\sharp}_{(X,\mathcal{E}')}$ to each ultrafilter q such that $Q \in q$. Hence, $||_{(X,\mathcal{E})} \neq ||_{(X,\mathcal{E}')}$.

To prove the second statement, it suffices to show that $(X,\mathcal{E})$ is coarsely discrete and apply Theorem 3.

We choose a representative from each class of parallel ultrafilter in $\mathcal{B}^{\sharp}_{(X,\mathcal{E})}$. Let $p_1, \dots, p_m$ be the set of obtained representatives. Let $E \in \mathcal{E}$. We choose $P_1 \in p_1, \dots, P_n \in p_n$ such that $E[P_i] \cap P_j = \varnothing$. Then $P_1 \cup, \dots, P_n$ is discrete and large in $(X,\mathcal{E})$, so $(X,\mathcal{E})$ is coarsely discrete. $\square$

Remark 2. Let $\mathcal{B}^\sharp_{(X,\mathcal{E})} = \mathcal{B}^\sharp_{(X,\mathcal{E}')}$. *Does* $\delta_{(X,\mathcal{E})} = \delta_{(X,\mathcal{E}')}$ *implies* $||_{(X,\mathcal{E})} = ||_{(X,\mathcal{E}')}$*?*
The answer to this question is negative. We take the balleans $(\omega, \mathcal{E}), (\omega, \mathcal{E}')$ from Example 11 and note that $p, q \in \omega^*$ are parallel in $\mathcal{B}^\sharp_{(\omega,\mathcal{E})}$ if and only if $p = f^\beta(q)$ for some bijection $f\colon \omega \longrightarrow \omega$. On the other hand, we partition $\omega = \bigcup_{n\in\omega} W_n$ so that $|W_n| = n+1$ and take a mapping $h\colon \omega \longrightarrow \omega$ such that $h(W_n) = \{a_n\}$, $a_n \in W_n$. We choose $p \in \omega^*$ such that p and $h^\beta(p)$ are not isomorphic. By the choice of h, p and $h^\beta(p)$ are parallel in $\mathcal{B}^\sharp_{(X,\mathcal{E}')}$ but not in $\mathcal{B}^\sharp_{(X,\mathcal{E})}$.

Theorem 9. *If* $(X, \mathcal{E})$, $(X, \mathcal{E}')$ *are finitary balleans and* $||_{(X,\mathcal{E})} = ||_{(X,\mathcal{E}')}$ *then* $\mathcal{E} = \mathcal{E}'$.

Proof. Let G, G' be groups of permutations of X defining $\mathcal{E}, \mathcal{E}'$. We assume that $\mathcal{E} \setminus \mathcal{E}' \neq \varnothing$. Then there is $H \in [G]^{<\omega}$ such that, for every $A \in [G']^{<\omega}$, the set $Y_A = \{x \in X\colon Hx \backslash Ax \neq \varnothing\}$ is infinite. We take $p \in \mathcal{B}^\sharp_{(X,\mathcal{E})}$ such that $Y_A \in p$ for each $A \in [G']^{<\omega}$. Let $H = \{h_1, \dots, h_n\}$. We take $g_1, \dots, g_n \in G'$ such that $h_1 p = g_1 p, \dots, h_n p = g_n p$ and put $B = \{g_1, \dots, g_n\}$. Then $\{x \in X\colon h_1 x = g_1 x, \dots, h_n x = g_n x\} \in p$ and $Y_B \notin p$, contradict the choice of p. □

Theorem 10. *For a normal ballean* $(X, \mathcal{E})$ *and* $p, q \in \mathcal{B}^\sharp_{(X,\mathcal{E})}$*, the following statements are equivalent:*

1. $p \sim q$;
2. $P \lambda Q$ *for each* $P \in p$, $Q \in q$;
3. $(p, q) \in cl||$.

Proof. (1) ⇒ (2). If P, Q are unbounded and asymptotically disjoint then, by the normality of $(X, \mathcal{E})$, there exists a slowly oscillating function $f\colon X \longrightarrow [0,1]$ such that $f|_P = 1$, $f|_Q = 0$, so $f^\beta(p) \neq f^\beta(q)$ and p, q are not equivalent.

(2) ⇒ (3). We choose $E \in \mathcal{E}$, unbounded subsets $P' \subseteq P$, $Q' \subseteq Q$ such that there is a bijection $h\colon P' \longrightarrow Q'$ satisfying $(x, h(x)) \in E$, $x \in P'$. If $r \in \mathcal{B}^\sharp_{(X,\mathcal{E})}$ and $P' \in r$ then $r || h^\beta(r)$ and $P \in r$, $Q \in h^\beta(r)$.

The implication (3) ⇒ (1) follows from definitions of $\sim$ and $||$. □

Theorem 11. *A ballean* $(X, \mathcal{E})$ *is normal if and only if* $\sim = cl||$.

Proof. By Theorem 10, if $(X, \mathcal{E})$ is normal then $\sim = cl||$. We assume that $\sim = cl||$ and show that any two unbounded asymptotically disjoint subsets P, Q of X have disjoint asymptotic neighborhoods.

Let $p, q \in \mathcal{B}^\sharp_{(X,\mathcal{E})}$, $P \in p$, $Q \in q$. Since P, Q are asymptotically disjoint, we have $(p, q) \notin cl||$, so there exists a slowly oscillating function $f\colon X \longrightarrow [0,1]$ such that $f^\beta(p) = 1$, $f^\beta(q) = 0$. We denote $P' = \{x \in P\colon f(x) > 3/4\}$, $Q' = \{x \in Q\colon f(x) < 1/2\}$. Then $P' \in p$, $Q' \in q$ and P', Q' have disjoint asymptotic neighborhoods.

We fix $p \in \mathcal{B}^\sharp_{(X,\mathcal{E})}$ such that $P \in p$ and apply above paragraph to find $q_1, \dots, q_n \in \mathcal{B}^\sharp_{(X,\mathcal{E})}$, $Q_1 \in q_1, \dots, Q_n \in q_n$ and $P' \in p$ such that P' and $Q_1 \cup \dots \cup Q_n$ have asymptotically disjoint neighborhoods and $Q \setminus (Q_1 \cup \dots \cup Q_n)$ is bounded. It follows that P', Q have disjoint asymptotic neighborhoods.

At last, we find $p_1, \dots, p_m \in \mathcal{B}^\sharp_{(X,\mathcal{E})}$ and $P_1 \in p_1, \dots, P_m \in p_m$ such that $p_1 \cup \dots \cup p_m$ and Q have disjoint asymptotic neighborhoods and $P \setminus (P_1 \cup \dots \cup P_m)$ is bounded. Hence, P and Q have disjoint asymptotic neighborhoods. □

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Faculty of Computer Science and Cybernetics
Kyiv University, Kyiv, Ukraine
i.v.protasov@gmail.com
ksuha@freenet.com.ua

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