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AVERAGING METHOD FOR IMPULSIVE DIFERENTIAL INCLUSIONS WITH FUZZY RIGHT-HAND SIDE

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In this paper the substantiation of the partial scheme of the averaging method for impulsive differential inclusions with fuzzy right-hand side in terms of R-solutions on the finite interval is considered. Consider the impulsive differential inclusion with the fuzzy right-hand side

$$\dot{x} \in \varepsilon F(t,x),\ t \neq t_i,\ x(0) \in X_0, \quad \Delta x\mid_{t=t_i} \in \varepsilon I_i(x), \tag{1}$$

where $t \in \mathbb{R}_+$ is time, $x \in \mathbb{R}^n$ is a phase variable, $\varepsilon > 0$ is a small parameter, $F\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{E}^n$, $I_i\colon \mathbb{R}^n \to \mathbb{E}^n$ are fuzzy mappings, moments t_i are enumerated in the increasing order. Associate with inclusion (1) the following partial averaged differential inclusion

$$\dot{\xi} \in \varepsilon \widetilde{F}(t,\xi),\ t \neq s_j,\ \xi(0) \in X_0, \quad \Delta\xi|_{t=s_j} \in \varepsilon K_j(\xi), \tag{2}$$

where the fuzzy mappings $\widetilde{F}\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{E}^n$; $K_j\colon \mathbb{R} \to \mathbb{E}^n$ satisfy the condition

$$\lim_{T\to\infty} \frac{1}{T} D\Big(\int\limits_t^{t+T} F(t,x)dt + \sum_{t\le t_i<t+T} I_i(x), \int\limits_t^{t+T} \widetilde{F}(t,x)dt + \sum_{t\le s_j<t+T} K_j(x)\Big) = 0, \tag{3}$$

moments s_j are enumerated in the increasing order. The following main theorem is proved: *Let in the domain $Q = \{t \ge 0, x \in G \subset \mathbb{R}^n\}$ the following conditions are fulfilled:*
1) *fuzzy mappings $F(t,x), \widetilde{F}(t,x), I_i(x), K_j(x)$ are continuous, uniformly bounded with constant M, concave in x, satisfy Lipschitz condition in x with a constant λ;*
2) *uniformly with respect to t, x limit (3) exists and $\frac{1}{T} i(t,t+T) \le d < \infty$, $\frac{1}{T} j(t,t+T) \le d < \infty$, where $i(t,t+T)$ and $j(t,t+T)$ are the quantities of impulse moments t_i and s_j on the interval $[t,t+T]$;*
3) *R-solutions of inclusion (2) for all $X_0 \subset G' \subset G$ for $t \in [0, L^*\varepsilon^{-1}]$ belong to the domain G with a ρ-neighborhood.*

Then for any $\eta > 0$ and $L \in (0, L^*]$ there exists $\varepsilon_0(\eta, L) \in (0,\sigma]$ such that for all $\varepsilon \in (0,\varepsilon_0]$ and $t \in [0, L\varepsilon^{-1}]$ the inequality $D(R(t,\varepsilon), \widetilde{R}(t,\varepsilon)) < \eta$ holds, where $R(t,\varepsilon), \widetilde{R}(t,\varepsilon)$ are the R-solutions of inclusions (1) and (2), $R(0,\varepsilon) = \widetilde{R}(0,\varepsilon)$.

1. Introduction. Fuzzy systems are very important either from the theoretical point of view or from the practical one. They are applied, for example, in the automotive, space and transport industries, in the engineering science, while creating of hydraulic and population models, in the sphere of finance, analysis and making administrative decisions, when forecasting different economic, political, elections situations, etc. Fuzzy systems are a natural way of dynamic systems modeling in the conditions of uncertainty. Formalization of fuzzy concepts allows to approximately describe the behavior of systems that are so complicated that the standard mathematical analysis can not be applied. In some cases such description is the only possible as in real situations the regularities, restrictions, choice criteria are

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mostly subjective and are distinctly not defined. Since 1965 when L. Zadeh [30] published his innovative work, hundreds of examples where the uncertainty's nature in the behavior of the system is rather fuzzy, than has stochastic character were considered.

