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NEW BIPARAMETRIC FAMILIES OF APOSTOL-FROBENIUS-EULER POLYNOMIALS OF LEVEL m

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We introduce two biparametric families of Apostol-Frobenius-Euler polynomials of level m. We give some algebraic properties, as well as some other identities which connect these polynomial class with the generalized λ-Stirling type numbers of the second kind, the generalized Apostol–Bernoulli polynomials, the generalized Apostol–Genocchi polynomials, the generalized Apostol–Euler polynomials and Jacobi polynomials. Finally, we will show the differential properties of this new family of polynomials.

1. Introduction. Throughout this paper, we use the following standard notions: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denotes the set of integers numbers, the set of real numbers and the set of complex numbers, respectively. Furthermore, $(\lambda)_0 = 1$ and $(\lambda)_k = \lambda(\lambda + 1)(\lambda + 2) \ldots (\lambda + k - 1)$, where $k \in \mathbb{N}, \lambda \in \mathbb{C}$. For the complex logarithm, we consider the principal branch, and $w = z^\alpha$ we denote the single branch of the a multiple-valued function $w = z^\alpha$ such that $1^\alpha = 1$. We take also $0^0 = 1$ and $0^n = 0$ if $n \in \mathbb{N}$.

The generating functions for the special polynomials are important from different view points and help in finding connection formulas, recursive relations, difference equations, and in solving problems in combinatorics and encoding their solutions. In particular, the Frobenius–Euler polynomials appear in the integral representation of differentiable periodic functions since they are employed for approximating such functions in terms of polynomials (see [10, 13, 16–20, 23]). Also, these polynomials play an important role in the number of theories and classical analysis. In this paper, we focus our attention on introducing two new biparametric class of Apostol–Frobenius–Euler polynomials of level m considering the works of [9, 14]. Then, we can prove that such a new polynomial class preserves some similar algebraic and differential properties as the generalized Apostol–type polynomials, that as an immediate consequence, we recover many known algebraic and differential properties of such polynomials.

For parameters $\lambda, u \in \mathbb{C}$ and $a, b, c \in \mathbb{R}^+$, with $a \neq b$, $b > 1$ and $a \geq 1$; the Apostol type Frobenius–Euler polynomials $H_n(x; \lambda; u)$, $n \geq 0$, and the generalized Apostol-type Frobenius–Euler polynomials $H_n^{(\alpha)}(x; a, b, c; \lambda; u)$, $n \geq 0$, are defined by means of the following generating functions (see [1, p. 2, Definition 2])

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$$\Big(\frac{1-u}{\lambda e^z-u}\Big)e^{xz}=\sum_{n=0}^{\infty}H_n(x;\lambda;u)\frac{z^n}{n!},\qquad |z|<\big|\ln\big(\lambda/u\big)\big|$$

and

$$\Big(\frac{a^z-u}{\lambda b^z-u}\Big)^{\alpha}c^{xz}=\sum_{n=0}^{\infty}H_n^{(\alpha)}(x;a,b,c;\lambda;u)\frac{z^n}{n!},\qquad |z|<\Big|\frac{\ln\big(\lambda/u\big)}{\ln b}\Big|. \tag{1}$$

Observe thet if $x=0$ and $\alpha=1$ in (1), we get $\frac{a^z-u}{\lambda b^z-u}=\sum_{n=0}^{\infty}H_n(a,b,c;\lambda;u)\frac{z^n}{n!}$, where $H_n(a,b,c;u;\lambda)$ denotes the generalized Apostol-type Frobenius–Euler numbers (cf. [15]). It is well-known that $H_n^{(\alpha)}(x;u):=H_n^{(\alpha)}(x;1,e,e;1;u)$, $|z|<|\ln(u)|$, is the generalized Frobenius–Euler polynomial of order α, where $u\in\mathbb{C}\setminus\{1\}$ and $\alpha\in\mathbb{Z}$. Observe that $H_n^{(1)}(x;u)=H_n(x;u)$, denotes the classical Frobenius–Euler polynomials, $H_n^{(\alpha)}(0;u)=H_n^{(\alpha)}(u)$, denotes the Frobenius–Euler numbers of order α, and $H_n(x;-1)=E_n(x)$ denotes the Euler polynomials (see [3,6,11,12,24]).

For real parameters x and y the Taylor series representation in $z=0$ of the following functions $e^{xz}\cos(yz)$ and $e^{xz}\sin(yz)$ are given by (see [7])

$$F_c(z;x;y)=e^{xz}\cos(yz)=\sum_{k=0}^{\infty}C_k(x,y)\frac{z^k}{k!},\quad F_s(z;x;y)=e^{xz}\sin(yz)=\sum_{k=0}^{\infty}S_k(x,y)\frac{z^k}{k!}. \tag{2}$$

The expressions $C_k(x,y)$ and $S_k(x,y)$ are given by

$$C_k(x;y)=\sum_{j=0}^{\left[\frac{k}{2}\right]}(-1)^j\binom{k}{2j}x^{k-2j}y^{2j},\quad S_k(x;y)=\sum_{j=0}^{\left[\frac{k-1}{2}\right]}(-1)^j\binom{k}{2j+1}x^{k-2j-1}y^{2j+1}.$$

Now, let us give some properties of the generalized Apostol–type Frobenius–Euler polynomials and generalized Apostol–type Frobenius–Euler polynomials of order α with parameters λ,a,b,c (cf. [6,8]).

