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COMMUTATIVE PERIODIC GROUP RINGS

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We find a satisfactory criterion when a commutative group ring $R(G)$ is periodic only in terms of R, G and their sections, provided that R is local.

1. Introduction and fundamentals. Throughout the text of this brief article, unless specified something else, all rings under consideration are assumed to be associative and commutative, containing identity element 1 which differs from the zero element 0. Our notions and notations are at all standard being mainly in agreement with [10] and [11], respectively. For example, for such a ring R, the symbol $J(R)$ denotes the Jacobson radical of R whereas the symbol $N(R)$ denotes the nil-radical of R; it is principally known that the inclusion $N(R) \subseteq J(R)$ is always true. Likewise, everywhere in the text, we shall assume that G is a multiplicative abelian group, and $R(G)$ is the group ring of G over R. As usual, for a prime p, G_p stands for the p-torsion component of the group G, and we shall say that the group G is a p-group whenever $G = G_p$. The torsion part of G is denoted by G_t which equals to $G_t = \coprod_p G_p$ saying that G is torsion, provided $G = G_t$. The more specific terminology will be stated explicitly below.

A ring R is known to be *regular* (in the sense of von Neumann), called in the commutative case also *strongly regular*, provided for any $r \in R$ the existence of an element $a \in R$ depending on r such that $r = rar = r^2a$. On a similar vein, a ring R is said to be π*-regular* if, for each $r \in R$, there are $i \in \mathbb{N}$ and $a \in R$ with the property $r^i = r^i a r^i = r^{2i}a$. Clearly, regular rings are π-regular and this implication is not reversible in general. About the "Group Ring Problem", it was established in [7, Theorem 2.4] (compare also with [3]) the following: *Suppose that R is a ring such that $p \cdot 1 \in N(R)$ for some prime number p, and suppose that G is a group. Then the group ring $R(G)$ is π-regular if, and only if, R is a π-regular ring and G is a torsion group.*

On the other side, a (not necessarily commutative) ring R is called *periodic* if, for every $r \in R$, there exist two natural numbers $m \neq n$ with the property $r^m = r^n$. A recent systematic study of these rings was done in [4] and [1, 2], respectively. In fact, by a simple restatement of [4, Theorem 3.4] in an equivalent way, it was shown there that *a ring R is periodic $\iff$ R is a π-regular ring whose units are torsion.* We will use exactly this necessary and sufficient condition for our applicable purposes in the sequel.

So, the goal for writing up this short paper is to strengthen the alluded to above assertions pertaining to commutative π-regular group rings by finding a suitable criterion when a commutative group ring is periodic in terms associated only with R, G and their sections.

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2. The criterion. Before proving our chief affirmation, we need two key technical claims. In fact, appealing to [8], one writes the formula

$$J(R(G)) = J(R)(G) + \langle r(g-1) \mid g \in G_p, pr \in J(R)\rangle.$$

In that aspect, consulting with [9], one writes the formula

$$N(R(G)) = N(R)(G) + \langle r(g-1) \mid g \in G_p, pr \in N(R)\rangle.$$

The following technicality is useful, too.

Lemma 1 ([5], Theorem 10)**.** *Suppose that F is a field of characteristic $p > 0$ and G is a group. Then $F(G)$ has only torsion units precisely when F has only torsion units and G is torsion.*

We now have all the machinery needed to prove the following chief statement, which gives a necessary and sufficient condition when a (commutative) group ring is periodic, provided additionally that the ring R must be local, that is, it possesses only a unique maximal ideal.

Theorem 1. *Let R be a local ring with $p \cdot 1 \in N(R)$ for some prime p and let G be a group. Then $R(G)$ is periodic if, and only if, R is periodic and G is torsion.*

Proof. Necessity. Since there is an epimorphism $R(G) \to R$, defined as the standard augmentation map, it easily follows that R is periodic, provided so is $R(G)$. Besides, since periodic rings are themselves π-regular, the aforementioned [7, Theorem 2.4] is applicable to get that G is torsion.

Sufficiency. First of all, let us mention that $J(R) = N(R)$ here. The natural map $R \to R/N(R)$ induces the element-wise defined surjective homomorphism $R(G) \to (R/N(R))(G)$ having the nil kernel $N(R)(G) \subseteq N(R(G))$ as showed the listed above formula of May from [9], whence the next isomorphism is fulfilled

$$R(G)/N(R)(G) \cong (R/N(R))(G).$$

On the other hand, since the units of $R/N(R)$ can be translated isomorphically to these of R modulo $1+N(R)$ and since all of the units of R are torsion, it is clear that the same holds for the units of $R/N(R)$ as well. Furthermore, as $R/N(R)$ is a field of characteristic p, Lemma 1 applies to deduce that $(R/N(R))(G)$, and hence $R(G)/N(R)(G)$, are simultaneously with torsion units. Utilizing a combination of [7, Theorem 2.4] and the noted above common restatement of [4, Theorem 3.4], one derives that $(R/N(R))(G)$ is a periodic ring. Applying now [4, Corollary 3.7], one finally concludes that $R(G)$ has to be periodic, as claimed. □

We close with one more helpful comment.

Remark 1. *We, however, conjecture that for arbitrary (commutative) R with $p \cdot 1 \in N(R)$, which is not necessarily local, the group ring $R(G)$ is periodic if, and only if, R is periodic and G is torsion, but the complete proof still eludes us.*

If R is a commutative periodic ring, then what be said in view of [4, Corollary 3.9] is that the quotient ring $R/N(R)$ is a potent ring, that is, each element in this factor-ring is a solution of the equation $x^n = x$ for some fixed $n \in \mathbb{N}$. However, any potent ring is necessarily strongly regular and thus it is a subdirect sum of fields, which are algebraic extensions of finite fields.

We close the work with the next two problems of some interest.

Problem 1. *Find a criterion when an arbitrary (not necessarily commutative) group ring is periodic.*

Problem 2. *Characterize those rings R for which there are two distinct fixed positive integers m, n such that, for each $r \in R$, $r^m = r^n$ or $r^m = -r^n$.*

It is worthwhile noticing that this class of rings properly contains both the classes of periodical rings and weakly $J(n)$-rings, where the latter one was completely described in [6].

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