O. M. Holovata, M. M. Sheremeta

ANALOGUES OF THE LOCKHART-STRAUS INEQUALITY AND TWO-MEMBER ASYMPTOTICS OF SERIES IN SYSTEMS OF FUNCTIONS

O. M. Holovata, M. M. Sheremeta. *Analogues of the Lockhart-Straus inequality and two-member asymptotics of series in systems of functions*, Mat. Stud. **64** (2025), 203–207.

For an entire transcendental function f and a sequence (λ_n) of positive numbers increasing to $+\infty$ let $A(z)=\sum_{n=1}^{\infty}a_n f(\lambda_n z)$ be a series in the system $f(\lambda_n z)$ regularly convergent in $\{z\colon |z|<R[A]\}$ that is $\mathfrak{M}(r,A)=\sum_{n=1}^{\infty}|a_n|M_f(r\lambda_n)<+\infty$ for $r\in[0,R[A])$, where $M_f(r)=\max\{|f(z)|\colon |z|=r\}$ $(r\in[0,+\infty))$, $\mu(r,A)=\max\{|a_n|M_f(r\lambda_n)\colon n\ge 1\}$ is the maximal term. Denote $n_\lambda(t)=\sum_{\lambda_n\le t}1$, and $\Gamma_f(r)=\frac{d\ln M_f(r)}{d\ln r}$ is the right-hand derivative.

Estimates of $\mathfrak{M}(r,A)$ by $\mu(r,A)$ are obtained, which are analogues of the Lockhart-Straus inequality (1985). The article proves, in particular, the following statement (Theorem 1): If $0<r<R[A]\le+\infty$, $0<2\varepsilon<R[A]-r$ and $\ln n_\lambda(t)=o(\Gamma_f(rt))$ as $t\to+\infty$ for every fixed $r>0$ then

$$\frac{\mathfrak{M}(r,A)}{\mu(r,A)}\le n_\lambda\Big(\Gamma_f^{-1}\Big(2\frac{\ln\mu(r+2\varepsilon,A)-\ln\mu(r+\varepsilon,A)}{\ln(r+2\varepsilon)-\ln(r+\varepsilon)}\Big)\Big)+2.$$

The application of the obtained results is indicated to study the relationship between the growth of $\mathfrak{M}(r,A)$ and $\mu(r,A)$ in terms of two-member asymptotics.

1. Introduction. For an entire transcendental function

$$g(z)=\sum_{k=0}^{\infty}g_k z^k$$

we denote

$$M_g(r)=\max\{|g(z)|\colon |z|=r\},\quad \mu_g(r)=\max\{|g_k|r^k\colon k\ge 0\}.$$

P. Lockhart and E.G. Straus [1] proved that

$$M_g(r)\le\frac{4r+\varepsilon}{\varepsilon}\mu_g(r)\Big(1+\ln\frac{\mu_g(r+\varepsilon)}{\mu_g(r)}\Big),\quad r<+\infty,\varepsilon>0.$$

Let $f(z)=\sum_{k=0}^{\infty}f_k z^k$ be an entire transcendental function and

$$A(z)=\sum_{n=1}^{\infty}a_n f(\lambda_n z)\tag{1}$$

be a series in the system $f(\lambda_n z)$, where $\Lambda=(\lambda_n)$ is a sequence of positive numbers increasing to $+\infty$. Let $R[A]$ be the radius of regular convergence of series (1), i.e.

$$\mathfrak{M}(r,A)=\sum_{n=1}^{\infty}|a_n|M_f(r\lambda_n)<+\infty\tag{2}$$

holds for $r<R[A]$ and not holds for $r>R[A]$.

2020 *Mathematics Subject Classification:* 30B50, 30D20.
Keywords: series in system of functions; regularly converging series; maximal term.
doi:10.30970/ms.64.2.203-207

Denote

$$\Gamma_f(r) = \frac{d\ln M_f(r)}{d\ln r}$$

(in points where the derivative does not exist, under $\frac{d\ln M_f(r)}{d\ln r}$ mean right-hand derivative). Since the function $\ln M_f(r)$ is logarithmically convex, we have $\Gamma_f(r)\nearrow+\infty$ as $r\to+\infty$.