Proposition 1. *Let* $\big(H_n^{(\alpha)}(x;u)\big)$ *and* $\big(H_n^{(\alpha)}(x;u;a,b,c;\lambda)\big)$ *be the sequences of generalized Apostol–type Frobenius–Euler polynomials and generalized Frobenius–Euler polynomials, respectively. The following statements hold:* 1. *(Special values)* $H_n^{(0)}(x;u)=x^n$ *for* $n\in\mathbb{N}_0$. 2. *(Summation formulas)* $H_n^{(\alpha)}(x;u;a,b,c;\lambda)=\sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(x;u;a,b,c;\lambda)(x\ln c)^{n-k}$,

$$H_n^{(\alpha+\beta)}(x+y;u;a,b,c;\lambda)=\sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(x;u;a,b,c;\lambda)H_{n-k}^{(\beta)}(y;u;a,b,c;\lambda),$$

$$((x+y)\ln c)^n=H_{n-k}^{(\alpha)}(y;u;a,b,c;\lambda)H_k^{(-\alpha)}(x;u;a,b,c;\lambda),$$

$$H_n^{(-\alpha)}(x;u^2;a^2,b^2,c^2;\lambda^2)=\sum_{k=0}^{n}\binom{n}{k}H_k^{(-\alpha)}(x;u;a,b,c;\lambda)H_{n-k}^{(-\alpha)}(x;-u;a,b,c;\lambda).$$

Consider $m\in\mathbb{N}$, $\alpha,\lambda,u\in\mathbb{C}$ and $a,c\in\mathbb{R}_+$. The generalized Apostol-type Frobenius–Euler polynomials of the variable x, parameters a,b,λ, order α and level m, are defined through the following generating function (see [14, p. 397, equation (3.1)])

$$\mathcal{F}^{[m-1,\alpha]}(z;x;a;c;\lambda;u)=\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda c^z-u^m}\right]^{\alpha}c^{xz}=\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,\alpha]}(x;a,c;\lambda;u)\frac{z^n}{n!}. \tag{3}$$

Let a,$b \in \mathbb{R}_+$, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{N}_0$. The gerenalized λ-Stirling type numbers of the second kind $S(n, \alpha; a, b; \lambda)$ are defined by means of the following function (see [15, p. 3, equation (1)])

$$f_{S,\alpha}(z; a, b; \lambda) = \frac{(\lambda b^z - a^z)^\alpha}{\alpha!} = \sum_{n=0}^{\infty} S(n, \alpha; a, b; \lambda)\frac{z^n}{n!}. \tag{4}$$

The Jacobi polynomials of the degree n and order (α, β), with $\alpha, \beta > -1$, the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ may be defined through Rodrigues' formula

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left\{(1-x)^{n+\alpha}(1+x)^{n+\alpha}\right\}, \tag{5}$$

and the values at the end points of the interval $[-1, 1]$ is given by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}, \qquad P_n^{(\alpha,\beta)}(-1) = (-1)^n\binom{n+\beta}{n}.$$

The relationship between the n-th monomial x^n and the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ may be written as (see [14, p. 395, (21)])

$$x^n = n!\sum_{k=0}^{n}\binom{n+\alpha}{n-k}(-1)^k\frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n+1}}P_k^{(\alpha,\beta)}(1-2x). \tag{6}$$

Let a, b, $c \in \mathbb{R}_+$ $(a \neq b)$ and $n \in \mathbb{N}_0$. Then the generalized Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{C}$ are defined for z, $|z| < |\frac{\ln(\lambda)}{\ln(a/b)}|$, $x \in \mathbb{R}$, by the following generating function (see [21, p. 254, (20)])

$$\mathcal{F}_{\mathcal{B}}^{\alpha}(z; x; a, b, c; \lambda; \alpha) = \left(\frac{z}{\lambda b^z - a^z}\right)^{\alpha} c^{xz} = \sum_{k=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x; \lambda; a, b, c)\frac{z^n}{n!}. \tag{7}$$

Let a, b, $c \in \mathbb{R}_+$ $(a \neq b)$ and $n \in \mathbb{N}_0$. Then the generalized Apostol–Euler polynomials $\mathcal{E}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{C}$ are defined for z, $|z| < |\frac{\ln(\lambda)}{\ln(a/b)}|$, $x \in \mathbb{R}$, by the following generating function (see [22, p. 254, (23)])

$$\mathcal{F}_{\mathcal{E}}^{\alpha}(z; x; a, b, c; \lambda; \alpha) = \left(\frac{2}{\lambda b^z + a^z}\right)^{\alpha} c^{xz} = \sum_{k=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x; \lambda; a, b, c)\frac{z^n}{n!}. \tag{8}$$

Let a, b, $c \in \mathbb{R}_+$ $(a \neq b)$ and $n \in \mathbb{N}_0$. Then the generalized Apostol–Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{C}$ are defined for z, $|z| < |\frac{\ln(\lambda)}{\ln(a/b)}|$, $x \in \mathbb{R}$, by the following generating function (see [22, p. 300, (70)])

$$\mathcal{F}_{\mathcal{G}}^{\alpha}(z; x; a, b, c; \lambda; \alpha) = \left(\frac{2z}{\lambda b^z + a^z}\right)^{\alpha} c^{xz} = \sum_{k=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x; \lambda; a, b, c)\frac{z^n}{n!}. \tag{9}$$

2. Biparametric families of the m-level Apostol-Frobenius-Euler polynomials $\mathcal{H}_{n,c}^{[m-1,\alpha]}(x; y; a; \lambda; u)$ and $\mathcal{H}_{n,s}^{[m-1,\alpha]}(x; y; a; \lambda; u)$. In view of the results in Section 1 and [9,14] we are going to introduce two new of level m biparametric families of Apostol-Frobenius-Euler polynomials.

