УДК 517.5

M. V. Dmytryshyn, C. Cesarano, O. Kondur, I.-A. Lutsiv

ON THE NUMERICAL STABILITY OF THE BRANCHED CONTINUED FRACTION EXPANSION OF THE RATIO $H_4(a, d+1; c, d; \mathbf{z})/H_4(a, d+2; c, d+1; \mathbf{z})$

M. V. Dmytryshyn, C. Cesarano, O. Kondur, I.-A. Lutsiv. *On the numerical stability of the branched continued fraction expansion of the ratio* $H_4(a, d+1; c, d; \mathbf{z})/H_4(a, d+2; c, d+1; \mathbf{z})$, Mat. Stud. **64** (2025), 133–143.

Continued fractions and their generalization, branched continued fractions, are the effective tools used to study special functions. In this aspect, an important problem of continued fractions and branched continued fractions is the study of their numerical stability. The backward recurrence algorithm is one of the main tools for computing approximants of both continued fraction and branched continued fractions. Like most recursive processes, it is prone to error growth. Each cycle of the recursive process not only generates its own rounding errors but also inherits the rounding errors made in all the previous cycles. This paper considers numerical stability of branched continued fraction expansion of the one ratio of Horn's hypergeometric functions H_4 in the special case, namely, $H_4(a, d+1; c, d; \mathbf{z})/H_4(a, d+2; c, d+1; \mathbf{z})$. For this purpose, the backward recurrence algorithm is investigated. It is proven that under certain conditions on the parameters a, c, and d the some open bi-disc is the set of numerical stability for branched continued fraction expansion, and it is found the estimate of relative rounding error, produced by the backward recurrence algorithm in calculating an nth approximant of this expansion.

The results of this paper provide a toolkit for analyzing the numerical stability of algorithms that use branched continued fractions of the studied structure. Error estimates can be used to choose computation parameters, control accuracy, and ensure the reliability of results in applied problems that will use the aforementioned branched continued fractions.

1. Introduction. Functions of one or several variables play an important role in almost all fields of science and engineering (see, for example, [16, 44, 55] and also books [8, 24, 58]). Various aspects of their study are considered, in particular, their integral representations ([28, 29, 30]), approximations by trigonometric polynomials ([51, 56, 59]), symmetric polynomials ([31, 42, 60]) and block-symmetric polynomials ([47, 48, 49]), linear summation methods of Fourier series (in particular, Taylor-Abel-Poisson method, Zygmund method, Fourier method [13, 14, 57]), generalized sampling series ([1, 2, 3]) and positive linear operators ([12, 45, 46]), representations by power series ([25, 26, 27]), as well as representations by continued fractions (see [4, 36, 54] and books [15, 43, 52]) and branched continued fractions (see, for example, [17, 39, 53] and also books [10, 11, 50]) are considered. Branched continued fractions as a special family of functions have proven to be one of the most effective tools for approximating

2020 *Mathematics Subject Classification:* 32A17, 33C65, 41A20, 65G50.
Keywords: branched continued fraction; Horn hypergeometric function; approximation by rational functions; roundoff error.
doi:10.30970/ms.64.2.133-143

the special functions ([5, 7, 41]). The vital problem here is the study of the convergence of the constructed expansions ([9, 22, 34]). Another essential direction in the investigation of branched continued fraction expansions of special functions, in particular hypergeometric functions, is the study of their numerical stability ([20, 33, 35]) and stability to perturbations ([32, 37, 40]).

This paper considers numerical stability of branched continued fraction expansion of the one ratio of Horn's hypergeometric functions H_4 in a special case constructed in [6].

Recall that the Horn's hypergeometric function H_4 is defined as (see [38])

$$H_4(a,b;c,d;\mathbf{z}) = \sum_{r,s=0}^{\infty} \frac{(a)_{2r+s}(b)_s}{(c)_r(d)_s} \frac{z_1^r}{r!} \frac{z_2^s}{s!}, \quad |z_1| < p, \ |z_2| < q, \tag{1}$$

where $a, b, c, d \in \mathbb{C}$; $c, d \notin \{0, -1, -2, \ldots\}$; p and q are positive numbers such that $4p = (q-1)^2$ and $q \neq 1$; $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$.

Theorem A ([6])**.** *The ratio*

$$\frac{H_4(a,d+1;c,d;\mathbf{z})}{H_4(a,d+2;c,d+1;\mathbf{z})}$$

has formal branched continued fraction expansion

$$1 + \cfrac{v_0 z_2}{1 - v_1 z_2 - \cfrac{u_1 z_1}{1 - v_2 z_2 - \cfrac{u_2 z_1}{1 - \ddots}}}, \tag{2}$$

where

$$v_0 = \frac{a}{d(d+1)}, \quad v_1 = 1 - \frac{a}{d+1}, \quad u_1 = \frac{2(a+1)}{c}, \tag{3}$$

$$v_k = 1, \quad u_k = \frac{(2c-a+k-3)(a+k)}{(c+k-2)(c+k-1)}, \quad k \geq 2. \tag{4}$$

The problem of convergence of (2) is considered in [19, 21]. In [6], branched continued fraction expansions for such ratios

$$\frac{H_4(a,b;c,b;\mathbf{z})}{H_4(a+1,b;c+1,b;\mathbf{z})}, \quad \frac{H_4(a,d+1;c,d;\mathbf{z})}{H_4(a+1,d+1;c,d+1;\mathbf{z})}$$

are also constructed. The domains of analytical continuation of these ratios were discussed in [6, 20, 23] and the truncation error bounds for their expansions were considered in [18]. Numerical aspects related to the backward recurrence algorithm for computing the approximants of branched continued fractions were considered in [20].

This paper proves that under certain conditions on the parameters of the function (1) some open bi-disc is the set of numerical stability for branched continued fraction expansion (2) and founds the estimate of relative rounding error, produced by the backward recurrence algorithm in calculating an nth approximant of this expansion.

2. Auxiliary results. We recall the necessary notations and definitions ([20]). Let n be an arbitrary natural number.

The backward recurrence algorithm for computing the nth approximant

$$f_n = 1 + \cfrac{c_{0,1}}{1 + c_{1,1} + \cfrac{c_{0,2}}{1 + \ddots + c_{1,n-2} + \cfrac{c_{0,n-1}}{1 + c_{1,n-1} + c_{0,n}}}}$$

of a branched continued fraction

$$1 + \cfrac{c_{0,1}}{1 + c_{1,1} + \cfrac{c_{0,2}}{1 + \ddots}} \tag{5}$$

consists of setting $U_n^{(n)} = 1$ and successively computing

$$U_k^{(n)} = 1 + c_{1,k} + \frac{c_{0,k+1}}{U_{k+1}^{(n)}}, \quad n-1 \geq k \geq 1,$$

for $n \geq 2$. And, therefore, $f_n = U_0^{(n)}$, where $U_0^{(n)} = 1 + c_{0,1}/U_1^{(n)}$.