It is known [2] that if $\Gamma_f(cr)\asymp\Gamma_f(r)$ as $r\to+\infty$ for each $c\in(0,+\infty)$ and $\ln n = o(\Gamma_f(\lambda_n))$ as $n\to\infty$ then

$$R[A] = \varliminf_{n\to\infty}\frac{1}{\lambda_n}M_f^{-1}\Big(\frac{1}{|a_n|}\Big).$$

For $r\in[0,R[A])$ let $\mu(r,A)=\max\{|a_n|M_f(r\lambda_n)\colon n\ge 1\}$ be the maximal term of series (2) and $\nu(r,A)=\max\{n\ge 1\colon |a_n|M_f(r\lambda_n)=\mu(r,A)\}$ be its central index.

Obviously $\mathfrak{M}(r,A)\ge\mu(r,A)$. In this note we will obtain upper bounds for $\mathfrak{M}(r,A)$ by $\mu(r,A)$ of the Lockhart-Straus inequality type.

2. Main theorem. Let $n_\lambda(t)=\sum_{\lambda_n\le t}1$ be the counting function of the sequence (λ_n). The following theorem is the main one.

Theorem 1. *If $0<r<R[A]\le+\infty$, $0<2\varepsilon<R[A]-r$ and $\ln n_\lambda(t)=o(\Gamma_f(rt))$ as $t\to+\infty$ for every fixed $r>0$ then*

$$\frac{\mathfrak{M}(r,A)}{\mu(r,A)}\le n_\lambda\Big(\Gamma_f^{-1}\Big(2\frac{\ln\mu(r+2\varepsilon,A)-\ln\mu(r+\varepsilon,A)}{\ln(r+2\varepsilon)-\ln(r+\varepsilon)}\Big)\Big)+2. \tag{3}$$

Proof. Since $\mu(r,A)=|a_{\nu(r,A)}|M_f(r\lambda_{\nu(r,A)})$, for $r+\varepsilon<R[A]$ we have

$$|a_n|M_f(r\lambda_n)=|a_n|M_f((r+\varepsilon)\lambda_n)\frac{M_f(r\lambda_n)}{M_f((r+\varepsilon)\lambda_n)}\le$$

$$\le\mu(r+\varepsilon,A)\frac{M_f(r\lambda_n)}{M_f((r+\varepsilon)\lambda_n)}=|a_{\nu(r+\varepsilon,A)}|M_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)})\frac{M_f(r\lambda_n)}{M_f((r+\varepsilon)\lambda_n)}=$$

$$=|a_{\nu(r+\varepsilon,A)}|M_f(r\lambda_{\nu(r+\varepsilon,A)})\frac{M_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)})}{M_f(r\lambda_{\nu(r+\varepsilon,A)})}\frac{M_f(r\lambda_n)}{M_f((r+\varepsilon)\lambda_n)}\le$$

$$\le\mu(r,A)\frac{M_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)})}{M_f(r\lambda_{\nu(r+\varepsilon,A)})}\frac{M_f(r\lambda_n)}{M_f((r+\varepsilon)\lambda_n)}. \tag{4}$$

But

$$\frac{M_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)})}{M_f(r\lambda_{\nu(r+\varepsilon,A)})}=\exp\{\ln M_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)})-\ln M_f(r\lambda_{\nu(r+\varepsilon,A)})\}=$$

$$=\exp\Big\{\int_{r\lambda_{\nu(r+\varepsilon,A)}}^{(r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)}}\Gamma_f(x)d\ln x\Big\}\le\exp\Big\{\Gamma_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)})\ln\frac{r+\varepsilon}{r}\Big\}$$

and similar

$$\frac{M_f(r\lambda_n)}{M_f((r+\varepsilon)\lambda_n)}=\exp\Big\{-\int_{r\lambda_n}^{(r+\varepsilon)\lambda_n}\Gamma_f(x)d\ln x\Big\}\le\exp\Big\{-\Gamma_f(r\lambda_n)\ln\frac{r+\varepsilon}{r}\Big\}.$$

Therefore, (5) implies

$$|a_n|M_f(r\lambda_n)\le\mu(r,A)\exp\Big\{-(\Gamma_f(r\lambda_n)-\Gamma_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)}))\ln\frac{r+\varepsilon}{r}\Big\}. \tag{5}$$

Let $n_0(r)=\min\{n\colon\Gamma_f(r\lambda_n)\ge 2\Gamma_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)})\}$. Then $\Gamma_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)})\le\Gamma_f(r\lambda_n)/2$ $(n\ge n_0(r))$ and (4) implies

$$|a_n|M_f(r\lambda_n)\le \mu(r,A)\exp\Big\{-\frac{\Gamma_f(r\lambda_n)}{2}\ln\frac{r+\varepsilon}{r}\Big\}.$$