Definition 1. For $m \in \mathbb{N}$, $\alpha, \lambda, u \in \mathbb{C}$, $u^m \neq \lambda$ and $a \in \mathbb{R}_+$, the *generalized Apostol-type Frobenius–Euler polynomials* $(\mathcal{H}_{n,c}^{[m-1,\alpha]}(x,y;a;\lambda;u))$, $(\mathcal{H}_{n,s}^{[m-1,\alpha]}(x,y;a;\lambda;u))$ in the variable x and y, with parameters a, λ, u order α and level m, are defined for z, $|z| < \left|\ln(u^m/\lambda)\right|$, by the following generating functions:

$$\mathcal{F}_c^{[m-1,\alpha]}(z;x;y;a;\lambda;u) = \left[\frac{\sum_{h=0}^{m-1} \frac{(z\ln a)^h}{h!} - u^m}{\lambda e^z - u^m}\right]^\alpha e^{xz}\cos(yz) = \sum_{n=0}^{\infty} \mathcal{H}_{n,c}^{[m-1,\alpha]}(x,y;a;\lambda;u)\frac{z^n}{n!}, \tag{10}$$

$$\mathcal{F}_s^{[m-1,\alpha]}(z;x;y;a;\lambda;u) = \left[\frac{\sum_{h=0}^{m-1} \frac{(z\ln a)^h}{h!} - u^m}{\lambda e^z - u^m}\right]^\alpha e^{xz}\sin(yz) = \sum_{n=0}^{\infty} \mathcal{H}_{n,s}^{[m-1,\alpha]}(x,y;a;\lambda;u)\frac{z^n}{n!}. \tag{11}$$

If in (10) $y = 0$, we obtain the generalized Apostol-Frobennius–Euler polynomials of parameters $\lambda, u \in \mathbb{C}$, $a \in \mathbb{R}_+$, order $\alpha \in \mathbb{C}$ and level m

$$\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;0;a;\lambda;u) := \mathcal{H}_n^{[m-1,\alpha]}(x;a;\lambda;u).$$

According to Definition 1, with $m = 1$ and $y = 0$, we have

$$\mathcal{H}_{n,c}^{[0,\alpha]}(x;0;1;\lambda;u) = H_n^{(\alpha)}(x;\lambda;u), \quad \mathcal{H}_{n,c}^{[0,1]}(x;0;1;\lambda;u) = H_n^{(1)}(x;\lambda;u).$$

Theorem 1. *Let* $\left(\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ *and* $\left(\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ *be the sequences of biparametric Apostol-type Frobenius–Euler polynomials of order* α *and level* m*. Then for* $m \in \mathbb{N}$ *the following statements hold*

1. *For* $\alpha, \lambda, u \in \mathbb{C}$ *and* $n \in \mathbb{N}_0$ $a \in \mathbb{R}_+$ *we have the relationship*

$$\mathcal{H}_n^{[m-1,\alpha]}(x+iy;a;\lambda;u) = \mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u) + i\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u). \tag{12}$$

2. *(Addition theorem of the argument) For* $\alpha, \lambda, u \in \mathbb{C}$ *and* $n, k \in \mathbb{N}_0$ $a \in \mathbb{R}_+$,

$$\mathcal{H}_{n,c}^{[m-1,\alpha]}(x+y;y,a;\lambda;u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)y^k, \tag{13}$$

$$\mathcal{H}_{n,s}^{[m-1,\alpha]}(x+y;y,a;\lambda;u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k,s}^{[m-1,\alpha]}(x;y;a,b;\lambda;u)y^k, \tag{14}$$

$$\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;x+iy,a;\lambda;u) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} \mathcal{H}_{n-2k,c}^{[m-1,\alpha]}(x;x;a;\lambda;u)y^{2k} - \\ - i\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} \mathcal{H}_{n-1-2k,s}^{[m-1,\alpha]}(x;x;a;\lambda;u)y^{2k+1}, \tag{15}$$

$$\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;x+iy,a;\lambda;u) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} \mathcal{H}_{n-2k,s}^{[m-1,\alpha]}(x;x;a;\lambda;u)y^{2k} - \\ - i\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} \mathcal{H}_{n-1-2k,c}^{[m-1,\alpha]}(x;x;a;\lambda;u)y^{2k+1}. \tag{16}$$

Proof of (12). From (10), we have

$$\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,\alpha]}(x+iy,a;\lambda;u)\frac{z^n}{n!}=\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda e^z-u^m}\right]^{\alpha}e^{(x+iy)z}=\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda e^z-u^m}\right]^{\alpha}\times$$

$$\times e^{xz}\left[\cos(yz)+i\sin(yz)\right]=\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}+i\sum_{n=0}^{\infty}\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}.$$

Proof of (1). Since $\cos((x+iy)z)=\cos(xz)\cosh(yz)-i\sin(xz)\sinh(yz)$, we successively obtain

$$\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,\alpha]}(x;x+iy;a;\lambda;u)\frac{z^n}{n!}=\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda e^z-u^m}\right]^{\alpha}e^{xz}\cos((x+iy)z)=$$

$$=\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda e^z-u^m}\right]^{\alpha}e^{xz}\cos(xz)\cosh(yz)-$$

$$-i\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda e^z-u^m}\right]^{\alpha}e^{xz}\sin(xz)\sinh(yz)=$$

$$=\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;x;a;\lambda;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\frac{y^{2n}z^{2n}}{2n!}-i\sum_{n=0}^{\infty}\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;x;a;\lambda;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\frac{y^{2n+1}z^{2n+1}}{(2n+1)!}=$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{2k}\mathcal{H}_{n-2k,c}^{[m-1,\alpha]}(x;x;a;\lambda;u)y^{2k}\frac{z^n}{n!}-$$

$$-i\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}\mathcal{H}_{n-1-2k,s}^{[m-1,\alpha]}(x;x;a;\lambda;u)y^{2k+1}\frac{z^n}{n!}.$$