Asymptotic methods for the research of nonlinear differential equations take the central place in nonlinear mechanics. The development of the general algorithm known as averaging method and the theorem of the proximity of the solutions of the initial and the averaged systems were proposed by N. M. Krylov and N. N. Bogolyubov [11]. The received results have gained further developments for nonlinear equations with slowly changing coefficients, multi-frequency systems, partial differential equations, equations with discontinuous right-hand sides, impulsive differential equations, equations with delay, stochastic equations, equations in the infinite spaces, differential inclusions, differential equations and inclusions with Hukuhara derivative, set-valued integral and integro-differential equations, quasidifferential equations, fuzzy equations and inclusions, etc. by Yu. A. Mitropolskiy, V. I. Arnold, J. K. Hale, M. A. Krasnoselskiy, S. G. Krein, N. N. Moiseev, N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko, J. A. Sanders and F. Verhulst, V. M. Volosov, etc. (see [4, 5, 6, 8, 10, 12, 13, 15, 20, 21, 22, 23, 24] and references herein).

In this paper the substantiation of the partial scheme of the averaging method for impulsive differential inclusions with fuzzy right-hand side in terms of R-solutions on the finite interval is considered.

2. Preliminaries. Let $\operatorname{conv}(\mathbb{R}^n)$ be the family of all nonempty compact convex subsets of $\mathbb{R}^n$ with the Hausdorff metric

$$h(A,B) = \max\{\max_{a\in A}\min_{b\in B}\|a-b\|,\ \max_{b\in B}\min_{a\in A}\|a-b\|\},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Let $\rho(x,A) = \inf_{y\in A}\|x-y\|$ be the distance from $x\in\mathbb{R}^n$ to the set $A\in\operatorname{conv}(\mathbb{R}^n)$.

Consider the fuzzy space $\mathbb{E}^n$ of the mappings $u\colon \mathbb{R}^n\to[0,1]$ that satisfy the following conditions:

1) u is upper semicontinuous, i.e. for any $\tilde{y}\in\mathbb{R}^n$ and $\varepsilon>0$ there exists $\delta(\tilde{y},\varepsilon)>0$ such that $u(y)<u(\tilde{y})+\varepsilon$ whenever $\|y-\tilde{y}\|<\delta$;

2) u is normal, i.e. there exists a vector $y_0\in\mathbb{R}^n$ such that $u(y_0)=1$;

3) u is fuzzy convex, i.e. $u(\lambda y_1+(1-\lambda)y_2)\ge\min\{u(y_1),u(y_2)\}$ for any $y_1,y_2\in\mathbb{R}^n$ and $\lambda\in[0,1]$;

4) the closure of the set $\{y\in\mathbb{R}^n\colon u(y)>0\}$ is compact.

Let $\hat{0}$ be the fuzzy number defined by

$$\hat{0}(y) = \begin{cases} 1, & y=0,\\ 0, & y\in\mathbb{R}^n\backslash\{0\}. \end{cases}$$

Definition 1. The set $\{y\in\mathbb{R}^n\colon u(y)\ge\alpha\}$ for $\alpha\in(0,1]$ and the closure of the set $\{y\in\mathbb{R}^n\colon u(y)>0\}$ for $\alpha=0$ is called the *α-level* of the fuzzy set $u\in\mathbb{E}^n$.

Theorem 1 ([14]). *If $u\in\mathbb{E}^n$ then:*

1) $[u]^\alpha\in\operatorname{conv}(\mathbb{R}^n)$ for all $\alpha\in[0,1]$;

2) $[u]^{\alpha_2}\subset[u]^{\alpha_1}$ for all $0\le\alpha_1\le\alpha_2\le1$;

3) if $\{\alpha_k\}_{k=1}^\infty$ is a non-decreasing sequence converging to α then $[u]^\alpha=\bigcap_{k\ge1}[u]^{\alpha_k}$.

Conversely, if the family $\{A_\alpha : \alpha \in [0,1]\}$ *of subsets of* $\operatorname{conv}(\mathbb{R}^n)$ *satisfy conditions 1)–3), then there exists* $u \in \mathbb{E}^n$ *such that* $[u]^\alpha = A_\alpha$ *for all* $\alpha \in (0,1]$ *and* $[u]^0 = \overline{\bigcup_{\alpha\in(0,1]} A_\alpha} \subset A_0$.

