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B. V. Zabavsky, O. M. Romaniv

ALMOST ZIP BEZOUT DOMAIN

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J. Zelmanowitz introduced the concept of a ring, which we call a zip ring. In this paper we characterize a commutative Bezout domain whose finite homomorphic images are zip rings modulo its nilradical.

1. General Appearance. All rings considered will be commutative with identity. A ring is a Bezout ring if every finitely generated ideal is principal. I. Kaplansky ([5]) defined the class of elementary divisor rings as rings R for which every matrix A over R admits a diagonal reduction, i.e. there exist invertible matrices P and Q such that PAD is a diagonal matrix $D=(d_i)$ with the property that every d_i is a divisor of d_{i+1}. B. Zabavsky defined fractionally regular rings as rings R such that for which every nonzero and nonunit element a from R the classical quotient ring $Q_{cl}(R/\mathrm{rad}(aR))$ is the regular, where $\mathrm{rad}(aR)$ is the nilradical of aR [9]. We say that the ring R has stable range 2 if whenever $aR+bR+cR=R$, then there are $\lambda,\mu\in R$ such that $(a+c\lambda)R+(b+c\mu)R=R$. We say R is semi-prime if $\mathrm{rad}(R)=\{0\}$, where $\mathrm{rad}(R)$ is the nilradical of the ring R. Obviously, rings in which nonzero principal ideal has only finitely many minimal primes are examples of fractionally regular rings ([1]).

An ideal I of a ring R is called a J-radical if it is an intersection of maximal ideals, or, equivalently, if R/I has zero Jacobson radical. We call R J-Noetherian if it satisfies the ascending chain condition on J-radical ideals.

The annihilator of a ideal I of a ring R is denoted by $I^{\perp}=\{x\in R\mid ix=0\ \forall i\in I\}$.

Following C. Faith ([4]) a ring R is zip if I is an ideal and if $I^{\perp}=\{0\}$ than $I_0^{\perp}=\{0\}$ for a finitely generated ideal $I_0\subset I$. An ideal I of a ring R is dense if its annihilator is zero. Thus I is a dense ideal if and only if it is a faithful R-module. A ring R is a Kasch ring if $I^{\perp}\neq\{0\}$ for any ideal $I\neq R$.

Let R be a ring. Then the ring R has finite Goldie dimension if it contains a direct sum of finite number of nonzero ideals. A ring R is called a Goldie ring if it has finite Goldie dimension and satisfies the ascending chain condition for annihilators ([4, 6, 10]). By [4] we have the following result.

Theorem 1 ([4]). *Semiprime commutative ring R is zip if and only if R is a Goldie ring.*

Proposition 1 ([4]). *A commutative Kasch ring is zip.*

Proposition 2 ([4]). *If $Q_{cl}(R)$ is a Kasch ring then R is zip.*

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For further research we will need the following results.

Theorem 2 ([4])**.** *A commutative ring R is zip if and only if its classical ring of quotients $Q_{cl}(R)$ is zip.*

Theorem 3. *Let R be a commutative Bezout domain and $0 \neq a \in R$, then R/aR is a Kasch ring if and only if R is a ring in which any maximal ideal is principal.*

Proof. First we will prove that the annihilator of any principal ideal of R/aR is a principal ideal. Suppose $b \in R$ and $aR \subseteq bR$. Then $(b\colon a) = \{r \in R \mid br \in aR\} = sR$, where $a = bs$, so $(b\colon a) = aR$. We can also show that every principal ideal of R/aR is an annihilator of a principal ideal. Moreover, if $I_1 = J_1^{\perp}$, $I_2 = J_2^{\perp}$, where I_i, J_i, $i = 1, 2$, are principal ideals, then

$$(I_1 \cap I_2)^{\perp} = (J_1^{\perp} \cap J_2^{\perp})^{\perp} = ((J_1 + J_2)^{\perp})^{\perp} = J_1 + J_2 = J_1^{\perp} + J_2^{\perp}.$$