□

Theorem 2. *For $m\in\mathbb{N}$, let $\left(\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ and $\left(\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ be the sequence of biparametric Apostol-type Frobenius–Euler polynomials, whit parameters $\lambda,u\in\mathbb{C}$ and $a\in\mathbb{R}_+$, of the order $\alpha\in\mathbb{N}_0$ and level m. Then the following statements hold*

$$\sum_{k=0}^{n}\binom{n}{k}C_k(x;y)\mathcal{H}_{n-k}^{[m-1,\alpha]}(0;a;0;u)=(-1)^{\alpha}\alpha!\sum_{k=0}^{n}\binom{n}{k}S\Big(n-k,\alpha,1,e;\frac{\lambda}{u^m}\Big)\mathcal{H}_{k,c}^{[m-1,\alpha]}(x;y;a;\lambda;u),\tag{17}$$

$$\sum_{k=0}^{n}\binom{n}{k}S_k(x;y)\mathcal{H}_{n-k}^{[m-1,\alpha]}(0;a;0;u)=(-1)^{\alpha}\alpha!\sum_{k=0}^{n}\binom{n}{k}S\Big(n-k,\alpha,1,e;\frac{\lambda}{u^m}\Big)\mathcal{H}_{k,s}^{[m-1,\alpha]}(x;y;a;\lambda;u).$$

Proof of (17). From (2), (4) and (10) we obtain the following equality

$$u^{m\alpha}\alpha!f_{S,\alpha}\left(z;1;e;\frac{\lambda}{u^m}\right)\mathcal{F}_c^{[m-1,\alpha]}(z;x;y;a;\lambda;u)=\left[\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m\right]^{\alpha}F_c(z;x;y).$$

Hence

$$\frac{(-1)^{\alpha}}{(-1)^{\alpha}}\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{u^m}\right]^{\alpha}\sum_{n=0}^{\infty}C_n(x;y)\frac{z^n}{n!}=$$

$$=\alpha!\sum_{n=0}^{\infty}S\left(n,\alpha,1,e;\frac{\lambda}{u^m}\right)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a,b;\lambda;u)\frac{z^n}{n!}.$$

Then

$$(-1)^{\alpha}\sum_{n=0}^{\infty}\mathcal{H}_{n}^{[m-1,\alpha]}(0;a;0;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}C_n(x;y)\frac{z^n}{n!}=$$

$$=\alpha!\sum_{n=0}^{\infty}S\left(n,\alpha,1,e;\frac{\lambda}{u^m}\right)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!},$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}C_k(x;y)\mathcal{H}_{n-k}^{[m-1,\alpha]}(0;a;0;u)\frac{z^n}{n!}=$$

$$=(-1)^{-\alpha}\alpha!\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}S\left(n-k,\alpha,1,e;\frac{\lambda}{u^m}\right)\mathcal{H}_{k,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}.$$

The proof of the second equality from Theorem 2 it is analogously. □

Theorem 3. *For $m\in\mathbb{N}$, let $\left(\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ and $\{\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a,b;\lambda;u)\}$ be the sequence of biparametric Apostol-type Frobenius–Euler polynomials, with parameters $\lambda,u\in\mathbb{C}$ and $a\in\mathbb{R}_+$, of order $\alpha\in\mathbb{C}$ and level m. Then the following statements hold*

$$\mathcal{H}_{n,s}^{[m-1,\alpha+\beta]}(2x;2y,a;\lambda;u)=2\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_{n-k,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\mathcal{H}_{k,s}^{[m-1,\beta]}(x;y;a;\lambda;u). \tag{18}$$

Proof. The following equality is constructed from (10) and (11)

$$\mathcal{F}_{s}^{[m-1,\alpha+\beta]}(z;2x;2y;a;\lambda;u)=2\mathcal{F}_{c}^{[m-1,\alpha]}(z;x;y;a;\lambda;u)\mathcal{F}_{s}^{[m-1,\beta]}(z;x;y;a;\lambda;u).$$

Therefore

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,s}^{[m-1,\alpha+\beta]}(2x;2y;a;\lambda;u)\frac{z^n}{n!}=2\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_{n,s}^{[m-1,\beta]}(x;y;a;\lambda;u)\frac{z^n}{n!}=$$

$$=2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_{n-k,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\mathcal{H}_{k,s}^{[m-1,\beta]}(x;y;a;\lambda;u)\frac{z^n}{n!}.$$

□

Theorem 4. *For $m\in\mathbb{N}$, let $\left(\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ and $\left(\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ be the sequence of biparametric Apostol-type Frobenius–Euler polynomials, with parameters $\lambda,u\in\mathbb{C}$ and $a\in\mathbb{R}_+$, of order $\alpha\in\mathbb{C}$ and level m. Then the following statements hold*

$$\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{2k}\mathcal{H}_{n-2k}^{[m-1,\alpha]}(x;a;\lambda;u)y^{2k}. \tag{19}$$

$$\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)=\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}\mathcal{H}_{n-1-2k}^{[m-1,\alpha]}(x;a;\lambda;u)y^{2k+1}. \tag{20}$$

Proof of (19). The following equality follows from (3) and (10).

$$\mathcal{F}_c^{[m-1,\alpha]}(z;x;y;a;\lambda;u)=\mathcal{F}^{[m-1,\alpha]}(z;x;a;\lambda;u)\cos(yz).$$

