УДК 517.5

Nabadwip Sarkar

BOUNDS ON HERMITIAN-TOEPLITZ DETERMINANT FOR STARLIKE, CONVEX AND BOUNDED TURNING FUNCTIONS ASSOCIATED WITH THE EXPONENTIAL FUNCTION

Nabadwip Sarkar. *Bounds on Hermitian-Toeplitz determinant for starlike, convex and bounded turning functions associated with the exponential function*, Mat. Stud. **64** (2025), 115–123.

In the article we derive sharp bounds for the third-order Hermitian-Toeplitz determinant of starlike and convex functions associated with the exponential function, as well as for their inverse classes. An analytic in the unit disk $\mathbb{D} = \{z\colon |z| < 1\}$ function f is said to be subordinate to an analytic in $\mathbb{D}$ function g (denoted by $f \prec g$), if there exists an analytic in $\mathbb{D}$ function w with $|w(z)| \le |z|$ and $w(0) = 0$ such that $f(z) = g(w(z))$. Let $\mathcal{A}$ be the class of analytic functions f in $\mathbb{D}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in \mathbb{D}$, and $\mathcal{S}_e^*$ be the class of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec e^z$.

In particular, the following statement has been proven (Theorem 1): If $f \in \mathcal{S}_e^*$, then $-\frac{1}{15} \le T_{3,1}(f) \le 1$, where $T_{3,1}(f)$ is the third-order Hermitian-Toeplitz determinant of the form $T_{3,1}(f) := 2\operatorname{Re}\big(a_2^2 \cdot \overline{a_3}\big) - 2|a_2|^2 - |a_3|^2 + 1$. The upper and lower bounds are sharp.

The article also obtained similar (sharp) estimates in the class

$$\mathcal{C}_e := \Big\{f \in \mathcal{A}\colon 1 + \frac{zf''(z)}{f'(z)} \prec e^z\Big\}\colon \quad \frac{9}{16} \le T_{3,1}(f) \le 1 \quad \text{(Theorem 2)},$$

and in the class

$$\mathcal{R}_e := \{f'(z) \prec e^z\}\colon \quad \frac{5}{9} \le T_{3,1}(f) \le 1 \quad \text{(Theorem 3)}.$$

1. Introduction. Let $\mathcal{A}$ denote the family of normalized analytic functions f in the open unit disc $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ having the Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

We write $\mathcal{S}$ for the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$. A function f is said to be *subordinate* to a function g, denoted by $f \prec g$, if there exists an analytic function w with $|w(z)| \le |z|$ and $w(0) = 0$ such that $f(z) = g(w(z))$. If g is univalent and $f(0) = g(0)$, then $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

Let φ be an analytic and univalent function in $\mathbb{D}$, starlike with respect to $\varphi(0) = 1$, satisfying $\varphi'(0) > 0$, and symmetric about the real axis. Ma and Minda ([5]) extended the classical families of bounded turning functions, starlike and convex functions by introducing the following classes: $\mathcal{R}(\varphi) := \{f'(z) \prec \varphi(z)\}$,

$$\mathcal{S}^*(\varphi) := \Big\{f \in \mathcal{A}\colon \frac{zf'(z)}{f(z)} \prec \varphi(z)\Big\}, \quad \mathcal{C}(\varphi) := \Big\{f \in \mathcal{A}\colon 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)\Big\},$$

2020 *Mathematics Subject Classification:* 30C50, 30C45.
Keywords: starlike functions; convex functions; coefficient problems; Hermitian-Toeplitz determinant.
doi:10.30970/ms.64.2.115-123

Recently, considerable attention has been devoted to subclasses of starlike and convex functions in which the superordinate function $\varphi(z)$ does not necessarily map onto the right half-plane. A natural choice is the exponential function, which gives rise to interesting and nontrivial problems.

The family of starlike functions related to the exponential function was introduced by R. Mendiratta, S. Nagpal and V. Ravichandran ([6]) and is given by

$$\mathcal{S}_e^* := \left\{ f \in \mathcal{A} \colon \frac{zf'(z)}{f(z)} \prec e^z \right\}.$$

Similarly, the corresponding family of convex functions and bounded turning functions associated with the exponential function are defined as

$$\mathcal{C}_e := \left\{ f \in \mathcal{A} \colon 1 + \frac{zf''(z)}{f'(z)} \prec e^z \right\}, \quad \mathcal{R}_e := \left\{ f'(z) \prec e^z \right\}.$$

M. F. Ali, D. K. Thomas and A. Vasudevarao ([1]) obtained sharp estimates of the symmetric Toeplitz determinants for univalent and typically real functions. The extensive research devoted to symmetric Toeplitz and Hankel determinants has motivated the study of bounds for Hermitian-Toeplitz determinants.