Define the metric $D\colon \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{R}_+$ in the fuzzy space $\mathbb{E}^n$ by the relation

$$D(u,v) = \sup\{h([u]^\alpha, [v]^\alpha)\colon 0 \le \alpha \le 1\}.$$

In 1989 J.-P. Aubin [1] and V. A. Baidosov [2, 3] first considered the differential inclusion with fuzzy right-hand side

$$\dot{x} \in F(t,x),\ x(t_0) \in X_0, \tag{1}$$

where $t \in I = [t_0, T]$ is time, $x\colon I \to \mathbb{R}^n$ is a phase variable, $\dot{x} = \frac{dx}{dt}$ is the derivative of the vector-function $x(\cdot)$, $F\colon I \times \mathbb{R}^n \to \mathbb{E}^n$ is a fuzzy mapping, $X_0 \in \mathbb{E}^n$ is a fuzzy set of initial states.

Definition 2 ([9])**.** The absolutely continuous function $x\colon I \to \mathbb{R}^n$ such that $x(t_0) \in [X_0]^\alpha$ and $\dot{x}(t) \in [F(t,x(t))]^\alpha$ almost everywhere on I is called an *α-solution* of inclusion (1).

Denote by $X_\alpha(t)$ the set of all α-solutions of inclusion (1) at the moment t. In case when the family $\{X_\alpha(t),\ \alpha \in [0,1]\}$ defines the fuzzy set $X(t)$, the fuzzy set $X(t)$ is called the *solution set* of inclusion (1) at the moment t.

The existence of the solutions set $X(t)$ and its properties were considered by S. Abbasbandy, T. Allahviranloo, P. Balasubramaniam, Y. Chalco-Cano, E. Hullermeier, V. Laksmikantham, O. Lopez- Pouso, K.K. Majumdar, R. N. Mohapatra, J. J. Nieto, J. Y. Park, A. V. Plotnikov, D. O'Regan, H. Roman-Flores, A. A. Tolstonogov etc.

Obviously the family $\{X_\alpha(t),\ \alpha \in [0,1]\}$ may not satisfy the conditions of Theorem 1, i.e. it does not define the fuzzy set $X(t)$. So in [16, 17] the notion of an R-solution of the differential inclusion with the fuzzy right-hand side was introduced.

Definition 3 ([16, 17])**.** The upper semicontinuous fuzzy mapping $R\colon I \to \mathbb{E}^n,\ R(t_0) = X_0$ such that

$$\lim_{\sigma\downarrow 0} \frac{1}{\sigma} \sup_{\alpha\in[0,1]} h\Big([R(t+\sigma)]^\alpha, \bigcup_{x\in[R(t)]^\alpha} \Big\{x + \int_t^{t+\sigma} [F(s,x)]^\alpha ds\Big\}\Big) = 0$$

is called an *R-solution* of differential inclusion with the fuzzy right-hand side (1).

Definition 4. The mapping $F\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{E}^n$ is called *concave* in x if

$$\beta[F(t,x)]^\alpha + (1-\beta)[F(t,y)]^\alpha \subset [F(t, \beta x + (1-\beta)y)]^\alpha$$

for all $\alpha, \beta \in [0,1]$.

Theorem 2 ([16, 17])**.** *Let the fuzzy mapping* $F\colon I \times \mathbb{R}^n \to \mathbb{E}^n$ *satisfy the following conditions:*

1) F is measurable it t for any fixed $x \in \mathbb{R}^n$;

2) F satisfies the Lipschitz condition in x with a constant λ for almost all $t \in I$;

3) is concave in x for almost all $t \in I$ and all $x \in \mathbb{R}^n$;

4) there exists a constant $\gamma > 0$ such that $D(F(t,x), \hat{0}) \le \gamma$ for almost all $t \in I$ and all $x \in \mathbb{R}^n$.

Then there exists a unique R-solution $R(\cdot)$ of the inclusion (1) defined on the interval $[t_0, t_0 + d] \subset I$.

Remark 1. Conditions 1), 2) and 4) guarantee the existence and uniqueness of the mapping $R(\cdot)$ and condition 3) guarantees that the mapping $R(\cdot)$ takes values in $\mathbb{E}^n$.