Therefore,

$$\frac{\mathfrak{M}(r,A)}{\mu(r,A)}\le\Bigg(\sum_{n=1}^{n_0(r)-1}+\sum_{n_0(r)}^{\infty}\Bigg)\frac{|a_n|M_f(r\lambda_n)}{\mu(r,A)}\le n_0(r)-1+\sum_{n=n_0(r)}^{\infty}\exp\Big\{-\frac{\Gamma_f(r\lambda_n)}{2}\ln\frac{r+\varepsilon}{r}\Big\}. \tag{6}$$

It is clear that (5) implies $\lambda_{n_0(r)-1}\le\Gamma_f^{-1}(2\Gamma_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)}))$, whence

$$n_0(r)-1\le n_\lambda(\Gamma_f^{-1}(2\Gamma_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)}))). \tag{7}$$

On the other hand, since $\ln n_\lambda(t)=o(\Gamma_f(rt))$ as $t\to+\infty$ for every fixed $r>0$, we can consider $n_0(r)$ so big that $\ln n_\lambda(t)\le\frac{\Gamma_f(rt)}{4}\ln\frac{r+\varepsilon}{r}$ for $t\ge\lambda_{n_0(r)}$. Then

$$\sum_{n=n_0(r)}^{\infty}\exp\Big\{-\frac{\Gamma_f(r\lambda_n)}{2}\ln\frac{r+\varepsilon}{r}\Big\}\le\int_{\lambda_{n_0(r)}}^{\infty}\exp\Big\{-\frac{\Gamma_f(rt)}{2}\ln\frac{r+\varepsilon}{r}\Big\}dn_\lambda(t)\le$$
$$\le\int_{\lambda_{n_0(r)}}^{\infty}n_\lambda(t)\exp\Big\{-\frac{\Gamma_f(rt)}{2}\ln\frac{r+\varepsilon}{r}\Big\}d\Big(\frac{\Gamma_f(rt)}{2}\ln\frac{r+\varepsilon}{r}\Big)\le$$
$$\le 2\exp\Big\{-\frac{\Gamma_f(r\lambda_{n_0(r)})}{4}\ln\frac{r+\varepsilon}{r}\Big\}\le 2. \tag{8}$$

From (6), (7) and (8) we get

$$\frac{\mathfrak{M}(r,A)}{\mu(r,A)}\le n_\lambda(\Gamma_f^{-1}(2\Gamma_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)})))+2. \tag{9}$$

In [3] it is proved that $\ln\mu(r,A)-\ln\mu(r_0,A)=\int_{r_0}^{r}\Gamma_f(t\lambda_{\nu(t,A)})d\ln t,\quad 0\le r_0\le r<+\infty$. Therefore,

$$\ln\frac{\mu(r+2\varepsilon,A)}{\mu(r+\varepsilon,A)}=\int\limits_{r+\varepsilon}^{r+2\varepsilon}\Gamma_f(t\lambda_{\nu(t,A)})d\ln t\ge\Gamma_f((r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)})\ln\frac{r+2\varepsilon}{r+\varepsilon}$$

and, thus,

$$(r+\varepsilon)\lambda_{\nu(r+\varepsilon,A)}\le\Gamma_f^{-1}\Big(\frac{\ln\mu(r+2\varepsilon,A)-\ln\mu(r+\varepsilon,A)}{\ln(r+2\varepsilon,A)-\ln(r+\varepsilon,A)}\Big).$$

From hence and (9) we obtain (3). □

Remark 1. Suppose that $hr\le\Gamma_f(r)\le Hr$ for all r, where $0<h\le H<+\infty$. Then $\ln n_\lambda(t)=o(\Gamma_f(rt))$ as $t\to+\infty$ for every fixed $r>0$ if and only if $\ln n_\lambda(t)=o(t)$ as $t\to+\infty$, $x/H\le\Gamma_f^{-1}(x)\le x/h$ and (3) holds if

$$\frac{\mathfrak{M}(r,A)}{\mu(r,A)}\le n_\lambda\Big(\frac{2}{h}\frac{\ln\mu(r+2\varepsilon,A)-\ln\mu(r+\varepsilon,A)}{\ln(r+2\varepsilon)-\ln(r+\varepsilon)}\Big)+2. \tag{10}$$