For $1 \leq k \leq n$, let $\widehat{c}_{1,k}$, $\widehat{c}_{0,k}$ be the rounded values of $c_{1,k}$, $c_{0,k}$ of the branched continued fraction (5), respectively, and

$$\widehat{f}_n = 1 + \cfrac{\widehat{c}_{0,1}}{1 + \widehat{c}_{1,1} + \cfrac{\widehat{c}_{0,2}}{1 + \ddots + \widehat{c}_{1,n-2} + \cfrac{\widehat{c}_{0,n-1}}{1 + \widehat{c}_{1,n-1} + \widehat{c}_{0,n}}}}$$

be the computed value of f_n.

Definition 1. A *numerical stability set* $\mathfrak{D}$ is a set to which for any $\varepsilon > 0$ there exists $\delta > 0$ depending only on ε and $\mathfrak{D}$ such that $|(\widehat{f}_n - f_n)/f_n| < \varepsilon$, $n \geq 1$, for every branched continued fraction (5) with $c_{1,k}, c_{0,k}, \widehat{c}_{1,k}, \widehat{c}_{0,k} \in \mathfrak{D}$, $k \geq 1$, such that $|(\widehat{c}_{1,k} - c_{1,k})/c_{1,k}| < \delta$, $|(\widehat{c}_{0,k} - c_{0,k})/c_{0,k}| < \delta$, $k \geq 1$.

Now, for branched continued fraction (2) we need to set

$$W_k^{(n)}(\mathbf{z}) = 1 - v_k z_2 - \cfrac{u_k z_1}{1 - v_{k+1} z_2 - \cfrac{u_{k+1} z_1}{1 - \ddots - v_{n-2} z_2 - \cfrac{u_{n-2} z_1}{1 - v_{n-1} z_2 - u_{n-1} z_1}}},$$

where $1 \leq k \leq n-1$ for $n \geq 2$. It gives us

$$W_k^{(n)}(\mathbf{z}) = 1 - v_k z_2 - \frac{u_k z_1}{W_{k+1}^{(n)}(\mathbf{z})}, \quad 1 \leq k \leq n-1, \tag{6}$$

for $n \geq 2$, and $f_n(\mathbf{z}) = W_0^{(n)}(\mathbf{z})$, where

$$W_0^{(n)}(\mathbf{z}) = 1 + \frac{v_0 z_2}{W_1^{(n)}(\mathbf{z})}, \quad W_n^{(n)}(\mathbf{z}) = 1. \tag{7}$$

Next, let α_1, α_2, γ_0, γ_1, and β_k, $1 \leq k \leq n$, be the relative errors in the rounded values $\widehat{z}_1$, $\widehat{z}_2$, $\widehat{v}_0$, $\widehat{v}_1$, $\widehat{u}_k$, $1 \leq k \leq n$, of z_1, z_2, v_0, v_1, and u_k, $1 \leq k \leq n$, respectively, so that

$$\widehat{z}_1 = z_1(1 + \alpha_1), \quad \widehat{z}_2 = z_2(1 + \alpha_2), \tag{8}$$

$$\widehat{v}_0 = v_0(1+\gamma_0), \quad \widehat{v}_1 = v_1(1+\gamma_1), \quad \widehat{u}_k = u_k(1+\beta_k), \quad 1 \le k \le n. \tag{9}$$

Similarly, for $0 \le k \le n$, let $\varepsilon_k^{(n)}$ be the relative error in $\widehat{W_k^{(n)}}(\widehat{\mathbf{z}})$ of computing $W_k^{(n)}(\mathbf{z})$ using $\widehat{z}_1$, $\widehat{z}_2$, $\widehat{v}_0$, $\widehat{v}_1$, and $\widehat{u}_k$, $1 \le k \le n$. And, therefore,

$$\widehat{W_k^{(n)}}(\widehat{\mathbf{z}}) = W_k^{(n)}(\mathbf{z})(1+\varepsilon_k^{(n)}), \quad 0 \le k \le n, \tag{10}$$

with initial conditions

$$\widehat{W_n^{(n)}}(\widehat{\mathbf{z}}) = W_n^{(n)}(\mathbf{z}) = 1, \quad \varepsilon_n^{(n)} = 0. \tag{11}$$

Also, let $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\gamma}_0$, $\widehat{\gamma}_1$, $\widehat{\beta}_k$, $1 \le k \le n$, and $\widehat{\varepsilon}_k^{(n)}$, $0 \le k \le n$, be the relative errors defined by

$$z_1 = \widehat{z}_1(1+\widehat{\alpha}_1), \quad z_2 = \widehat{z}_2(1+\widehat{\alpha}_2), \quad v_0 = \widehat{v}_0(1+\widehat{\gamma}_0), \quad v_1 = \widehat{v}_1(1+\widehat{\gamma}_1),$$
$$u_k = \widehat{u}_k(1+\widehat{\beta}_k), \quad 1 \le k \le n, \quad W_k^{(n)}(\mathbf{z}) = \widehat{W_k^{(n)}}(\widehat{\mathbf{z}})(1+\widehat{\varepsilon}_k^{(n)}), \quad 0 \le k \le n,$$

respectively.

In what follows, we will establish the recurrence relations for $\varepsilon_k^{(n)}$, $0 \le k \le n-1$. For $k=0$ we have

$$\varepsilon_0^{(n)} = \frac{\widehat{W_0^{(n)}}(\widehat{\mathbf{z}}) - W_0^{(n)}(\mathbf{z})}{W_0^{(n)}(\mathbf{z})} = \frac{1}{W_0^{(n)}(\mathbf{z})}\left(1 + \frac{\widehat{v}_0\widehat{z}_2}{\widehat{W_1^{(n)}}(\widehat{\mathbf{z}})}\right) - 1 =$$
$$= \frac{1}{W_0^{(n)}(\mathbf{z})}\left(1 + \frac{v_0(1+\gamma_0)z_2(1+\alpha_2)}{W_1^{(n)}(\mathbf{z})(1+\varepsilon_1^{(n)})}\right) - 1 = \frac{1}{W_0^{(n)}(\mathbf{z})} + \frac{v_0(1+\gamma_0)z_2(1+\alpha_2)(1+\widehat{\varepsilon}_1^{(n)})}{W_0^{(n)}(\mathbf{z})W_1^{(n)}(\mathbf{z})} - 1.$$

Since $\dfrac{1}{W_0^{(n)}(\mathbf{z})} = 1 - \dfrac{v_0 z_2}{W_0^{(n)}(\mathbf{z})W_1^{(n)}(\mathbf{z})}$, then

$$\varepsilon_0^{(n)} = \frac{v_0 z_2}{W_0^{(n)}(\mathbf{z})W_1^{(n)}(\mathbf{z})}((1+\gamma_0)(1+\alpha_2)(1+\widehat{\varepsilon}_1^{(n)}) - 1) =$$
$$= \frac{v_0 z_2}{W_0^{(n)}(\mathbf{z})\widehat{W_1^{(n)}}(\widehat{\mathbf{z}})}(\gamma_0 + \alpha_2 + \gamma_0\alpha_2) + \frac{v_0 z_2}{W_0^{(n)}(\mathbf{z})W_1^{(n)}(\mathbf{z})}\widehat{\varepsilon}_1^{(n)}. \tag{12}$$