Many processes in biology, control theory, electronics are described by impulsive differential inclusions with fuzzy right-hand side [7]:

$$\dot{x} \in F(t,x),\ t \neq t_i,\ x(0) = x_0,\quad \Delta x|_{t=t_i} \in I_i(x), \tag{2}$$

where $t_i \in I,\ i \in \overline{1,m}$ are the moments of impulses enumerated in the increasing order, $\Delta x|_{t=t_i} = x(t_i+0) - x(t_i-0) = x(t_i+0) - x(t_i)$ is the jump of the phase vector in the impulse moment t_i, $I_i \colon \mathbb{R}^n \to \mathbb{E}^n$ are fuzzy sets.

Let us introduce the notion of an R-solution of differential inclusion with fuzzy right-hand side (2):

Definition 5 ([29])**.** The fuzzy mapping $R\colon I \to \mathbb{E}^n,\ R(t_0) = X_0$ satisfying the following conditions: 1) on the intervals between impulses $R(\cdot)$ is upper semicontinuous and

$$\lim_{\sigma\downarrow 0}\frac{1}{\sigma}\sup_{\alpha\in[0,1]} h\Big([R(t+\sigma)]^\alpha, \bigcup_{x\in[R(t)]^\alpha}\Big\{x + \int_t^{t+\sigma}[F(s,x)]^\alpha ds\Big\}\Big) = 0;$$

2) $R(\cdot)$ is left-continuous at the moments of impulses t_i and $R(t_i+0) = \bigcup_{x\in R(t_i)}\{x + I_i(x)\}$ is called an *R-solution* of impulsive differential inclusion with the fuzzy right-hand side (2).

Obviously the existence and uniqueness of the R-solution of inclusion (2) holds if the fuzzy mapping $F(t,x)$ satisfies the assumptions of Theorem 2 on the intervals between impulses and the fuzzy mappings $I_i(x)$ are bounded, concave and satisfy the Lipschitz condition.

In [18, 19] the substantiation of the averaging method on the finite interval for differential inclusions with the fuzzy right-hand side with a small parameter is proposed. In the proof of the obtained theorems the proximity of the α-solutions is shown and therefore the proximity of solutions sets of the initial and averaged inclusions is proved. The similar results for impulsive differential inclusions with fuzzy right–hand side are obtained in [25, 26, 27, 28].

In [16, 17] the possibility of application of the averaging method in terms of an R-solution for differential inclusions with fuzzy right-hand side is considered. In [29] the results of [16, 17] were expanded to impulsive case for full averaging scheme. In this paper we consider the partial averaging scheme for impulsive differential inclusions with fuzzy right-hand side.

3. Averaging of impulsive differential inclusions with fuzzy right-hand side. Consider the impulsive differential inclusion with the fuzzy right-hand side

$$\dot{x} \in \varepsilon F(t,x),\ t \neq t_i,\ x(0) \in X_0,\quad \Delta x\,|_{t=t_i} \in \varepsilon I_i(x), \tag{3}$$

where $t \in \mathbb{R}_+$ is time, $x \in \mathbb{R}^n$ is a phase variable, $\varepsilon > 0$ is a small parameter, $F\colon \mathbb{R}_+\times\mathbb{R}^n \to \mathbb{E}^n$, $I_i\colon \mathbb{R}^n \to \mathbb{E}^n$ are fuzzy mappings, moments t_i are enumerated in the increasing order.

Associate with inclusion (3) the following partial averaged differential inclusion

$$\dot{\xi} \in \varepsilon \widetilde{F}(t,\xi),\ t \neq s_j,\ \xi(0) \in X_0,\quad \Delta\xi|_{t=s_j} \in \varepsilon K_j(\xi), \tag{4}$$

where the fuzzy mappings $\widetilde{F}\colon \mathbb{R}_+\times\mathbb{R}^n \to \mathbb{E}^n;\ K_j\colon \mathbb{R} \to \mathbb{E}^n$ satisfy the condition

$$\lim_{T\to\infty}\frac{1}{T}D\Bigg(\int\limits_t^{t+T} F(t,x)dt + \sum_{t\le t_i<t+T} I_i(x),\ \int\limits_t^{t+T}\widetilde{F}(t,x)dt + \sum_{t\le s_j<t+T} K_j(x)\Bigg) = 0, \tag{5}$$

moments s_j are enumerated in the increasing order.