Let R/aR be a Kasch ring. Let $\overline{M}$ be a maximal ideal in R/aR. Denote $R/aR = \overline{R}$. Then $\overline{M}^{\perp} = \overline{H}$, where $\overline{H}$ is an ideal in $\overline{R} = R/aR$ and $\overline{H} = \{\overline{0}\}$. Since $\overline{H}$ annihilates the maximal ideal $\overline{M}$ then $\overline{H} \cdot \overline{M} = \{\overline{0}\}$. Since the maximal ideal $\overline{M}$ belongs to $\overline{H}^{\perp}$, by maximality of $\overline{M}$, $\overline{M} = \overline{H}^{\perp} \neq R/aR$.

Since $\overline{M}$ is a maximal ideal, for every element $\overline{d} \neq \overline{0}$ belonging to $\overline{H}$ we have the equality $\overline{dM} = \{\overline{0}\}$. Thus, the maximal ideal $\overline{M}$ belongs to $\overline{d}^{\perp}$, where $\overline{d}$ is a nonunit.

Hence $\overline{M} = \overline{d}^{\perp} = \overline{bR}$. Therefore, $\overline{M} = \overline{bR}$ and $M = bR + aR = cR$, because R is a commutative Bezout domain for some $c \in R$. Hence M is a maximal ideal which is a principal ideal.

Suppose that a maximal ideal M contains an element a, is a principal one considering its homomorphic image, and we have $\overline{M} = \overline{m}\overline{R} = (\overline{n}\overline{R})^{\perp}$. Since $\overline{m} \notin U(\overline{R})$, we have $(\overline{n}\overline{R})^{\perp} \neq \overline{R}$ and hence $\overline{n}\overline{R} \neq \{\overline{0}\}$.

As a result $\overline{M}^{\perp} = ((\overline{n}\overline{R})^{\perp})^{\perp} = \overline{n}\overline{R} \neq (\overline{0})$. Therefore, $\overline{M}^{\perp}$ is a nonzero principal ideal. This proves the fact that $\overline{R}$ is a Kasch ring. □

2. Almost zip Bezout domain. We start this section with the following statements.

Proposition 3. *Let R be a Bezout ring. Then R is zip if and only if every dense ideal contains a regular element.*

Proof. Suppose I is a dense ideal of a zip ring, and if I is a principal dense ideal contained in I, hence I is generated by a regular element. □

Theorem 4. *Let R be a semiprime commutative Bezout ring which is a Goldie ring. Then any minimal prime ideal of R is principal, generated by an idempotent, and there are only finitely many minimal prime ideals.*

Proof. The restrictions on R imply that the classical quotient ring $Q_{cl}(R)$ is an Artinian regular ring with finitely many minimal prime ideals. Let P be a minimal prime ideal of R. Consider the ideal $P_Q = \{\frac{p}{s} \mid p \in P\}$. It is obvious that P_Q is a prime ideal of $Q_{cl}(R)$. Since $Q_{cl}(R)$ is an Artinian regular ring, there exists an idempotent $e \in Q_{cl}(R)$ such that $P_Q = eQ_{cl}(R)$. Since R is an arithemical ring, we have $e \in R$ ([9]). For any $p \in P$ we obtain that $p = er$, where r is a von Neumann regular element, i.e. $rxr = r$ for some $x \in R$. Hence $ep = e^2r = er = p$, we have $P \subset eR$, $e \in P$, so $eR \subset P$ and $P = eR$. Since any minimal prime ideal of R is principal by [1], we have that R has finitely many minimal prime ideals. □

Definition 1. Let R be a commutative Bezout domain. A nonzero and nonunit element $a \in R$ is said to be an *almost zip element* if $R/\mathrm{rad}(aR)$ is a zip ring. A commutative Bezout domain is said to be an *almost zip ring* if any nonzero nonunit element of R is an almost zip element.