For the sequence $\{a_k\}$ of coefficients of a normalized analytic function f, the Hermitian-Toeplitz determinant of order n is defined by ([3, 7])

$$T_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \overline{a_{n+1}} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{n+q-1}} & \overline{a_{n+q-2}} & \cdots & a_n \end{vmatrix}. \tag{2}$$

From equation (2), it follows that the third-order Hermitian-Toeplitz determinant $T_{3,1}(f)$ reduces to the functional form

$$T_{3,1}(f) := 2\ \mathrm{Re}\big(a_2^2 \cdot \overline{a_3}\big) - 2|a_2|^2 - |a_3|^2 + 1. \tag{3}$$

If $f \in \mathcal{S}$, then the inverse function $F := f^{-1}$ is well-defined and analytic in $D_{r(f)}$, where $r(f) := \sup\{r > 0 \colon D_r \subset f(\mathbb{D})\}$. Thus, we can write

$$F(w) = w + \sum_{n=2}^{\infty} A_n w^n, \quad w \in D_{r(f)}, \tag{4}$$

where $A_n := a_n(F)$.

It follows from equation (4) (e.g., [4, V.I, p. 57]) that

$$A_2 = -a_2, \qquad A_3 = -a_3 + 2a_2^2, \qquad a_n := a_n(f). \tag{5}$$

Then from (3), (5), we deduce that

$$T_{3,1}(f^{-1}) := 2\ \mathrm{Re}\big(a_2^2 \cdot \overline{a_3}\big) - 2|A_2|^2 - |A_3|^2 + 1 = 2\ \mathrm{Re}\big(a_2^2 \cdot \overline{a_3}\big) - 2|a_2|^2 - |a_3|^2 + 1 = T_{3,1}(f).$$

In paper [2] were the first to establish sharp lower and upper bounds for the second and third-order Hermitian–Toeplitz determinants for the classes of starlike and convex functions of order α. V. Kumar, R. Srivastava and N. E. Cho ([8]) derived the sharp bounds for second and third-order Hermitian-Toeplitz determinants in the Janowski starlike and convex function classes. More recently, S. Kumar, R. K. Pandey and P. Rai ([9]) determined sharp bounds on Hermitian-Toeplitz determinants of associated Sakaguchi functions.

Inspired by the previous studies on Hermitian-Toeplitz determinants, this paper focuses on deriving sharp upper and lower bounds for the third-order Hermitian-Toeplitz determinant corresponding to functions belonging to the classes $\mathcal{S}_e^*$, $\mathcal{C}_e$ and $\mathcal{R}_e$.

We recall the following result due to R. J. Libera and E. J. Zlotkiewicz ([10]).

Lemma 1 ([10], Lemma 3, p.254)**.** *Let $\mathcal{P}$ denote the class of analytic functions with the Taylor expansion*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots , \tag{6}$$

satisfying $\mathrm{Re}\{p(z)\} > 0$ *for* $z \in \mathbb{D}$. *Then* $2p_2 = p_1^2 + (4 - p_1^2)\,\xi$ *for some* $\xi \in \overline{\mathbb{D}}$.

2. Main results.

Theorem 1. *Let $f \in \mathcal{S}_e^*$ and be given by (1). Then*

$$-\frac{1}{15} \le T_{3,1}(f) \le 1. \tag{7}$$

The upper bound and lower bound are sharp for the functions f_1 and f_2 defined by

$$f_1(z) = z \exp\Big(\int_0^z \frac{e^{t^3} - 1}{t}\, dt\Big) = z + \frac{1}{3} z^4 + \ldots,$$
$$f_2(z) = \exp \int_0^z \exp\Big(\big(t\sqrt{56/15}t - 2t^2\big)\big(2 - t\sqrt{56/15}\big)\Big)\frac{dt}{t} = z + \sqrt{14/15}z^2 + \frac{2}{3}z^3 + \ldots,$$

respectively.