For $k=1$ we obtain

$$\varepsilon_1^{(n)} = \frac{\widehat{W_1^{(n)}}(\widehat{\mathbf{z}}) - W_1^{(n)}(\mathbf{z})}{W_1^{(n)}(\mathbf{z})} = \frac{1}{W_1^{(n)}(\mathbf{z})}\left(1 - \widehat{v}_1\widehat{z}_2 - \frac{\widehat{u}_1\widehat{z}_1}{\widehat{W_2^{(n)}}(\widehat{\mathbf{z}})}\right) - 1 =$$
$$= \frac{1}{W_1^{(n)}(\mathbf{z})}\left(1 - v_1(1+\gamma_1)z_2(1+\alpha_2) - \frac{u_1(1+\beta_1)z_1(1+\alpha_1)}{W_2^{(n)}(\mathbf{z})(1+\varepsilon_2^{(n)})}\right) - 1 =$$
$$= \frac{1}{W_1^{(n)}(\mathbf{z})} - \frac{v_1(1+\gamma_1)z_2(1+\alpha_2)}{W_1^{(n)}(\mathbf{z})} - \frac{u_1(1+\beta_1)z_1(1+\alpha_1)(1+\widehat{\varepsilon}_2^{(n)})}{W_1^{(n)}(\mathbf{z})W_2^{(n)}(\mathbf{z})} - 1.$$

Since

$$\frac{1}{W_1^{(n)}(\mathbf{z})} = 1 + \frac{v_1 z_2}{W_1^{(n)}(\mathbf{z})} + \frac{u_1 z_1}{W_1^{(n)}(\mathbf{z})W_2^{(n)}(\mathbf{z})},$$

then

$$\varepsilon_1^{(n)} = -\frac{v_1 z_2}{W_1^{(n)}(\mathbf{z})}(\gamma_1 + \alpha_2 + \gamma_1\alpha_2) - \frac{u_1 z_1}{W_1^{(n)}(\mathbf{z})W_2^{(n)}(\mathbf{z})}((1+\beta_1)(1+\alpha_1)(1+\widehat{\varepsilon}_2^{(n)}) - 1) =$$

$$= -\frac{v_1 z_2(\gamma_1+\alpha_2+\gamma_1\alpha_2)}{W_1^{(n)}(\mathbf{z})} - \frac{u_1 z_1(\beta_1+\alpha_1+\beta_1\alpha_1)}{W_1^{(n)}(\mathbf{z})\widehat{W}_2^{(n)}(\widehat{\mathbf{z}})} - \frac{u_1 z_1 \widehat{\varepsilon}_2^{(n)}}{W_1^{(n)}(\mathbf{z})W_2^{(n)}(\mathbf{z})}. \tag{13}$$

Similarly,

$$\widehat{\varepsilon}_1^{(n)} = -\frac{\widehat{v}_1 \widehat{z}_2(\widehat{\gamma}_1+\widehat{\alpha}_2+\widehat{\gamma}_1\widehat{\alpha}_2)}{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} - \frac{\widehat{u}_1 \widehat{z}_1(\widehat{\beta}_1+\widehat{\alpha}_1+\widehat{\beta}_1\widehat{\alpha}_1)}{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})W_2^{(n)}(\mathbf{z})} - \frac{\widehat{u}_1 \widehat{z}_1 \varepsilon_2^{(n)}}{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})\widehat{W}_2^{(n)}(\widehat{\mathbf{z}})}. \tag{14}$$

For k, $2 \le k \le n-1$, and for $n \ge 3$, we have

$$\begin{aligned}
\varepsilon_k^{(n)} &= \frac{\widehat{W}_k^{(n)}(\widehat{\mathbf{z}}) - W_k^{(n)}(\mathbf{z})}{W_k^{(n)}(\mathbf{z})} = \frac{1}{W_k^{(n)}(\mathbf{z})}\Big(1-\widehat{z}_2 - \frac{\widehat{u}_k\widehat{z}_1}{\widehat{W}_{k+1}^{(n)}(\widehat{\mathbf{z}})}\Big) - 1 = \\
&= \frac{1}{W_k^{(n)}(\mathbf{z})}\Big(1 - z_2(1+\alpha_2) - \frac{u_k(1+\beta_k)z_1(1+\alpha_1)}{W_{k+1}^{(n)}(\mathbf{z})(1+\varepsilon_{k+1}^{(n)})}\Big) - 1 = \\
&= \frac{1}{W_k^{(n)}(\mathbf{z})} - \frac{z_2(1+\alpha_2)}{W_k^{(n)}(\mathbf{z})} - \frac{u_k(1+\beta_k)z_1(1+\alpha_1)(1+\widehat{\varepsilon}_{k+1}^{(n)})}{W_k^{(n)}(\mathbf{z})W_{k+1}^{(n)}(\mathbf{z})} - 1.
\end{aligned}$$

Since $\frac{1}{W_k^{(n)}(\mathbf{z})} = 1 + \frac{z_2}{W_k^{(n)}(\mathbf{z})} + \frac{u_k z_1}{W_k^{(n)}(\mathbf{z})W_{k+1}^{(n)}(\mathbf{z})}$, then

$$\begin{aligned}
\varepsilon_k^{(n)} &= \frac{z_2}{W_k^{(n)}(\mathbf{z})} - \frac{z_2(1+\alpha_2)}{W_k^{(n)}(\mathbf{z})} - \frac{u_k z_1}{W_k^{(n)}(\mathbf{z})W_{k+1}^{(n)}(\mathbf{z})}((1+\beta_k)(1+\alpha_1)(1+\widehat{\varepsilon}_{k+1}^{(n)}) - 1) = \\
&= -\frac{z_2\alpha_2}{W_k^{(n)}(\mathbf{z})} - \frac{u_k z_1}{W_k^{(n)}(\mathbf{z})\widehat{W}_{k+1}^{(n)}(\widehat{\mathbf{z}})}(\beta_k+\alpha_1+\beta_k\alpha_1) - \frac{u_k z_1}{W_k^{(n)}(\mathbf{z})W_{k+1}^{(n)}(\mathbf{z})}\widehat{\varepsilon}_{k+1}^{(n)}.
\end{aligned} \tag{15}$$

Similarly, for $\widehat{\varepsilon}_k^{(n)}$, $2 \le k \le n-1$, and for $n \ge 3$, we obtain

$$\widehat{\varepsilon}_k^{(n)} = -\frac{\widehat{z}_2\widehat{\alpha}_2}{\widehat{W}_k^{(n)}(\widehat{\mathbf{z}})} - \frac{\widehat{u}_k \widehat{z}_1}{\widehat{W}_k^{(n)}(\widehat{\mathbf{z}})W_{k+1}^{(n)}(\mathbf{z})}(\widehat{\beta}_k+\widehat{\alpha}_1+\widehat{\beta}_k\widehat{\alpha}_1) - \frac{\widehat{u}_k \widehat{z}_1}{\widehat{W}_k^{(n)}(\widehat{\mathbf{z}})\widehat{W}_{k+1}^{(n)}(\widehat{\mathbf{z}})}\varepsilon_{k+1}^{(n)}. \tag{16}$$