Theorem 5. *Let R be a commutative Bezout domain and let a be an almost zip element of R. Then there are only finitely many prime ideals minimal over aR.*

Proof. Since $R/\mathrm{rad}(aR)$ is a semiprime zip ring, by Theorem 1 we have that $R/\mathrm{rad}(aR)$ is a Goldie Bezout ring. By Theorem 4 we have that any minimal prime ideal of $R/\mathrm{rad}(aR)$ is principal and generated by an idempotent. Then there are only finitely many minimal prime ideals. Therefore, aR has finitely many minimal prime ideals. □

Consequently we have the following results.

Theorem 6. *An almost zip commutative Bezout domain is a J-Noetherian domain (i.e. Noetherian maximal spectrum).*

Proof. By Theorem 6 we have that any nonzero and nonunit element has finitely many minimal ideals. By [2] R is a J-Noetherian domain. □

Since a commutative J-Noetherian Bezout domain ([8]) is an elementary divisor ring by Theorem 6, we have the following results.

Theorem 7. *A commutative almost zip Bezout domain is an elementary divisor domain.*

Since a J-Noetherian Bezout domain is fractionally regular ring, we have the following result.

Theorem 8. *An almost zip Bezout domain is a fractionally regular domain.*

In the future we will consider the Bezout ring in which any maximal ideal is projective.

Theorem 9. *Let R be a commutative reduced Bezout ring in which any maximal ideal is projective. Then R is zip if and only if any maximal ideal is principal.*

Proof. Let R be a zip ring and M be any maximal ideal of R.

1) If $M^\perp \neq (0)$ then $M^\perp \cap M = \{0\}$. Really, if $m \in M \cap M^\perp$, and $m \neq 0$ then $m^2 = 0$. Since R is a reduced ring, we have $m = 0$ and this is a contradiction. Since M is a maximal ideal and $M \cap M^\perp = \{0\}$, we obtain $M + M^\perp = R$, i.e. $1 = m + n$ for some $m \in M$ and $n \in M^\perp$. From here, we have $m = m(m+n) = m^2 + mn = m^2$, i.e. $m^2 = m$. Then for any $k \in M$ one has $k = k(m+n) = km$, i.e. $M \subseteq mR$. Since $m \in M$ we have $M = mR$, where $m^2 = m$.

2) Let $M^\perp = \{0\}$. Since R is a zip Bezout ring, there exists a principal ideal sR, such that $sR^\perp = \{0\}$. Since R is a commutative Bezout ring, we obtain that s is a regular element (i.e. non zero divisor of R). By [3] M is a principal ideal.

Let R be a ring in which any maximal ideal is principle. According to the restrictions imposed on the ring, R is a zip ring. □

Definition 2. Let R be a commutative ring. We say that an ideal $I \subset R$ is *pure* if the quotient ring R/I is flat over R.

By [3] we have the following result.

Proposition 4. *An ideal I of a commutative ring R is pure if and only if for every $a \in I$ there exists an element $b \in I$ such that $ab = a$.*

By Proposition 4 we have that if I is a pure ideal and $I \subseteq J(R)$, then $I = (0)$, i.e. $J(R)$ is pure if and only if $J(R) = 0$.

Let R be a commutative Bezout domain and $a \in R\backslash\{0\}$. By [4], Example 3, R/aR is a ring in which every injective module is flat. It is clear that R/aR is a ring in which every R/aR-module can be embedded in a flat R/aR-module. By [7], Proposition 2.6, we have if $J(R/aR)$ is a flat ideal of R/aR, then $R/aR/J(R/aR)$ is flat, i.e. by Proposition 4 $J(R/aR)$ is a pure ideal.

Since R/aR is a ring in which every injective module is flat, we have the following result.

Theorem 10. *Let R be a commutative Bezout domain and $a \in R\backslash\{0\}$. Then $J(R/aR)$ is pure flat or injective R/aR-module if and only if for any decomposition $a = bc$ where $b, c \notin U(R)$ we have $bR + cR = R$.*

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Department of Algebra and Logic
Ivan Franko National University Lviv, Ukraine
zabavskii@gmail.com
oleh.romaniv@lnu.edu.ua

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