3. Corollaries. For entire functions (1) the following statement is correct.

Corollary 1. *If $R[A]=+\infty$, $0<h,H<+\infty$, $hr\le\Gamma_f(r)\le Hr$ $(\forall r)$, $\ln n_\lambda(t)\le q\ln t$ for some $q\in(0,+\infty)$ and all $t\ge t_0$ then for every $\alpha\in(0,+\infty)$ and all r enough large*

$$\ln\mathfrak{M}(r,A)\le\ln\mu(r,A)+q\ln\ln\frac{\mu((1+2\alpha)r,A)}{\mu((1+\alpha)r,A)}+q\ln\frac{4(1+\alpha)}{h\alpha}+2\ln 2. \tag{11}$$

Proof. Since $\ln\frac{r+2\varepsilon}{r+\varepsilon}\ge\frac{\varepsilon}{2(r+\varepsilon)}$ and $n_\lambda(t)\le t^q$, for $\varepsilon=\alpha r$ from (10) we get

$$\mathfrak{M}(r,A)\le\mu(r,A)\Big(\Big(\frac{4(1+\alpha)}{h\alpha}\Big)^q\ln^q\frac{\mu((1+2\alpha)r,A)}{\mu((1+\alpha)r,A)}+2\Big),$$

whence (11) follows. □

If series (1) has a finite radius of regular convergence, then the situation is somewhat different. If $R[A]<+\infty$ then the function $\mu(r,A)$ can be bounded on $[0,R[A])$. The following statement is correct.

Proposition 1. *In order that $\mu(r,A)\to+\infty$ as $r\to R[A]$, it is necessary and sufficient that $\varlimsup\limits_{n\to\infty}|a_n|M_f(R[A]\lambda_n)=+\infty$.*

Proof. If $\varlimsup\limits_{n\to\infty}|a_n|M_f(R[A]\lambda_n)<+\infty$ then $(\exists K)(\forall n\ge1)\colon\ |a_n|M_f(R[A]\lambda_n)\le K<+\infty$. Therefore, $|a_n|M_f(r\lambda_n)\le K$ for all $n\ge1$ and $r\in[0,R[A])$, i.e. , $\mu(r,A)\le K$ for all $r\in[0,R[A])$. On the contrary, if $\mu(r,A)\le K$ for all $r\in[0,R[A])$ then $|a_n|M_f(r\lambda_n)\le K$ for all $n\ge1$ and $r\in[0,R[A])$. Fixing n and directing $r\to R[A]$ from here we get $|a_n|M_f(R[A]\lambda_n)\le K$ □

In what follows, we will assume that the condition $\varlimsup\limits_{n\to\infty}|a_n|M_f(R[A]\lambda_n)=+\infty$ is satisfied and we prove the following statement.

Corollary 2. *If $0<R[A]<+\infty$, $0<h,H<+\infty$, $hr\le\Gamma_f(r)\le Hr$ for all r and $\ln n_\lambda(t)\le q\ln t$ for some $q\in(0,+\infty)$ and all $t\ge t_0$ then*

$$\ln\mathfrak{M}(r,A)\le\ln\mu(r,A)+q\ln\ln\frac{\mu((R[A]+r)/2,A)}{\mu((R[A]+3r)/4,A)}+q\ln\frac{1}{R[A]-r}+q\ln\frac{16R[A]}{h}+2\ln2. \tag{12}$$

Proof. Indeed, if we choose $\varepsilon=(R[A]-r)/4$ then $r+2\varepsilon=(R]A]+r)/2<R]A]$ for $r<R]A]$ and

$$\ln\frac{r+2\varepsilon}{r+\varepsilon}\ge\frac{\varepsilon}{2(r+\varepsilon)}=\frac{(R[A]-r)/4}{2(r+(R[A]-r)/4)}=\frac{R[A]-r}{2(R[A]+3r)}>\frac{R[A]-r}{8R[A]}.$$