Proof. Let $f \in \mathcal{S}_e^*$. By definition of the class $\mathcal{S}_e^*$, there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < |z|$ in $\mathbb{D}$ such that

$$\frac{zf'(z)}{f(z)} = e^{w(z)}. \tag{8}$$

Let $p \in \mathcal{P}$. Then, using the definition of subordination, we can write

$$w(z) = \frac{p(z) - 1}{p(z) + 1}. \tag{9}$$

Let p be given by (6). From (8) and (9), by equating coefficients we obtain

$$a_2 = \frac{1}{2} p_1, \qquad a_3 = \frac{1}{16} p_1^2 + \frac{1}{4} p_2. \tag{10}$$

Before proceeding with the estimates, it is important to note that both the class $\mathcal{P}$ of functions with positive real part and the class $\mathcal{S}_e^*$ are invariant under rotations. Hence, without loss of generality, since $|p_n| \le 2$, we may assume $0 \le p_1 \le 2$.

In view of Lemma 1 together with (10), we obtain, for some $\xi \in \overline{\mathbb{D}}$, that

$$2\,\mathrm{Re}(a_2^2 \cdot \overline{a_3}) = 2\,\mathrm{Re}\left(\frac{p_1^2}{4}\right)\left(\frac{1}{16}p_1^2 + \frac{1}{4}\overline{p}_2\right) = \frac{p_1^2}{32}\left(p_1^2 + 2(p_1^2 + (4-p_1^2)\ \mathrm{Re}\ \overline{\xi}\right) =$$
$$= \frac{3p_1^4}{32} + \frac{p_1^2(4-p_1^2)\ \mathrm{Re}\ \overline{\xi}}{16}, \qquad -2|a_2|^2 = -\frac{1}{2}p_1^2, \tag{11}$$
$$-|a_3|^2 = -\left|\frac{1}{16}p_1^2 + \frac{1}{4}p_2\right|^2 = -\left|\frac{3}{16}p_1^2 + \frac{1}{8}(4-p_1^2)\,\xi\right|^2 =$$
$$= -\frac{9}{256}p_1^4 - \frac{3}{64}p_1^2(4-p_1^2)\ \mathrm{Re}\ \xi - \frac{1}{64}(4-p_1^2)^2\,|\xi|^2. \tag{12}$$

By applying equations (11), (12), equation (3) can be written as

$$T_{3,1}(f) = 1 + \frac{3p_1^4}{32} + \frac{p_1^2(4-p_1^2)\ \mathrm{Re}\ \overline{\xi}}{16} - \frac{1}{2}p_1^2 - \frac{9}{256}p_1^4 - \frac{3}{64}p_1^2(4-p_1^2)\ \mathrm{Re}\ \xi - \frac{(4-p_1^2)^2\,|\xi|^2}{64} =$$
$$= \frac{1}{256}\left(256 + 15p_1^4 - 128p_1^2 + 4p_1^2(4-p_1^2)\ \mathrm{Re}\ \xi - 4(4-p_1^2)^2\,|\xi|^2\right). \tag{13}$$

Next, we aim to maximize the right-hand side of (13). Since $\mathrm{Re}\,\xi \le |\xi|$, it follows from (13) that

$$T_{3,1}(f) \le \frac{1}{256}\left(256 + 15p_1^4 - 128p_1^2 + 4p_1^2(4-p_1^2)|\xi| - 4(4-p_1^2)^2\,|\xi|^2\right) = \frac{1}{256}F(p_1^2, |\xi|). \tag{14}$$

Setting $p_1^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$, then $F(p_1^2, |\xi|)$ can be written as follows

$$F(x,y) = 256 + 15x^2 - 128x + 4x(4-x)y - 4(4-x)^2y^2. \tag{15}$$

By simple standard calculation we obtain

$$T_{3,1}(f) \le \frac{1}{256}\max\{F(x,y)\colon x \in [0,4],\ y \in [0,1]\} = 1.$$

Next, for obtain a lower estimate of $T_{3,1}(f)$ we apply the two inequalities $\mathrm{Re}\,\xi \ge -|\xi|$ and $|\xi| \le 1$ in (13). Set again $p_1^2 =: x \in [0,4]$. Therefore,

$$T_{3,1}(f) \ge \frac{1}{256}\left(256 + 15p_1^4 - 128p_1^2 - 4p_1^2(4-p_1^2)|\xi| - 4(4-p_1^2)^2\,|\xi|^2\right) \ge$$
$$\ge \frac{1}{256}\min\{256 + 15x^2 - 128x - 4x(4-x) - 4(4-x)^2)\colon x \in [0,4]\} =$$
$$= \frac{1}{256}\min\{15x^2 - 112x + 192\colon x \in [0,4]\} = -\frac{1}{15}.$$

Equality for the upper bound in inequality (7) is attained for the function f_1, where

$$f_1(z) = z\exp\Big(\int_0^z \frac{e^{t^3}-1}{t}\,dt\Big) = z + \frac{1}{3}z^4 + \cdots.$$

We now present the extremal functions corresponding to the attainment of the lower bounds in the various cases.