Thus

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}=\sum_{n=0}^{\infty}\mathcal{H}_{n}^{[m-1,\alpha]}(x;a;\lambda;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}z^{2n}}{(2n)!}.$$

Hence

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{2k}\mathcal{H}_{n-2k}^{[m-1,\alpha]}(x;a;\lambda;u)y^{2k}\frac{z^n}{n!}.$$

The proof of (20) is analogous to (19), using the fact that

$$\mathcal{F}_s^{[m-1,\alpha]}(z;x;y;a;\lambda;u)=\mathcal{F}^{[m-1,\alpha]}(z;x;a,e;\lambda;u)\sin(yz)$$

from the relationships given in (3) and (11). □

Theorem 5. *For $m\in\mathbb{N}$ the biparametric Apostol-type Frobenius–Euler polynomials $\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y,a;\lambda;u)$, $\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)$ of level m are related to the generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda;a,c)$ by means of the following identities*

$$\mathcal{H}_{n-\alpha,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)= \tag{21}$$

$$=\frac{(-1)^\alpha}{\alpha!\binom{n}{\alpha}}\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{j}{2}\right]}(-1)^k\binom{n}{j}\binom{j}{2k}\mathcal{H}_{n-j}^{[m-1,\alpha]}(0;a;0;u)\mathcal{B}_{j-2k}^{(\alpha)}\left(x;\frac{\lambda}{u^m};a,e\right)y^{2k}.$$

$$\mathcal{H}_{n-\alpha,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)= \tag{22}$$

$$=\frac{(-1)^\alpha}{\alpha!\binom{n}{\alpha}}\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{j-1}{2}\right]}(-1)^k\binom{n}{j}\binom{j}{2k+1}\mathcal{H}_{n-j}^{[m-1,\alpha]}(0;a;0;u)\mathcal{B}_{j-1-2k}^{(\alpha)}\left(x;\frac{\lambda}{u^m};a,e\right)y^{2k+1}.$$

Proof of (21)*.* The following equality follows from (3), (7) and (10)

$$\mathcal{F}_c^{[m-1,\alpha]}(z;x;y;a;\lambda;u)=\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{zu^m}\right]^\alpha\mathcal{F}_{\mathcal{B}}^{\alpha}\left(z;x;1,e,e;\frac{\lambda}{u^m};\alpha\right)\cos(yz).$$

Thus, we have

$$z^\alpha\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}=$$

$$=\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{u^m}\right]^\alpha\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}\left(x;\frac{\lambda}{u^m},a,e,e\right)\frac{z^n}{n!}\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}z^{2n}}{(2n)!},$$

$$z^\alpha\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}=$$

$$= (-1)^{\alpha}\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{-u^m}\right]^{\alpha}\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}\Big(x;\frac{\lambda}{u^m},a,e,e\Big)\frac{z^n}{n!}\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}z^{2n}}{(2n)!},$$

$$\sum_{n=0}^{\infty}\binom{n}{\alpha}\alpha!\mathcal{H}_{n-\alpha,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}=$$

$$= (-1)^{\alpha}\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,\alpha]}(0;a,e;0;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}\Big(x;\frac{\lambda}{u^m},a,e,e\Big)\frac{z^n}{n!}\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}z^{2n}}{(2n)!},$$

$$\sum_{n=0}^{\infty}\binom{n}{\alpha}\alpha!\mathcal{H}_{n-\alpha,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}=$$

$$= (-1)^{\alpha}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{[\frac{j}{2}]}(-1)^k\binom{n}{j}\binom{j}{2k}\mathcal{H}_{n-j}^{[m-1,\alpha]}(0;a;e;0;u)\mathcal{B}_{j-2k}^{(\alpha)}\Big(x;\frac{\lambda}{u^m},a,e,e\Big)y^{2k}\frac{z^n}{n!}.$$

The proof of (22) is similar to that of (21), using the fact that

$$\mathcal{F}_s^{[m-1,\alpha]}(z;x;y;a;\lambda;u)=\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{zu^m}\right]^{\alpha}\mathcal{F}_{\mathcal{B}}^{\alpha}(z;x;1,e,e;\frac{\lambda}{u^m};\alpha)\sin(yz).$$

□

Theorem 6. *For $m\in\mathbb{N}$ the biparametric Apostol-type Frobenius–Euler polynomials of level m $\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y,a;\lambda;u)$ and $\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)$ are related to the generalized Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda;a,b,c)$ by means of the following identities*

$$\mathcal{H}_{n-\alpha,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)= \tag{23}$$
$$= 2^{-\alpha}\frac{1}{\alpha!\binom{n}{\alpha}}\sum_{j=0}^{n}\sum_{k=0}^{[\frac{j}{2}]}(-1)^k\binom{n}{j}\binom{j}{2k}\mathcal{H}_{n-j}^{[m-1,\alpha]}(0;a;e;0;u)\mathcal{G}_{j-2k}^{(\alpha)}\Big(x;-\frac{\lambda}{u^m};a,e,e\Big)y^{2k},$$

$$\mathcal{H}_{n-\alpha,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)= \tag{24}$$
$$= 2^{-\alpha}\frac{1}{\alpha!\binom{n}{\alpha}}\sum_{j=0}^{n}\sum_{k=0}^{[\frac{j-1}{2}]}(-1)^k\binom{n}{j}\binom{j}{2k+1}\mathcal{H}_{n-j}^{[m-1,\alpha]}(0;a;e;0;u)\mathcal{G}_{j-1-2k}^{(\alpha)}\Big(x;-\frac{\lambda}{u^m};a,e,e\Big)y^{2k+1}.$$