Now, from (12)–(16) for $n \ge 3$ we get the following

$$\begin{aligned}
\varepsilon_n = \varepsilon_0^{(n)} &= \frac{v_0 z_2(\gamma_0+\alpha_2+\gamma_0\alpha_2)}{W_0^{(n)}(\mathbf{z})\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} + \frac{v_0 z_2 \widehat{\varepsilon}_1^{(n)}}{W_0^{(n)}(\mathbf{z})W_1^{(n)}(\mathbf{z})} = \frac{v_0 z_2(\gamma_0+\alpha_2+\gamma_0\alpha_2)}{W_0^{(n)}(\mathbf{z})\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} + \\
&+ \Big(\widehat{v}_1\widehat{z}_2(\widehat{\gamma}_1+\widehat{\alpha}_2+\widehat{\gamma}_1\widehat{\alpha}_2) + \frac{\widehat{u}_1\widehat{z}_1(\widehat{\beta}_1+\widehat{\alpha}_1+\widehat{\beta}_1\widehat{\alpha}_1)}{W_2^{(n)}(\mathbf{z})}\Big)\frac{\pi_0^{(n)}(\mathbf{z})}{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} - \pi_0^{(n)}(\mathbf{z})\widehat{\pi}_1^{(n)}(\widehat{\mathbf{z}})\varepsilon_2^{(n)}
\end{aligned}$$

and

$$\begin{aligned}
&-\pi_0^{(n)}(\mathbf{z})\widehat{\pi}_1^{(n)}(\widehat{\mathbf{z}})\varepsilon_2^{(n)} = \\
&= \frac{\pi_0^{(n)}(\mathbf{z})\widehat{\pi}_1^{(n)}(\widehat{\mathbf{z}})}{W_2^{(n)}(\mathbf{z})}\Big(z_2\alpha_2 + \frac{u_2 z_1(\beta_2+\alpha_1+\beta_2\alpha_1)}{\widehat{W}_3^{(n)}(\widehat{\mathbf{z}})}\Big) + \frac{u_2 z_1 \pi_0^{(n)}(\mathbf{z})\widehat{\pi}_1^{(n)}(\widehat{\mathbf{z}})\widehat{\varepsilon}_3^{(n)}}{W_2^{(n)}(\mathbf{z})W_3^{(n)}(\mathbf{z})} = \\
&= \frac{\pi_0^{(n)}(\mathbf{z})\widehat{\pi}_1^{(n)}(\widehat{\mathbf{z}})}{W_2^{(n)}(\mathbf{z})}\Big(z_2\alpha_2 + \frac{u_2 z_1(\beta_2+\alpha_1+\beta_2\alpha_1)}{\widehat{W}_3^{(n)}(\widehat{\mathbf{z}})}\Big) + \\
&+ \frac{\pi_0^{(n)}(\mathbf{z})\widehat{\pi}_1^{(n)}(\widehat{\mathbf{z}})\pi_2^{(n)}(\mathbf{z})}{\widehat{W}_3^{(n)}(\widehat{\mathbf{z}})}\Big(\widehat{z}_2\widehat{\alpha}_2 + \frac{\widehat{u}_3\widehat{z}_1(\widehat{\beta}_3+\widehat{\alpha}_1+\widehat{\beta}_3\widehat{\alpha}_1)}{W_4^{(n)}(\mathbf{z})}\Big) - \pi_0^{(n)}(\mathbf{z})\widehat{\pi}_1^{(n)}(\widehat{\mathbf{z}})\pi_2^{(n)}(\mathbf{z})\widehat{\pi}_3^{(n)}(\widehat{\mathbf{z}})\varepsilon_4^{(n)} =
\end{aligned}$$

$$= \sum_{k=2}^{n-1} \frac{\pi_0^{(n)}(\mathbf{z})}{\widetilde{W}_k^{(n)}} \Big(z_{2,k}\alpha_{2,k} + \frac{\widetilde{u}_k z_{1,k}(\widetilde{\beta}_k + \alpha_{1,k} + \widetilde{\beta}_k \alpha_{1,k})}{\widetilde{W}_{k+1}^{(n)}} \Big) \prod_{r=1}^{k-1} \widetilde{\pi}_r^{(n)},$$

where

$$z_{r,k} = \begin{cases} z_r, & \text{if } k \text{ even;} \\ \widehat{z}_r, & \text{if } k \text{ odd,} \end{cases} \qquad \alpha_{r,k} = \begin{cases} \alpha_r, & \text{if } k \text{ even;} \\ \widehat{\alpha}_r, & \text{if } k \text{ odd,} \end{cases} \qquad r = 1, 2, \quad \widetilde{\beta}_k = \begin{cases} \beta_k, & \text{if } k \text{ even;} \\ \widehat{\beta}_k, & \text{if } k \text{ odd,} \end{cases}$$

$$\widetilde{u}_k = \begin{cases} u_k, & \text{if } k \text{ even;} \\ \widehat{u}_k, & \text{if } k \text{ odd,} \end{cases} \qquad \widetilde{\pi}_k^{(n)} = \begin{cases} \pi_k^{(n)}(\mathbf{z}), & \text{if } k \text{ even;} \\ \widehat{\pi}_k^{(n)}(\widehat{\mathbf{z}}), & \text{if } k \text{ odd,} \end{cases} \qquad \widetilde{W}_k^{(n)} = \begin{cases} W_k^{(n)}(\mathbf{z}), & \text{if } k \text{ even;} \\ \widehat{W}_k^{(n)}(\widehat{\mathbf{z}}), & \text{if } k \text{ odd,} \end{cases}$$

$$\pi_0^{(n)}(\mathbf{z}) = -\frac{v_0 z_2}{W_0^{(n)}(\mathbf{z}) W_1^{(n)}(\mathbf{z})}, \tag{17}$$

$$\pi_{2k}^{(n)}(\mathbf{z}) = -\frac{u_{2k} z_1}{W_{2k}^{(n)}(\mathbf{z}) W_{2k+1}^{(n)}(\mathbf{z})}, \quad \widehat{\pi}_{2k-1}^{(n)}(\widehat{\mathbf{z}}) = -\frac{\widehat{u}_{2k-1} \widehat{z}_1}{\widehat{W}_{2k-1}^{(n)}(\widehat{\mathbf{z}}) \widehat{W}_{2k}^{(n)}(\widehat{\mathbf{z}})}, \quad 1 \le k \le \Big[\frac{n-1}{2}\Big], \tag{18}$$

(here $[\xi]$ is the integer part of the number ξ).