Theorem 3. *Let in the domain* $Q = \{t \geq 0, x \in G \subset \mathbb{R}^n\}$ *the following conditions are fulfilled:*
1) *fuzzy mappings* $F(t,x), \widetilde{F}(t,x), I_i(x), K_j(x)$ *are continuous, uniformly bounded with constant* M, *concave in* x, *satisfy the Lipschitz condition in* x *with a constant* λ;
2) *uniformly with respect to* t, x *limit (5) exists and*

$$\frac{1}{T} i(t, t+T) \leq d < \infty, \ \frac{1}{T} j(t, t+T) \leq d < \infty,$$

where $i(t, t+T)$ *and* $j(t, t+T)$ *are the quantities of impulse moments* t_i *and* s_j *on the interval* $[t, t+T]$;
3) *R-solutions of inclusion (4) for all* $X_0 \subset G' \subset G$ *for* $t \in [0, L^*\varepsilon^{-1}]$ *belong to the domain* G *with a* ρ*-neighborhood.*

Then for any $\eta > 0$ *and* $L \in (0, L^*]$ *there exists* $\varepsilon_0(\eta, L) \in (0, \sigma]$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in [0, L\varepsilon^{-1}]$ *the inequality*

$$D(R(t,\varepsilon), \widetilde{R}(t,\varepsilon)) < \eta \tag{6}$$

holds, where $R(t,\varepsilon), \widetilde{R}(t,\varepsilon)$ *are R-solutions of inclusions (3) and (4),* $R(0,\varepsilon) = \widetilde{R}(0,\varepsilon)$.

Proof. For any $m > 1$ divide the interval $[0, L\varepsilon^{-1}]$ on m equal parts by the points $\tau_k = \frac{kL}{\varepsilon m}, k = \overline{0,m}$. Construct the fuzzy mappings $R^m(t,\varepsilon)$ and $\widetilde{R}^m(t,\varepsilon)$ such that

$$[R^m(t,\varepsilon)]^\alpha = \bigcup_{x \in [R^m(\tau_k,\varepsilon)]^\alpha} \left\{ x + \varepsilon \int_{\tau_k}^{t} [F(s,x)]^\alpha ds + \varepsilon \sum_{\tau_k \leq t_i < t} [I_i(x)]^\alpha \right\}, \ [R^m(0,\varepsilon)]^\alpha = [X_0]^\alpha,$$

$$[\widetilde{R}^m(t,\varepsilon)]^\alpha = \bigcup_{y \in [\widetilde{R}^m(\tau_k,\varepsilon)]^\alpha} \left\{ y + \varepsilon \int_{\tau_k}^{t} [\widetilde{F}(s,y)]^\alpha ds + \varepsilon \sum_{\tau_k \leq s_j < t} [K_j(y)]^\alpha \right\}, \ [\widetilde{R}^m(0,\varepsilon)]^\alpha = [X_0]^\alpha, \tag{7}$$

for all $t \in (\tau_k, \tau_{k+1}], \ k = \overline{0, m-1}, \ \alpha \in [0,1]$.

For $t \in (\tau_k, \tau_{k+1}]$ the inequality holds

$$D\big(R^m(\tau_k,\varepsilon), R^m(t,\varepsilon)\big) =$$

$$= \sup_{\alpha \in [0,1]} h\Bigg([R^m(\tau_k,\varepsilon)]^\alpha, \bigcup_{x \in [R^m(\tau_k,\varepsilon)]^\alpha} \left\{ x + \varepsilon \int_{\tau_k}^{t} [F(s,x)]^\alpha ds + \varepsilon \sum_{\tau_k \leq t_i < t} [I_i(x)]^\alpha \right\} \Bigg) \leq$$

$$\leq \varepsilon M (t - \tau_k) + \varepsilon M d (t - \tau_k) \leq \frac{ML(1+d)}{m}. \tag{8}$$