Proof of (23). Equalities (3), (9) and (10) imply the following equalities

$$\mathcal{F}_c^{[m-1,\alpha]}(z;x;y;a;\lambda;u)=\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{-2u^m z}\right]^{\alpha}\mathcal{F}_{\mathcal{G}}^{\alpha}\Big(z;x;1,e,e;-\frac{\lambda}{u^m};\alpha\Big)\cos(yz),$$

$$2^{\alpha}z^{\alpha}\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}=$$

$$=\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,\alpha]}(0;a;e;0;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}\Big(x;-\frac{\lambda}{u^m},a,e,e\Big)\frac{z^n}{n!}\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}z^{2n}}{(2n)!},$$

$$\sum_{n=0}^{\infty}\binom{n}{\alpha}\alpha!\mathcal{H}_{n-\alpha,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}=$$

$$=(2)^{-\alpha}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{j}{2}\right]}(-1)^k\binom{n}{j}\binom{j}{2k}\mathcal{H}_{n-j}^{[m-1,\alpha]}(0;a;e;0;u)\mathcal{G}_{j-2k}^{(\alpha).}\left(x;-\frac{\lambda}{u^m},a,e,e\right)y^{2k}\frac{z^n}{n!}.$$

The proof of (23) is similar to that of (24), using the fact that

$$\mathcal{F}_{s}^{[m-1,\alpha]}(z;x;y;a;\lambda;u)=\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{-2u^m z}\right]^{\alpha}\mathcal{F}_{\mathcal{G}}^{\alpha}(z;x;1,e,e;\frac{\lambda}{u^m};\alpha)\sin(yz).$$

□

Theorem 7. *For $m\in\mathbb{N}$ the biparametric Apostol-type Frobenius–Euler polynomials of level m $\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y,a;\lambda;u)$ and $\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)$ are related with the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ by means of the following identities*

$$\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)= \tag{25}$$

$$=2^{-\alpha}\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{j}{2}\right]}(-1)^k\binom{n}{j}\binom{j}{2k}\mathcal{H}_{n-j}^{[m-1,\alpha]}(0;a;e;0;u)\mathcal{E}_{j-2k}^{(\alpha)}\left(x;-\frac{\lambda}{u^m};a,e,e\right)y^{2k},$$

$$\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)= \tag{26}$$

$$=2^{-\alpha}\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{j-1}{2}\right]}(-1)^k\binom{n}{j}\binom{j}{2k+1}\mathcal{H}_{n-j}^{[m-1,\alpha]}(0;a;e;0;u)\mathcal{E}_{j-1-2k}^{(\alpha)}\left(x;-\frac{\lambda}{u^m};a,e,e\right)y^{2k+1}.$$

Proof of (25). The following equalities follow from (3), (8) and (10)

$$\mathcal{F}_{c}^{[m-1,\alpha]}(z;x;y;a;\lambda;u)=\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{-2u^m}\right]^{\alpha}\mathcal{F}_{\mathcal{E}}^{\alpha}\left(z;x;1,e,e;-\frac{\lambda}{u^m};\alpha\right)\cos(yz),$$

$$2^{\alpha}\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a,e;\lambda;u)\frac{z^n}{n!}=$$

$$=\sum_{n=0}^{\infty}\mathcal{H}_{n}^{[m-1,\alpha]}(0;a;e;0;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{E}^{(\alpha)}\left(x;-\frac{\lambda}{u^m},a,e,e\right)\frac{z^n}{n!}\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}z^{2n}}{(2n)!},$$

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\frac{z^n}{n!}=$$

$$=(2)^{-\alpha}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{j}{2}\right]}(-1)^k\binom{n}{j}\binom{j}{2k}\mathcal{H}_{n-j}^{[m-1,\alpha]}(0;a;e;0;u)\mathcal{E}_{j-2k}^{(\alpha)}\left(x;-\frac{\lambda}{u^m},a,e,e\right)y^{2k}\frac{z^n}{n!}.$$

The proof of (26) is similar to that of (25) in view of the equality

$$\mathcal{F}_s^{[m-1,\alpha]}(z;x;y;a;\lambda;u)=\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{-2u^m}\right]^{\alpha}\mathcal{F}_{\mathcal{E}}^{\alpha}(z;x;1,e,e;\frac{\lambda}{u^m};\alpha)\sin(yz).$$

□

Theorem 8. *For $m\in\mathbb{N}$ the generalized Apostol-type Frobenius–Euler polynomials $\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)$ and $\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)$ of level m are related to the Jacobi polynomials $P_n^{(\alpha,\beta)}(y)$ by means of the identities*

$$\mathcal{H}_{n,c}^{[m-1,\alpha]}(x+y;y;a;\lambda;u)= \tag{27}$$
$$=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!(\ln c)^j\binom{j+\alpha}{j-k}\binom{n}{j}\frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{j+1}}\mathcal{H}_{n-j,c}^{[m-1,\alpha]}(x;y;a;\lambda;u))P_k^{(\alpha,\beta)}(1-2y),$$

$$\mathcal{H}_{n,s}^{[m-1,\alpha]}(x+y;y;a;\lambda;u)= \tag{28}$$
$$=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!(\ln c)^j\binom{j+\alpha}{j-k}\binom{n}{j}\frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{j+1}}\mathcal{H}_{n-j,s}^{[m-1,\alpha]}(x;y;a;\lambda;u))P_k^{(\alpha,\beta)}(1-2y).$$