Thus, for $n \ge 3$,

$$\varepsilon_n = \frac{v_0 z_2 (\gamma_0 + \alpha_2 + \gamma_0 \alpha_2)}{W_0^{(n)}(\mathbf{z}) \widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} + \Big(\widehat{v}_1 \widehat{z}_2 (\widehat{\gamma}_1 + \widehat{\alpha}_2 + \widehat{\gamma}_1 \widehat{\alpha}_2) + \frac{\widehat{u}_1 \widehat{z}_1 (\widehat{\beta}_1 + \widehat{\alpha}_1 + \widehat{\beta}_1 \widehat{\alpha}_1)}{W_2^{(n)}(\mathbf{z})} \Big) \frac{\pi_0^{(n)}(\mathbf{z})}{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} +$$
$$+ \sum_{k=2}^{n-1} \frac{\pi_0^{(n)}(\mathbf{z})}{\widetilde{W}_k^{(n)}} \Big(z_{2,k}\alpha_{2,k} + \frac{\widetilde{u}_k z_{1,k}(\widetilde{\beta}_k + \alpha_{1,k} + \widetilde{\beta}_k \alpha_{1,k})}{\widetilde{W}_{k+1}^{(n)}} \Big) \prod_{r=1}^{k-1} \widetilde{\pi}_r^{(n)}. \tag{19}$$

2. Main results. The following theorem holds.

Theorem 1. *Let there exists a constant α, $0 < \alpha < 1$, such that*

$$|\alpha_1| \le \alpha, \quad |\alpha_2| \le \alpha, \quad |\gamma_0| \le \alpha, \quad |\gamma_1| \le \alpha, \quad |\beta_k| \le \alpha \quad k \ge 1, \tag{20}$$

where α_1, α_2, γ_0, γ_1, β_k, $k \ge 1$, are relative errors of z_1, z_2, v_0, v_1, u_k, $k \ge 1$, respectively, herewith v_0, v_1, u_k, $k \ge 1$, are defined in (3) and (4), $\mathbf{z} \in \mathfrak{D}_{u,v,\tau}$,

$$\mathfrak{D}_{u,v,\tau} = \Big\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{\tau(1-\tau)}{2u}, \ |z_2| < \frac{1-\tau}{2v} \Big\}, \tag{21}$$

$$u = \max\Big\{ \sup_{k \in \mathbb{N}} |u_k|, \ \sup_{k \in \mathbb{N}} |\widehat{u}_k| \Big\}, \quad v = \max\{v_1, \ 1\}, \quad \tau \in (0,1) \setminus \{1/3\}, \tag{22}$$

$\widehat{u}_k$, $k \ge 1$, are defined in (9), and $|v_0| < v$, $v_1 \ge 0$. Then the set (21) forms the numerical stability set of the branched continued fraction (2), and, in addition, if ε_n denotes the relative errors of nth approximant of (2), then, for $n \ge 3$,

$$|\varepsilon_n| < \frac{2|v_0|(1-\tau)\alpha}{1+\tau+|1-3\tau|} \Big(\frac{2+\alpha}{v - |v_0|} + \frac{4(1-\tau)}{v(1+\tau+|1-3\tau|) - 2|v_0|(1-\tau)} \times$$
$$\times \Big(\frac{3+\alpha}{2} + \frac{4\tau(2+\alpha)}{1+\tau+|1-3\tau|} \Big) \frac{\eta_\tau - \eta_\tau^{n-1}}{1-\eta_\tau} \Big), \tag{23}$$

where

$$\eta_\tau = \begin{cases} 2\tau/(1-\tau), & \text{if} \quad 0 < \tau < 1/3; \\ (1-\tau)/(2\tau), & \text{if} \quad 1/3 < \tau < 1. \end{cases} \tag{24}$$

Note that that for complex constants a, c, and d such that

$$|u_k| - \mathrm{Re}(u_k) \le rs(1-s), \quad k \ge 1, \quad v_1 \ge 0,$$

the branched continued fraction (2) converges uniformly on every compact subset of the domain (21), where u_k, $k \ge 1$, v_1 are defined in (3) and (4) herewith $c, d \notin \{0, -1, -2, \ldots\}$, and $r > 0$, $0 < s < 1$, (see [19, Theorem 3]).

Proof of Theorem 1. In [20] the continued fraction

$$(1+\tau)/2 - \cfrac{\tau(1-\tau)/2}{(1+\tau)/2 - \cfrac{\tau(1-\tau)/2}{(1+\tau)/2 - \ddots}} \tag{25}$$

is considered and it is shown that (25) converges to

$$f = \frac{1+\tau+|1-3\tau|}{4} \tag{26}$$

for $0 < \tau < 1$ (here the elements of the continued fraction (25) satisfy the Theorem 3.2 in [43]). It is also noted that its approximants f_n, $n \ge 1$, forms a monotonically decreasing sequence (which is easy to verify).

Let n be an arbitrary natural number such that $n \ge 3$. In what follows, we will prove that

$$|W_k^{(n)}(\mathbf{z})| > f_{n-k}, \quad 1 \le k \le n-1, \tag{27}$$

where $W_k^{(n)}(\mathbf{z})$, $1 \le k \le n-1$, are defined in (6) and (7).

For $k = n-1$ we have

$$|W_{n-1}^{(n)}(\mathbf{z})| \ge 1 - v_{n-1}|z_2| - |u_{n-1}||z_1| > \frac{1+\tau}{2} - \frac{\tau(1-\tau)}{2} > (1+\tau)/2 - \frac{\tau(1-\tau)/2}{(1+\tau)/2} = f_1.$$

Let (27) holds for $k = s+1 \le n-1$. Then, using the induction assumption, for $k = s$ from (6) we obtain

$$|W_s^{(n)}(\mathbf{z})| = \left|1 - v_s z_2 - \frac{u_s z_1}{W_{s+1}^{(n)}(\mathbf{z})}\right| \ge 1 - v_s|z_2| - \frac{|u_s||z_1|}{|W_{s+1}^{(n)}(\mathbf{z})|} > \frac{(1+\tau)}{2} - \frac{\tau(1-\tau)/2}{f_{n-s-1}} = f_{n-s},$$

which proves (27).

Since the sequence $\{f_n\}$ is monotonically decreasing, then $f_n > f$, $n \ge 1$, and, therefore, $|W_k^{(n)}(\mathbf{z})| > f$, $1 \le k \le n-1$, where f is defined by (26). Now, we estimate the following

$$|W_0^{(n)}(\mathbf{z})| = \left|1 + \frac{v_0 z_2}{W_1^{(n)}(\mathbf{z})}\right| \ge 1 - \frac{|v_0||z_2|}{|W_1^{(n)}(\mathbf{z})|} > 1 - \frac{2(1-\tau)}{1+\tau+|1-3\tau|} \ge 1 - \frac{|v_0|}{v}. \tag{28}$$