Similarly

$$D\big(\widetilde{R}^m(\tau_k,\varepsilon), \widetilde{R}^m(t,\varepsilon)\big) \leq \frac{ML(1+d)}{m}, \quad D\big(R(\tau_k,\varepsilon), R(t,\varepsilon)\big) \leq \frac{ML(1+d)}{m},$$

$$D\big(\widetilde{R}(\tau_k,\varepsilon), \widetilde{R}(t,\varepsilon)\big) \leq \frac{ML(1+d)}{m}.$$

Let us prove that

$$\lim_{m \to \infty} D\big(R^m(t,\varepsilon), R(t,\varepsilon)\big) = 0, \quad \lim_{m \to \infty} D\big(\widetilde{R}^m(t,\varepsilon), \widetilde{R}(t,\varepsilon)\big) = 0.$$

Let $t \in (\tau_k, \tau_{k+1}]$. Denote by $t_1^k, t_2^k, \dots, t_p^k$ the moments of impulses t_i on the interval $[\tau_k, \tau_{k+1}]$. Then

$$[R(t,\varepsilon)]^\alpha = \bigcup_{x \in [R(\tau_k,\varepsilon)]^\alpha} \; \bigcup_{v(s) \in [F(s,y(s))]^\alpha} \left\{ y(t) = x + \varepsilon \int_{\tau_k}^{t} v(s)ds \right\}, \; \tau_k \le t < t_1^k,$$

$$[R(t,\varepsilon)]^\alpha = \bigcup_{x \in [R(t_q^k,\varepsilon)]^\alpha} \; \bigcup_{v(\cdot),r} \left\{ y(t) = z + \varepsilon \int_{t_q^k}^{t} v(s)ds \colon \; \begin{array}{c} v(s) \in [F(s,y(s))]^\alpha, \\ z = x + \varepsilon r, \quad r \in [I_{t_q^k}(x)]^\alpha \end{array} \right\}, \tag{9}$$

$$t_q^k < t \le t_{q+1}^k, \; q = \overline{1,p}, \; t_{p+1}^k = \tau_{k+1}, \; \alpha \in [0,1].$$

Let $\delta_k = D\big(R(\tau_k,\varepsilon), R^m(\tau_k,\varepsilon)\big)$. Then for $t \in [\tau_k, t_1^k]$, $\alpha \in [0,1]$ we have

$$h\big([R(t,\varepsilon)]^\alpha, [R^m(t,\varepsilon)]^\alpha\big) =$$

$$= h\Bigg(\bigcup_{x \in [R(\tau_k,\varepsilon)]^\alpha} \; \bigcup_{v(s) \in [F(s,y(s))]^\alpha} \left\{ y(t) = x + \varepsilon \int_{\tau_k}^{t} v(s)ds \right\}, \bigcup_{z \in [R^m(\tau_k,\varepsilon)]^\alpha} \left\{ z + \varepsilon \int_{\tau_k}^{t} [F(s,z)]^\alpha ds \right\} \Bigg) \le$$

$$\le \sup_{x,z,v(\cdot)} \rho\left(x + \varepsilon \int_{\tau_k}^{t} v(s)ds, z + \varepsilon \int_{\tau_k}^{t} [F(s,z)]^\alpha ds \right) \le$$

$$\le \sup_{x,z,v(\cdot)} \left(\|x - z\| + \varepsilon \int_{\tau_k}^{t} \rho(v(s), [F(s,z)]^\alpha) ds \right) \le$$

$$\le \delta_k + \varepsilon \int_{\tau_k}^{t} \sup_{z,y} h\Big([F(s,y(s))]^\alpha, [F(s,z)]^\alpha\Big) ds \le \delta_k + \varepsilon\lambda \int_{\tau_k}^{t} \left[\sup_{x,y} \|y(s) - x\| + \delta_k \right] ds \le$$

$$\le \delta_k + \varepsilon\lambda \left[\varepsilon M \frac{L}{\varepsilon m} + \delta_k \right] \frac{L}{\varepsilon m} = \delta_k \left(1 + \frac{\lambda L}{m} \right) + \frac{\lambda M L^2}{m^2}.$$