Proof. By substituting (6) into the right-hand side of (13) and using appropriate binomial coefficient identities (see, for instance [2, 4, 5]), we have

$$\mathcal{H}_{n,c}^{[m-1,\alpha]}(x+y;y;a,b;\lambda;u)=\sum_{j=0}^{n}\binom{n}{j}\mathcal{H}_{j,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)(n-j)!(\ln c)^{n-j}\times$$
$$\times\sum_{k=0}^{n-j}(-1)^k\binom{n-j+\alpha}{n-j-k}\frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n-j+1}}P_k^{(\alpha,\beta)}(1-2y)=\sum_{j=0}^{n}\sum_{k=0}^{n-j}(-1)^k\binom{n}{j}(n-j)!\times$$
$$\times\mathcal{H}_{j,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)(\ln c)^{n-j}\binom{n-j+\alpha}{n-j-k}\frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n-j+1}}P_k^{(\alpha,\beta)}(1-2y)=$$
$$=\sum_{k=0}^{n}(-1)^k\sum_{j=0}^{n-k}\binom{n}{j}\binom{n-j+\alpha}{n-j-k}\mathcal{H}_{j,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)(n-j)!(\ln c)^{n-j}\times$$
$$\times\frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n-j+1}}P_k^{(\alpha,\beta)}(1-2y)=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!(\ln c)^j\times$$
$$\times\binom{j+\alpha}{j-k}\binom{n}{j}\frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{j+1}}\mathcal{H}_{n-j,c}^{[m-1,\alpha]}(x;y;a,b;\lambda;u)P_k^{(\alpha,\beta)}(1-2y).$$

Therefore, (27) holds. The proof (28) is similar. □

3. Partial Derivative biparametric Apostol–Frobenius-Euler polynomials $\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)$ and $\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)$. In this section, by applying partial derivative operator to equations (10) and (11), we give some derivative formulae and finite combinatorial sums for the two biparametric Apostol–Frobenius-Euler polynomials.

Theorem 9. *Let $\left(\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ and $\left(\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ be the sequence of biparametric Apostol-type Frobenius–Euler polynomials, wits parameters $\lambda,u\in\mathbb{C}$ and $a\in\mathbb{R}_+$, of order $\alpha\in\mathbb{C}$ and level m. Then for $n,m,k\in\mathbb{N}$ the following identities hold*

$$\frac{\partial^k}{\partial x^k}\left\{\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right\}=$$

$$= (-1)^k \sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{B}_j^{(k)}\Big(\frac{\lambda}{u^m};1,e,e\Big)\mathcal{H}_{r-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u), \quad (29)$$

$$\frac{\partial^k}{\partial x^k}\left\{\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right\} =$$

$$= (-1)^k \sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{B}_j^{(k)}\Big(\frac{\lambda}{u^m};1,e,e\Big)\mathcal{H}_{r-j,s}^{[m-1,\alpha-k]}(x;y;a;\lambda;u). \quad (30)$$

Proof. Differentiating (10) with respect to x, we have

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\left\{\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right\}\frac{z^n}{n!} =$$

$$= (-1)^k\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!} - u^m}{-u^m}\right]^k \sum_{n=0}^{\infty}\mathcal{B}_n^{(k)}\Big(\frac{\lambda}{u^m};1,e,e\Big)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)\frac{z^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\left\{\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right\}\frac{z^n}{n!} =$$

$$=(-1)^k\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,k]}(0;a,e;0;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{B}_n^{(k)}\Big(\frac{\lambda}{u^m};1,e,e\Big)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)\frac{z^n}{n!} =$$

$$=(-1)^k\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,k]}(0;a,e;0;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathcal{B}_j^{(k)}\Big(\frac{\lambda}{u^m};1,e,e\Big)\mathcal{H}_{n-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)\frac{z^n}{n!} =$$

$$=(-1)^k\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{B}_r^{(k)}\Big(\frac{\lambda}{u^m};1,e,e\Big)\mathcal{H}_{r-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)\frac{z^n}{n!}.$$

By uniqueness of the series we obtain the first statement of Theorem 9. The proof of (30) is similar. □

Theorem 10. *Let* $\left(\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ *and* $\left(\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ *be the sequences of biparametric Apostol-type Frobenius–Euler polynomials wits parameters* $\lambda, u \in \mathbb{C}$ *and* $a \in \mathbb{R}_+$ *of order* $\alpha \in \mathbb{C}$ *and level* m*. Then for* $n, m, k \in \mathbb{N}$ *the following identities hold*

$$\frac{\partial^k}{\partial x^k}\left\{\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right\} = \quad (31)$$

$$= \Big(\frac{1}{2}\Big)^k \sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{G}_j^{(k)}\Big(\frac{\lambda}{-u^m};1,e,e\Big)\mathcal{H}_{r-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u),$$

$$\frac{\partial^k}{\partial x^k}\left\{\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right\} = \quad (32)$$

$$= \Big(\frac{1}{2}\Big)^k \sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{G}_j^{(k)}\Big(\frac{\lambda}{-u^m};1,e,e\Big)\mathcal{H}_{r-j,s}^{[m-1,\alpha-k]}(x;y;a;\lambda;u).$$

Proof. Applying derivate in (10) on x, we have

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\left\{\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right\}\frac{z^n}{n!}=$$

$$=\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{-2u^m}\right]^k\sum_{n=0}^{\infty}\mathcal{G}_n^{(k)}\left(\frac{\lambda}{-u^m};1,e,e\right)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)\frac{z^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\left\{\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right\}\frac{z^n}{n!}=$$

$$=\frac{1}{2^k}\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,k]}(0;a,e;0;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k)}\left(\frac{\lambda}{-u^m};1,e,e\right)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_{n,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)\frac{z^n}{n!}=$$

$$=\frac{1}{2^k}\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,k]}(0;a,e;0;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathcal{G}_j^{(k)}\left(\frac{\lambda}{-u^m};1,e,e\right)\mathcal{H}_{n-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)\frac{z^n}{n!}=$$

$$=\frac{1}{2^k}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{G}_r^{(k)}\left(\frac{\lambda}{-u^m};1,e,e\right)\mathcal{H}_{r-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)\frac{z^n}{n!}.$$

By uniqueness of the series, the first statement of Theorem 10 is proved. The proof of (32) is similar. □

The following Theorem 11 can be proved similarly to Theorem 10.