Consider the function $p_1 \colon \mathbb{D} \to \mathbb{C}$ defined by

$$p_1(z) := \frac{1-z^2}{1-\sqrt{\frac{56}{15}}\, z + z^2}, \quad z \in \mathbb{D}.$$

The function p_1 is analytic in $\mathbb{D}$ and $p_1 \in \mathcal{P}$. Also p_1 has only simple poles at the points $z_{\pm} = \frac{\sqrt{14} \pm i}{\sqrt{15}} \in \partial\mathbb{D}$.

The equality

$$\frac{zf'(z)}{f(z)} = e^{\frac{p_1(z)-1}{p_1(z)+1}} = e^{\frac{\sqrt{\frac{56}{15}}z - 2z^2}{2-\sqrt{\frac{56}{15}}z}}$$

holds for the function f_2, which give

$$f_2(z) = \exp\left(\int_0^z \frac{1}{t} \exp\left(\frac{\sqrt{\frac{56}{15}}t - 2t^2}{2-\sqrt{\frac{56}{15}}t}\right) dt\right) = z + \frac{1}{2}\sqrt{\frac{56}{15}}\, z^2 + \frac{2}{3} z^3 + \cdots$$

and so $a_2 = \frac{1}{2}\sqrt{\frac{56}{15}}$, $a_3 = \frac{2}{3}$. It is easy to verify that $T_{3,1}(f) = -\frac{1}{15}$. $\square$

Theorem 2. *Let $f \in C_e$ and be given by (1). Then* $\dfrac{9}{16} \leq T_{3,1}(f) \leq 1$.

The upper bound and lower bound are sharp for the functions f_3 and f_4 defined by

$$f_3(z) = \int_0^z \exp\left(\int_0^x \frac{e^{t^3}-1}{t}\, dt\right) dx = z + \frac{1}{12} z^4 + \ldots,$$
$$f_4(z) = \int_0^z \exp\left(\int_0^x \frac{e^{t}-1}{t}\, dt\right) dx = z + \frac{1}{2} z^2 + \frac{1}{4} z^3 + \ldots,$$

respectively.

Proof. Let $f \in \mathcal{C}_e$. There exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < |z|$ in $\mathbb{D}$ such that

$$1 + \frac{zf''(z)}{f'(z)} = e^{w(z)}. \tag{16}$$

Let p be given by (6). From (16) and (9), by equating coefficients we obtain

$$a_2 = \frac{1}{4} p_1, \qquad a_3 = \frac{1}{48} p_1^2 + \frac{1}{12} p_2. \tag{17}$$

Prior to carrying out the estimates, we remark that both the class $\mathcal{P}$ of functions with positive real part and the class $\mathcal{C}_e$ are preserved under rotations. Consequently, without any loss of generality, and since $|p_n| \leq 2$, we may restrict ourselves to the case $0 \leq p_1 \leq 2$.

Applying Lemma 1, (17), we get, for some $\xi \in \overline{\mathbb{D}}$, that

$$2\operatorname{Re}(a_2^2 \cdot \overline{a_3}) = 2\operatorname{Re}\left(\frac{p_1^2}{16}\right)\left(\frac{1}{48}p_1^2 + \frac{1}{16}\overline{p}_2\right) = \frac{p_1^2}{384}\left(p_1^2 + 2(p_1^2 + (4-p_1^2)\ \operatorname{Re}\ \overline{\xi}\right) =$$
$$= \frac{p_1^4}{128} + \frac{p_1^2(4-p_1^2)\ \operatorname{Re}\ \overline{\xi}}{192}, \quad -2|a_2|^2 = -\frac{1}{8}p_1^2, \tag{18}$$

$$-|a_3|^2 = -\left|\frac{1}{48}p_1^2 + \frac{1}{12}p_2\right|^2 = -\left|\frac{1}{16}p_1^2 + \frac{1}{24}(4-p_1^2)\,\xi\right|^2 =$$
$$= -\frac{1}{256}p_1^4 - \frac{1}{192}p_1^2(4-p_1^2)\ \mathrm{Re}\ \xi - \frac{1}{576}(4-p_1^2)^2\,|\xi|^2. \tag{19}$$