For $t \in (t_q^k, t_q^{k+1}]$ and $\alpha \in [0,1]$ we get

$$h\big([R(t,\varepsilon)]^\alpha, [R^m(t,\varepsilon)]^\alpha\big) =$$

$$= h\Bigg(\bigcup_{x \in [R(\tau_k,\varepsilon)]^\alpha} \; \bigcup_{v(s) \in [F(s,y(s))]^\alpha} \left\{ y(t) = x + \varepsilon \int_{\tau_k}^{t} v(s)ds + \varepsilon \sum_{\tau_k \le t_i < t} \Delta_i, \; \Delta_i = \Delta y|_{t_i} \in [I_i(y(t_i))]^\alpha \right\},$$

$$\bigcup_{w \in [R^m(\tau_k,\varepsilon)]^\alpha} \left\{ w + \varepsilon \int_{\tau_k}^{t} [F(s,w)]^\alpha ds + \varepsilon \sum_{\tau_k \le t_i < t} [I_i(w)]^\alpha \right\} \Bigg) \le$$

$$\le \sup_{x,w,v(\cdot)} \rho\left(x + \varepsilon \int_{\tau_k}^{t} v(s)ds + \varepsilon \sum_{\tau_k \le t_i < t} \Delta_i, w + \varepsilon \int_{\tau_k}^{t} [F(s,w)]^\alpha ds + \varepsilon \sum_{\tau_k \le t_i < t} [I_i(w)]^\alpha \right) \le$$

$$\le \sup_{x,w,v(\cdot)} \left(\|x - w\| + \varepsilon \int_{\tau_k}^{t} \rho(v(s), [F(s,w)]^\alpha) ds + \varepsilon \sum_{\tau_k \le t_i < t} \rho(\Delta_i, [I_i(w)]^\alpha) \right) \le$$

$$\leq \delta_k + \varepsilon\lambda \int\limits_{\tau_k}^{t} \Big(\sup_{x,y} \|y(s) - x\| + \delta_k \Big) ds + \frac{\lambda d L}{m} \Big(\sup_{x,y} \|y(t_i) - x\| + \delta_k \Big) \leq$$

$$\leq \delta_k \Big(1 + \frac{\lambda L}{m} + \frac{\lambda d L}{m} \Big) + \varepsilon\lambda \frac{L}{\varepsilon m} \Big(\varepsilon M \frac{L}{\varepsilon m} + \varepsilon d M \frac{L}{\varepsilon m} \Big) + \frac{\lambda d L}{m} \Big(\varepsilon M \frac{L}{\varepsilon m} + \varepsilon d M \frac{L}{\varepsilon m} \Big) =$$

$$= \delta_k \Big(1 + \frac{\lambda L(1+d)}{m} \Big) + \frac{\lambda M L^2 (1+d)^2}{m^2}.$$

Then $\delta_{k+1} \leq \delta_k (1 + \frac{\lambda L(1+d)}{m}) + \frac{\lambda M L^2 (1+d)^2}{m^2}$, $\delta_0 = 0$. So

$$\delta_k \leq \frac{\lambda M L^2 (1+d)^2}{m^2} \cdot \frac{\big(1 + \frac{\lambda L(1+d)}{m}\big)^k - 1}{\frac{\lambda L(1+d)}{m}} \leq \frac{ML(1+d)}{m} \big(e^{\lambda L(1+d)} - 1 \big).$$

But

$$D(R(t,\varepsilon), R^m(t,\varepsilon)) \leq D(R(t,\varepsilon), R(\tau_k,\varepsilon)) + D(R(\tau_k,\varepsilon), R^m(\tau_k,\varepsilon)) + D(R^m(\tau_k,\varepsilon), R^m(t,\varepsilon)).$$

Therefore

$$D(R(t,\varepsilon), R^m(t,\varepsilon)) \leq \frac{ML(1+d)}{m} \big(e^{\lambda L(1+d)} + 1 \big). \tag{10}$$

Similarly we can get the estimate

$$D\big(\widetilde{R}(t,\varepsilon), \widetilde{R}^m(t,\varepsilon)\big) \leq \frac{ML(1+d)}{m} \big(e^{\lambda L(1+d)} + 1 \big). \tag{11}$$