Theorem 11. *Let* $\left(\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$, $\left(\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ *be the sequences of biparametric Apostol-type Frobenius–Euler polynomials with parameters* $\lambda,u\in\mathbb{C}$ *and* $a\in\mathbb{R}_+$, *of order* $\alpha\in\mathbb{C}$ *and level* m. *Then for* $n,m,k\in\mathbb{N}$, $n>k$ *the following identities hold:*

$$\frac{\partial^k}{\partial x^k}\left\{\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right\}=\left(\frac{1}{2}\right)^k k!\binom{n}{k}\times \tag{33}$$

$$\times\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n-k}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{E}_r^{(k)}\left(\frac{\lambda}{-u^m};1,e,e\right)\mathcal{H}_{r-k-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u),$$

$$\frac{\partial^k}{\partial x^k}\left\{\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right\}=\left(\frac{1}{2}\right)^k k!\binom{n}{k}\times \tag{34}$$

$$\times\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-L}^{[m-1,k]}(0;a,e;0;u)\mathcal{E}_j^{(k)}\left(\frac{\lambda}{-u^m};1,e,e\right)\mathcal{H}_{r-k-j,s}^{[m-1,\alpha-k]}(x;y;a;\lambda;u).$$

Combining (29) and (31) with (33), we obtain the statement of Theorem 12.

Theorem 12. *Let* $\left(\mathcal{H}_{n,c}^{[m-1,\alpha]}(x;y;a;\lambda;u)\right)$ *be the sequence of biparametric Apostol-type Frobenius–Euler polynomials, with parameters* $\lambda,u\in\mathbb{C}$ *and* $a\in\mathbb{R}_+$, *of order* $\alpha\in\mathbb{C}$ *and level* m. *Then for* $n,m,k\in\mathbb{N}$, $n>k$ *the following identities hold:*

$$\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{B}_j^{(k)}\left(\frac{\lambda}{u^m};1,e,e\right)\mathcal{H}_{r-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)=$$

$$=\frac{1}{(-2)^k}\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{G}_j^{(k)}\left(\frac{\lambda}{-u^m};1,e,e\right)\mathcal{H}_{r-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u),$$

$$\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{B}_{j}^{(k)}\Big(\frac{\lambda}{u^{m}};1,e,e\Big)\mathcal{H}_{r-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)=$$

$$=\frac{k!}{(-2)^{k}}\binom{n}{k}\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{E}_{j}^{(k)}\Big(\frac{\lambda}{-u^{m}};1,e,e\Big)\mathcal{H}_{r-k-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u),$$

$$\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{G}_{j}^{(k)}\Big(\frac{\lambda}{-u^{m}};1,e,e\Big)\mathcal{H}_{r-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)=$$

$$=k!\binom{n}{k}\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{E}_{j}^{(k)}\Big(\frac{\lambda}{-u^{m}};1,e,e\Big)\mathcal{H}_{r-k-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u).$$

Combining (30) and (32) with (34), we obtain the statement of Theorem 13.

Theorem 13. *Let* $\big(\mathcal{H}_{n,s}^{[m-1,\alpha]}(x;y;a;\lambda;u)\big)$ *be the sequence of biparametric Apostol-type Frobenius–Euler polynomials, wits parameters* $\lambda,u\in\mathbb{C}$ *and* $a\in\mathbb{R}_{+}$*, of order* $\alpha\in\mathbb{C}$ *and level* m*. Then for* $n,m,k\in\mathbb{N}$, $n>k$ *the following identities hold:*

$$\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{B}_{j}^{(k)}\Big(\frac{\lambda}{u^{m}};1,e,e\Big)\mathcal{H}_{r-j,c}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)=$$

$$=\frac{1}{(-2)^{k}}\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{G}_{j}^{(k)}\Big(\frac{\lambda}{-u^{m}};1,e,e\Big)\mathcal{H}_{r-j,s}^{[m-1,\alpha-k]}(x;y;a;\lambda;u),$$

$$\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{B}_{j}^{(k)}\Big(\frac{\lambda}{u^{m}};1,e,e\Big)\mathcal{H}_{r-j,s}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)=$$

$$=\frac{k!}{(-2)^{k}}\binom{n}{k}\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{E}_{j}^{(k)}\Big(\frac{\lambda}{-u^{m}};1,e,e\Big)\mathcal{H}_{r-k-j,s}^{[m-1,\alpha-k]}(x;y;a;\lambda;u),$$

$$\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{G}_{j}^{k}\Big(\frac{\lambda}{-u^{m}};1,e,e\Big)\mathcal{H}_{r-j,s}^{[m-1,\alpha-k]}(x;y;a;\lambda;u)=$$

$$=k!\binom{n}{k}\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\mathcal{H}_{n-r}^{[m-1,k]}(0;a,e;0;u)\mathcal{E}_{j}^{k}\Big(\frac{\lambda}{-u^{m}};1,e,e\Big)\mathcal{H}_{r-k-j,s}^{[m-1,\alpha-k]}(x;y;a;\lambda;u).$$

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