Next, we estimate the values $\pi_0^{(n)}(\mathbf{z})$, $\pi_{2k}^{(n)}(\mathbf{z})$, $1 \le k \le [(n-1)/2]$, which are defined in (17) and (18). We have

$$|\pi_0^{(n)}(\mathbf{z})| = \left|\frac{v_0 z_2}{W_0^{(n)}(\mathbf{z}) W_1^{(n)}(\mathbf{z})}\right| \le \frac{\dfrac{|v_0||z_2|}{|W_1^{(n)}(\mathbf{z})|}}{1 - \dfrac{|v_0||z_2|}{|W_1^{(n)}(\mathbf{z})|}} < \frac{2|v_0|(1-\tau)}{v(1+\tau+|1-3\tau|) - 2|v_0|(1-\tau)}, \tag{29}$$

and for any k, $1 \le k \le [(n-1)/2]$, we get

$$|\pi_{2k}^{(n)}(\mathbf{z})| = \Big|\frac{u_{2k}z_1}{W_{2k}^{(n)}(\mathbf{z})W_{2k+1}^{(n)}(\mathbf{z})}\Big| \le \frac{\dfrac{|u_{2k}||z_1|}{|W_{2k+1}^{(n)}(\mathbf{z})|}}{1-v_{2k}|z_2|-\dfrac{|u_{2k}||z_1|}{|W_{2k+1}^{(n)}(\mathbf{z})|}} <$$
$$< \frac{\tau(1-\tau)}{(1+\tau)\dfrac{1+\tau+|1-3\tau|}{4}-\tau(1-\tau)} = \eta_\tau, \tag{30}$$

where η_τ is defined by (24). Now, since $\widehat{\mathbf{z}} \in \mathfrak{D}_{u,v,\tau}$, where $\mathfrak{D}_{u,v,\tau}$ is defined by (21), then

$$|\widehat{W}_k^{(n)}(\widehat{\mathbf{z}})| > f_{n-k}, \quad 1 \le k \le n-1, \tag{31}$$

and, therefore,

$$\widehat{\pi}_{2k-1}^{(n)}(\widehat{\mathbf{z}}) < \eta_\tau, \quad 1 \le k \le [(n-1)/2], \tag{32}$$

where $\widehat{W}_k^{(n)}(\widehat{\mathbf{z}})$, $1 \le k \le n-1$, and $\widehat{\pi}_{2k-1}^{(n)}(\widehat{\mathbf{z}})$, $1 \le k \le [(n-1)/2]$, are defined in (10) and (18). In addition, from the conditions of this theorem it follows

$$|z_{1,k}| < \frac{\tau(1-\tau)}{2u}, \quad |z_{2,k}| < \frac{1-\tau}{2v}, \quad |\widetilde{u}_k| \le u, \quad 1 \le k \le n.$$

Now, from (19) we have

$$|\varepsilon_n| \le \frac{|v_0||z_2|}{|W_0^{(n)}(\mathbf{z})||\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})|}(|\gamma_0|+|\alpha_2|+|\gamma_0||\alpha_2|)+$$
$$+\Big(|\widehat{v}_1||\widehat{z}_2|(|\widehat{\gamma}_1|+|\widehat{\alpha}_2|+|\widehat{\gamma}_1||\widehat{\alpha}_2|)+\frac{|\widehat{u}_1||\widehat{z}_1|(|\widehat{\beta}_1|+|\widehat{\alpha}_1|+|\widehat{\beta}_1||\widehat{\alpha}_1|)}{|W_2^{(n)}(\mathbf{z})|}\Big)\frac{|\pi_0^{(n)}(\mathbf{z})|}{|\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})|}+$$
$$+\sum_{k=2}^{n-1}\frac{|\pi_0^{(n)}(\mathbf{z})|}{|\widetilde{W}_k^{(n)}|}\Big(|z_{2,k}||\alpha_{2,k}|+\frac{|\widetilde{u}_k||z_{1,k}|(|\widetilde{\beta}_k|+|\alpha_{1,k}|+|\widetilde{\beta}_k|\alpha_{1,k}|)}{|\widetilde{W}_{k+1}^{(n)}|}\Big)\prod_{r=1}^{k-1}|\widetilde{\pi}_r^{(n)}|.$$

Using (20)–(22) and (26)–(32), we get

$$|\varepsilon_n| < \frac{2|v_0|(1-\tau)\alpha}{1+\tau+|1-3\tau|}\times$$
$$\times\Big(\frac{2+\alpha}{v-|v_0|}+\frac{4(1-\tau)}{v(1+\tau+|1-3\tau|)-2|v_0|(1-\tau)}\Big(\frac{3+\alpha}{2}+\frac{4\tau(2+\alpha)}{1+\tau+|1-3\tau|}\Big)\sum_{k=2}^{n-1}\eta_\tau^{k-1}\Big).$$

Applying the equality $\sum_{k=2}^{n-1}\eta^{k-1} = \eta\cdot\frac{1-\eta^{n-2}}{1-\eta}$, $\eta \ne 1$, we obtain (23).

Let $\tau \in (0,1)\setminus\{1/3\}$. Then we consider the function $\varphi(\alpha)$ defined by the right side of (23). Since $\lim\limits_{\alpha\to 0^+}\varphi(\alpha) = 0$, then for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that for any $0 < \alpha < \delta_\epsilon$, $\varphi(\alpha) < \epsilon$. Thus, if

$$|\alpha_1| \le \alpha < \delta_\epsilon, \quad |\alpha_2| \le \alpha < \delta_\epsilon, \quad |\gamma_0| \le \alpha < \delta_\epsilon, \quad |\gamma_1| \le \alpha < \delta_\epsilon, \quad |\beta_k| \le \alpha < \delta_\epsilon \quad k \ge 1,$$

then $|\epsilon_n| \le \varphi(\alpha) < \epsilon$, $n \ge 3$, proving numerical stability of branched continued fraction (2). □

Remark 1. If $\tau = 1/3$ in (22), then from (30) it follows that $\eta_\tau = 1$ and then ε depends on n, which contradicts Definition 1.