Therefore, from (10) we get

$$\frac{\mathfrak{M}(r,A)}{\mu(r,A)}\le\Big(\frac{16R[A]}{h(R[A]-r)}\ln\frac{\mu((R[A]+r)/2,A)}{\mu((R[A]+3r)/4,A)}\Big)^q+2,$$

whence (12) follows. □

4. Two-term asymptotics. Suppose that $R[A]=+\infty$ and

$$\ln\mu(r,A)\le Te^{\varrho r}+(1+o(1))\tau e^{\varrho_1 r}\quad(r\to+\infty), \tag{13}$$

where $0<\varrho_1<\varrho<+\infty$, $T\in(0,+\infty)$ and $\tau\in\mathbb{R}\setminus0$. Then for every $\alpha\in(0,+\infty)$ and all r enough large we have

$$\ln\ln\frac{\mu((1+2\alpha)r,A)}{\mu((1+\alpha)r,A)}\le\ln\ln\mu((1+2\alpha)r,A)\le\ln T+\varrho(1+2\alpha)r+o(1)=o(e^{\varrho_1 r})\ (r\to+\infty).$$

Therefore, if $hr\le\Gamma_f(r)\le Hr$ for all r and $\ln n_\lambda(t)=O(\ln t)$ as $t\to+\infty$ then by Corollary 1 we get

$$\ln\mathfrak{M}(r,A)\le Te^{\varrho r}+(1+o(1))\tau e^{\varrho_1 r}+q\ln\ln\frac{\mu((1+2\alpha)r,A)}{\mu((1+\alpha)r,A)}+o(e^{\varrho_1 r})\ (r\to+\infty),$$

i.e.

$$\ln\mathfrak{M}(r,A)\le Te^{\varrho r}+(1+o(1))\tau e^{\varrho_1 r}\ (r\to+\infty). \tag{14}$$

On the other hand, since $\mu(r,A)\le\mathfrak{M}(r,A)$, (14) implies (13). Thus, the following statement is correct.

Proposition 2. *Let $R[A]=+\infty$, $0<h,H<+\infty$, $0<\varrho_1<\varrho<+\infty$, $T\in(0,+\infty)$ and $\tau\in\mathbb{R}\setminus 0$. If $hr\le\Gamma_f(r)\le Hr$ for all r and $\ln n_\lambda(t)=O(\ln t)$ as $t\to+\infty$ then* (14) *holds if and only if* (13) *holds.*

Now let $0<R[A]<+\infty$ and
$$\ln\mu(r,A)\le\frac{T}{(R[A]-r)^{\varrho}}+\frac{(1+o(1))\tau}{(R[A]-r)^{\varrho_1}},\quad r\uparrow R[A]. \tag{15}$$
Then
$$\ln\ln\frac{\mu((R[A]+r)/2,A)}{\mu((R[A]+3r)/4,A)}\le\ln\ln\mu((R[A]+r)/2,A)\le$$
$$\le\ln\frac{(1+o(1))T}{(R[A]-(R[A]+r)/2)^{\varrho}}=\ln\frac{(1+o(1))T2^{\varrho}}{(R[A]-r)^{\varrho}}=(1+o(1))\varrho\ln\frac{1}{R[A]-r},\quad r\uparrow R[A].$$
Therefore, by Corollary 2 we get
$$\ln\mathfrak{M}(r,A)\le\ln\mu(r,A)+O\Big(\ln\frac{1}{R[A]-r}\Big)=\ln\mu(r,A)+o\Big(\frac{1}{(R[A]-r)^{\varrho_1}}\Big),\quad r\uparrow R[A],$$
i.e.
$$\ln\mathfrak{M}(r,A)\le\frac{T}{(R[A]-r)^{\varrho}}+\frac{(1+o(1))\tau}{(R[A]-r)^{\varrho_1}},\quad r\uparrow R[A]. \tag{16}$$
On the other hand, since $\mu(r,A)\le\mathfrak{M}(r,A)$, (16) implies (15). Thus, the following statement is correct.

Proposition 3. *Let $0<R[A]<+\infty$, $0<h,H<+\infty$, $0<\varrho_1<\varrho<+\infty$, $T\in(0,+\infty)$ and $\tau\in\mathbb{R}\setminus 0$. If $hr\le\Gamma_f(r)\le Hr$ for all r and $\ln n_\lambda(t)=O(\ln t)$ as $t\to+\infty$ then* (15) *holds if and only if* (16) *holds.*

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https://doi.org/10.30970/ms.59.1.46-59

Ivan Franko National University of Lviv
Lviv, Ukraine

oksana.holovata@lnu.edu.ua
m.m.sheremeta@gmail.com

Received 26.06.2025
Revised 01.12.2025