Denote by $\sigma_k = D(R^m(\tau_k,\varepsilon), \widetilde{R}^m(\tau_k,\varepsilon))$. For $t \in (\tau_k, \tau_{k+1}]$, $k = \overline{0, m-1}$ and any $\alpha \in [0,1]$ we have

$$h([R^m(t,\varepsilon)]^\alpha, [\widetilde{R}^m(t,\varepsilon)]^\alpha) = h\Bigg(\bigcup_{x \in [R^m(\tau_k,\varepsilon)]^\alpha} \Big\{ x + \varepsilon \int\limits_{\tau_k}^{t} [F(s,x)]^\alpha ds +$$

$$+ \varepsilon \sum_{\tau_k \leq t_i < t} [I_i(x)]^\alpha \Big\}, \bigcup_{y \in [\widetilde{R}^m(\tau_k,\varepsilon)]^\alpha} \Big\{ y + \varepsilon \int\limits_{\tau_k}^{t} [\widetilde{F}(s,y)]^\alpha ds + \varepsilon \sum_{\tau_k \leq s_j < t} [K_j(y)]^\alpha \Big\} \Bigg) \leq$$

$$\leq \sup_{x,y} \Bigg(\|x - y\| + \varepsilon h\Big(\int\limits_{\tau_k}^{t} [F(s,x)]^\alpha ds + \sum_{\tau_k \leq t_i < t} [I_i(x)]^\alpha, \int\limits_{\tau_k}^{t} [\widetilde{F}(s,y)]^\alpha ds + \sum_{\tau_k \leq s_j < t} [K_j(y)]^\alpha \Big) \Bigg) \leq$$

$$\leq \sup_{x,y} \Bigg(\|x - y\| + \varepsilon h\Big(\int\limits_{\tau_k}^{t} [F(s,x)]^\alpha ds + \sum_{\tau_k \leq t_i < t} [I_i(x)]^\alpha, \int\limits_{\tau_k}^{t} [F(s,y)]^\alpha ds + \sum_{\tau_k \leq t_i < t} [I_i(y)]^\alpha \Big) +$$

$$+ \varepsilon h\Big(\int\limits_{\tau_k}^{t} \widetilde{[}F(s,y)]^\alpha ds + \sum_{\tau_k \leq t_i < t} [I_i(y)]^\alpha, \int\limits_{\tau_k}^{t} [\widetilde{F}(s,y)]^\alpha ds + \sum_{\tau_k \leq s_j < t} [K_j(y)]^\alpha \Big) \Bigg) \leq$$

$$\leq \sup_{x,y} \Bigg(\|x - y\| + \varepsilon\lambda \int\limits_{\tau_k}^{t} \|x - y\| ds + \varepsilon\lambda d \frac{L}{\varepsilon m} \|x - y\| \Bigg) + \varepsilon \frac{L}{\varepsilon m} \eta_1 \leq$$

$$\leq \sigma_k\big(1+\frac{\lambda L(1+d)}{m}\big)+\frac{L\eta_1}{m}\leq \frac{L\eta_1}{m}\cdot\frac{\big(1+\frac{\lambda L(1+d)}{m}\big)^k-1}{\frac{\lambda L(1+d)}{m}}\leq \frac{\eta_1\big(e^{\lambda L(1+d)}-1\big)}{\lambda(1+d)}. \tag{12}$$

From (10)–(12) we get

$$D\big(R(t,\varepsilon),\widetilde{R}(t,\varepsilon)\big)\leq \frac{\eta_1\big(e^{\lambda L(1+d)}-1\big)}{\lambda(1+d)}+\frac{2ML(1+d)}{m}\big(e^{\lambda L(1+d)}+1\big). \tag{13}$$

Choosing $m > 4ML(1+d)(e^{\lambda L(1+d)}+1)/\eta$ and $\eta_1 < \frac{\lambda(1+d)\eta}{2(e^{\lambda L(1+d)}-1)}$, from (13) we get the statement of the theorem. $\square$

4. Conclusion. We conclude with a few remarks.

Remark 2. In case when $\widetilde{F}(t,x)\equiv\widetilde{F}(x),\ K_j(x)\equiv\hat{0}$, Theorem 3 substantiates the scheme of the full averaging for impulsive differential inclusions with the fuzzy right-hand side [29].

Remark 3. If inclusions (3), (4) are periodic in t then estimate (6) can be improved

$$D\big(R(t,\varepsilon),\widetilde{R}(t,\varepsilon)\big)\leq C\varepsilon.$$

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