REFERENCES

1. T. Acar, O. Alagoz, A. Aral, D. Costarelli, M. Turgay, G. Vinti, *Approximation by sampling Kantorovich series in weighted spaces of functions*, Turkish J. Math., **46** (2022). http://dx.doi.org/2663–2676.10.55730/1300-0098.3293
2. T. Acar, O. Alagoz, A. Aral, D. Costarelli, M. Turgay, G. Vinti, *Convergence of generalized sampling series in weighted spaces*, Demonstr. Math., **55** (2022), 153–162. http://dx.doi.org/10.1515/dema-2022-0014
3. O. Alagoz, *Weighted approximations by sampling type operators: recent and new results*, Constr. Math. Anal., **7** (2024), 114–125. http://dx.doi.org/10.33205/cma.1528004
4. T. Antonova, R. Dmytryshyn, P. Kril, S. Sharyn, *Representation of some ratios of Horn's hypergeometric functions* H_7 *by continued fractions*, Axioms, **12** (2023), 738. http://dx.doi.org/10.3390/math11214487
5. T. Antonova, R. Dmytryshyn, S. Sharyn, *Generalized hypergeometric function* ${}_3F_2$ *ratios and branched continued fraction expansions*, Axioms, **10** (2021), 310. http://dx.doi.org/10.3390/axioms10040310
6. T. Antonova, R. Dmytryshyn, I.-A. Lutsiv, S. Sharyn, *On some branched continued fraction expansions for Horn's hypergeometric function* $H_4(a,b;c,d;z_1,z_2)$ *ratios*, Axioms, **12** (2023), 299. http://dx.doi.org/10.3390/axioms12030299
7. T. Antonova, R. Dmytryshyn, S. Sharyn, *Branched continued fraction representations of ratios of Horn's confluent function* H_6, Constr. Math. Anal., **6** (2023), 22–37. http://dx.doi.org/10.33205/cma.1243021
8. W.W. Bell, Special functions for scientists and engineers, D. Van Nostrand Ltd, London, 1968.
9. D.I. Bodnar, O.S. Bodnar, M.V. Dmytryshyn, M.M. Popov, M.V. Martsinkiv, O.B. Salamakha, *Research on the convergence of some types of functional branched continued fractions*, Carpathian Math. Publ., **16** (2024), 448–460. http://doi.org/10.15330/cmp.16.2.448-460
10. D.I. Bodnar, Branched continued fractions, Naukova Dumka, Kyiv, 1986. (in Russian)
11. P.I. Bodnarchuk, V.Y. Skorobohatko, Branched continued fractions and their applications, Naukova Dumka, Kyiv, 1974. (in Ukrainian)
12. J. Bustamante, *Approximation of bounded functions by positive linear operators*, Modern Math. Methods, **2** (2024), 117–131. http://dx.doi.org/10.64700/mmm.30
13. S. Chaichenko, V. Savchuk, A. Shidlich, *Approximation of functions by linear summation methods in the Orlicz-type spaces*, J. Math. Sci., **249** (2020), 705–719. http://dx.doi.org/10.1007/s10958-020-04967-y
14. S.O. Chaichenko, A.L. Shidlich, *Approximation of functions by linear methods in weighted Orlicz type spaces with variable exponent*, Res. Math., **32** (2024), 70–87. http://doi.org/10.15421/242420
15. A. Cuyt, P. Van der Cruyssen, *Rounding error analysis for forward continued fraction algorithms*, Comput. Math. Appl., **11** (1985), 541–564. http://dx.doi.org/10.1016/0898-1221(85)90037-9
16. M. Dmytryshyn, V. Hladun, *On the sets of stability to perturbations of some continued fraction with applications*, Symmetry, **17(9)** (2025), 1442. http://doi.org/10.3390/sym17091442
17. R. Dmytryshyn, T. Antonova, M. Dmytryshyn, *On the analytic extension of the Horn's confluent function* H_6 *on domain in the space* $\mathbb{C}^2$, Constr. Math. Anal., **7** (2024), 11–26. http://dx.doi.org/10.33205/cma.1545452
18. R. Dmytryshyn, C. Cesarano, M. Dmytryshyn, I.-A. Lutsiv, *A priori bounds for truncation error of branched continued fraction expansions of Horn's hypergeometric functions* H_4 *and their ratios*, Res. Math., **33** (2025), 13–22. http://dx.doi.org/10.15421/242502
19. R. Dmytryshyn, C. Cesarano, I.-A. Lutsiv, *On the analytical continuation of the ratio* $H_4(\alpha,\delta+1;\gamma,\delta;-\mathbf{z})/H_4(\alpha,\delta+2;\gamma,\delta+1;-\mathbf{z})$, Res. Math., **33** (2025), 65–76. https://dx.doi.org/10.15421/242515
20. R. Dmytryshyn, C. Cesarano, I.-A. Lutsiv, M. Dmytryshyn, *Numerical stability of the branched continued fraction expansion of Horn's hypergeometric function* H_4, Mat. Stud., **61** (2024), №1, 51–60. http://dx.doi.org/10.30970/ms.61.1.51-60
21. R. Dmytryshyn, I.-A. Lutsiv, O. Bodnar, *On the domains of convergence of the branched continued fraction expansion of ratio* $H_4(a,d+1;c,d;\mathbf{z})/H_4(a,d+2;c,d+1;\mathbf{z})$, Res. Math., **31** (2023), 19–26. http://dx.doi.org/10.15421/242311
22. R. Dmytryshyn, I.-A. Lutsiv, M. Dmytryshyn, C. Cesarano, *On some domains of convergence of branched continued fraction expansions of the ratios of Horn hypergeometric functions* H_4, Ukr. Math. J., **76** (2024), 559–565. http://dx.doi.org/10.1007/s11253-024-02338-3

23. R. Dmytryshyn, I.-A. Lutsiv, M. Dmytryshyn, *On the analytic extension of the Horn's hypergeometric function H_4*, Carpathian Math. Publ., **16** (2024), 32–39. http://dx.doi.org/10.15330/cmp.16.1.32-39
24. H. Exton, Multiple hypergeometric functions and applications, Halsted Press, Chichester, 1976.
25. C. Ferreira, J.L. López, E. Pérez Sinusía, *Convergent and asymptotic expansions of the displacement elastodynamic integral in terms of known functions*, J. Comput. Appl. Math., **460** (2025), 116395. http://dx.doi.org/10.1016/j.cam.2024.116395.
26. C. Ferreira, J.L. López, E. Pérez Sinusía, *New series expansions for the $\mathscr{H}$-function of communication theory*, Integral Transforms Spec. Funct., **34** (2023), 879–890. http://dx.doi.org/10.1080/10652469.2023.2234556
27. C. Ferreira, J.L. López, E. Pérez Sinusía, *Uniform convergent expansions of the error function in terms of elementary functions*, Mediterr. J. Math., **20** (2023), 117. http://dx.doi.org/10.1007/s00009-023-02297-2
28. P.-Ch. Hang, M. Henkel, M.-J. Luo, *Asymptotics of the Humbert functions Ψ_1 and Ψ_2*, J. Approx. Theory, **314** (2026), 106233. http://dx.doi.org/10.1016/j.jat.2025.106233
29. P.-Ch. Hang, M.-J. Luo, *Asymptotics of Saran's hypergeometric function F_K*, J. Math. Anal. Appl., **541** (2025), 128707. http://dx.doi.org/10.1016/j.jmaa.2024.128707
30. P.-Ch. Hang, M.-J. Luo, *Asymptotics of the Humbert functions Ψ_1 for two large arguments*, SIGMA, **20** (2024), 074. http://dx.doi.org/10.3842/SIGMA.2024.074
31. O. Handera-Kalynovska, V. Kravtsiv, *The Waring-Girard formulas for symmetric polynomials on spaces l_p*, Carpathian Math. Publ., **16** (2024), 407–413. http://dx.doi.org/10.15330/cmp.16.2.407-413
32. V.R. Hladun, D.I. Bodnar, R.S. Rusyn, *Convergence sets and relative stability to perturbations of a branched continued fraction with positive elements*, Carpathian Math. Publ., **16** (2024), 16–31. http://dx.doi.org/10.15330/cmp.16.1.16-31
33. V.R. Hladun, M.V. Dmytryshyn, V.V. Kravtsiv, R.S. Rusyn, *Numerical stability of the branched continued fraction expansions of the ratios of Horn's confluent hypergeometric functions* H_6, Math. Model. Comput., **11** (2024), 1152–1166. http://doi.org/10.23939/mmc2024.04.1152
34. V.R. Hladun, N.P. Hoyenko, O.S. Manzij, L. Ventyk, *On convergence of function $F_4(1,2;2,2;z_1,z_2)$ expansion into a branched continued fraction*, Math. Model. Comput., **9** (2022), 767–778. http://dx.doi.org/10.23939/mmc2022.03.767
35. V. Hladun, V. Kravtsiv, M. Dmytryshyn, R. Rusyn, *On numerical stability of continued fractions*, Mat. Stud., **62** (2024), №2, 168–183. http://doi.org/10.30970/ms.62.2.168-183
36. V. Hladun, R. Rusyn, M. Dmytryshyn, *On the analytic extension of three ratios of Horn's confluent hypergeometric function* H_7, Res. Math., **32** (2024), 60–70. http://dx.doi.org/10.15421/242405
37. V.R. Hladun, *Some sets of relative stability under perturbations of branched continued fractions with complex elements and a variable number of branches*, Mat. method. and fiz.-mech. polya, **57** (2014), 14–24. (in Ukrainian); Engl. transl.: J. Math. Sci., **215** (2016), 11–25. http://dx.doi.org/10.1007/s10958-016-2818-x
38. J. Horn, *Hypergeometrische funktionen zweier veränderlichen*, Math. Ann., **105** (1931), 381–407. 20230135. https://doi.org/10.1515/dema-2023-013510.1007/BF01455825
39. N. Hoyenko, T. Antonova, S. Rakintsev, *Approximation for ratios of Lauricella–Saran fuctions F_S with real parameters by a branched continued fractions*, Math. Bul. Shevchenko Sci. Soc., **8** (2011), 28–42. (in Ukrainian)
40. N.P. Hoyenko, V.R. Hladun, O.S. Manzij, *On the infinite remains of the Nörlund branched continued fraction for Appell hypergeometric functions*, Carpathian Math. Publ., **6** (2014), 11–25. (in Ukrainian) http://dx.doi.org/10.15330/cmp.6.1.11-25
41. N.P. Hoyenko, O.S. Manzij, *Expansion of Appel F_1 and Lauricella $F_D^{(N)}$ hypergeometric functions into branched continued fractions*, Visnyk Lviv. Univ. Ser. Mech.-Math., **48** (1997), 17–26. (in Ukrainian)
42. R. Hryniv, V. Kravtsiv, T. Vasylyshyn, A. Zagorodnyuk, *Symmetric and supersymmetric polynomials on l_p and partition functions in quantum statistical physics*, Phys. Scr., **100** (2025), 075208. http://dx.doi.org/10.1088/1402-4896/adde1e
43. W.B. Jones, W.J. Thron, Continued fractions: analytic theory and applications, Addison-Wesley Pub. Co., Reading, 1980.
44. D. Kaliuzhnyi-Verbovetskyi, V. Pivovarchik, *Recovering the shape of a quantum caterpillar tree by two spectra*, Mech. Math. Methods, **5** (2023), №1, 14–24. http://dx.doi.org/10.31650/2618-0650-2023-5-1-14-24
45. H. Karsli, *Approximation process of a positive linear operator of hypergeometric type*, Demonstr. Math., **57** (2024), 20230135. 20230135. https://doi.org/10.1515/dema-2023-013510.1515/dema-2023-0135

46. H. Karsli, *Historical Backround of wavelets and orthonormal systems: Recent results on positive linear operators reconstructed via wavelets: Recent results on positive linear operators reconstructed via wavelets*, Modern Math. Methods, **2** (2024), 132–154. http://dx.doi.org/10.64700/mmm.44
47. V. Kravtsiv, *Block-supersymmetric polynomials on spaces of absolutely convergent series*, Symmetry, **16** (2024), 179. http://dx.doi.org/10.3390/sym16020179
48. V. Kravtsiv, A. Zagorodnyuk, *Descriptions of spectra of algebras of bounded-type block-symmetric analytic functions*, Symmetry, **17** (2025), 1974. http://dx.doi.org/10.3390/sym17111974
49. V. Kravtsiv, A.V. Zagorodnyuk, *Spectra of algebras of block-symmetric analytic functions of bounded type*, Mat. Stud., **58** (2022), №1, 69–81. http://dx.doi.org/10.30970/ms.58.1.69-81
50. Kh.Yo. Kuchminska, Two-dimensional continued fractions, Pidstryhach Institute for Appl. Probl. in Mech. and Math., NAS of Ukraine, Lviv, 2010. (in Ukrainian)
51. D. Leviatan, O.V. Motorna, I.O. Shevchuk, *More on fast decreasing trigonometric polynomials*, Res. Math., **32** (2024), 101–114. https://doi.org/10.15421/242422
52. L. Lorentzen, H. Waadeland, Continued fractions, V.1, Convergence Theory, Atlantis Press, Amsterdam, 2008.
53. Y. Lutsiv, T. Antonova, R. Dmytryshyn, M. Dmytryshyn, *On the branched continued fraction expansions of the complete group of ratios of the generalized hypergeometric function* ${}_4F_3$, Res. Math., **32** (2024), 115–132. http://dx.doi.org/10.15421/242423
54. O. Manziy, V. Hladun, L. Ventyk, *The algorithms of constructing the continued fractions for any rations of the hypergeometric Gaussian functions*, Math. Model. Comput., **4** (2017), 48–58. http://dx.doi.org/10.23939/mmc2017.01.048
55. A. Pinti, O. Tulai, Y. Chaikovskyi, M. Stetsko, M. Dmytryshyn, L. Alekseyenko, *International Communication and Innovation in Investment in Flagship, Mortgage and Social Projects to Guarantee Security and Minimize Risks of Public Finances*, In: R.K. Hamdan (eds) Integrating Big Data and IoT for Enhanced Decision-Making Systems in Business, Studies in Big Data, **177**, Springer, Cham, 2026. http://doi.org/10.1007/978-3-031-97609-4_35
56. K.V. Pozharska, A.S. Romanyuk, S.Ya. Yanchenko, *Estimates of the characteristics of nonlinear approximation of classes of periodic functions of many variables*, Carpathian Math. Publ., **17** (2025), 447–460. http://dx.doi.org/10.15330/cmp.17.2.447-460.
57. J. Prestin, V. Savchuk, A. Shidlich, *Approximation on hexagonal domains by Taylor-Abel-Poisson means*, J. Math. Anal. Appl., **529** (2024), 12753674. http://dx.doi.org/10.1016/j.jmaa.2023.127536
58. J.B. Seaborn, Hypergeometric functions and their applications, Springer, New York, 1991.
59. A.S. Romanyuk, S.Y. Yanchenko, *Approximation of the classes of periodic functions of one and many variables from the Nikol'skii–Besov and Sobolev spaces*, Ukr. Math. J., **74** (2022), 967–980. http://dx.doi.org/10.1007/s11253-022-02110-5
60. A. Zagorodnyuk, N. Baziv, Y. Chopyuk, T. Vasylyshyn, I. Burtnyak, V. Kravtsiv, *Symmetric and supersymmetric polynomials and their applications in the blockchain technology and neural networks*, Proceedings of 2023 IEEE World Conference on Applied Intelligence and Computing (AIC), Sonbhadra, India, (2023), 508–513. http://dx.doi.org/10.1109/AIC57670.2023.10263936

West Ukrainian National University
Ternopil, Ukraine
m.dmytryshyn@wunu.edu.ua

International Telematic University UNINETTUNO
Roma, Italy
c.cesarano@uninettunouniversity.net

Vasyl Stefanyk Carpathian National University
Ivano-Frankivsk, Ukraine
oksana.kondur@cnu.edu.ua
lutsiv.ilona@gmail.com

Received 05.03.2025
Revised 02.12.2025