By applying equations (18), (19), equation (3) can be written as

$$T_{3,1}(f) = 1 + \frac{p_1^4}{128} + \frac{p_1^2(4-p_1^2)\ \mathrm{Re}\ \overline{\xi}}{192} - \frac{1}{8}p_1^2 =$$
$$= -\frac{1}{256}p_1^4 - \frac{1}{192}p_1^2(4-p_1^2)\,\mathrm{Re}\ \overline{\xi} - \frac{1}{576}(4-p_1^2)^2\,|\xi|^2 =$$
$$= \frac{1}{2304}\left(2304 + 9p_1^4 - 288p_1^2 - 4(4-p_1^2)^2\,|\xi|^2\right) = \frac{1}{2304}I(p_1^2,|\xi|). \tag{20}$$

Setting $p_1^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$, then $I(p_1^2,1)$ can be written as follows

$$I(x,y) = 2304 + 9x^2 - 288x - 4(4-x)^2y^2. \tag{21}$$

Now, differentiating partially (21) with respect to x and y we obtain

$$\frac{\partial I(x,y)}{\partial x} = 18x - 288 + 8(4-x)y^2, \quad \frac{\partial I(x,y)}{\partial y} = -8(4-x)^2y.$$

Solving an equations $\frac{\partial}{\partial x}J(x,y) = 0$ and $\frac{\partial}{\partial y}J(x,y) = 0$, we obtain that the only critical point is $(16,0) \notin [0,4]\times[0,1]$. Therefore, the maximum value of $I(x,y)$ is attained on the boundary of $[0,4]\times[0,1]$.

On the boundary of the rectangular region $[0,4]\times[0,1]$, the function $F(x,y)$ takes the following forms:

$$I(0,y) = 2304 - 64y^2 \le 2304, \quad I(4,y) = 1296 \ \text{ for all } y \in [0,1]$$

and

$$I(x,0) = 2304 + 9x^2 - 288x, \quad I(x,1) = 5x^2 - 256x + 2240 \ \text{ for all } x \in [0,4].$$

We see that $I'(x,0) < 0$ and $I'(x,1) < 0$ for all $x \in [0,4]$. Therefore $I(x,0) \le 1296$ and $I(x,1) \le 1296$ for all $x \in [0,4]$. From above discussion, we deduce that

$$T_{3,1}(f) \le \frac{1}{2304}\max\{2304, 1296\} = 1.$$

Next, we aim to minimize the right-hand side of (20). Now from (20) we have

$$T_{3,1}(f) = \frac{1}{2304}\left(2304 + 9p_1^4 - 288p_1^2 - 4(4-p_1^2)^2\,|\xi|^2\right) \ge \frac{1}{2304}I(p_1^2,1).$$

Setting $p_1^2 =: x \in [0,4]$. then $I(p_1^2,|\xi|)$ can be written as follows

$$I(x,1) = 2304 + 9p_1^4 - 288p_1^2 - 4(4-p_1^2)^2 = 5x^2 - 256x + 2240.$$

Here, it is easy to clear that $I'(x,1) < 0$ for $x \in [0,4]$. Moreover, since $I''(x,1) = 10 > 0$, it follows that $I(x,1)$ attains its minimum at $x = 4$. Therefore $I(x,1) \ge 1296$ for all $x \in [0,4]$.

From above discussion, we deduce that

$$T_{3,1}(f) \geq \frac{1296}{2304} = \frac{9}{16}.$$

Hence proved.

The equality $T_3(f) = 1$ is attained for the function

$$f(z) = f_3(z) = \int_0^z \exp\left(\int_0^x \frac{e^{t^3} - 1}{t}\, dt\right) dx = z + \frac{1}{12}z^4 + \cdots .$$

We now present the extremal functions corresponding to the attainment of the lower bounds in the various cases.

Consider the function $p_2 \colon \mathbb{D} \to \mathbb{C}$ defined by $p_2(z) := \frac{1+z}{1-z}$, $z \in \mathbb{D}$. Since, $\frac{p_2(z)-1}{p_2(z)+1} = z$, we define the function f_4 by the conditions

$$1 + \frac{zf_4''(z)}{f_4'(z)} = e^{\frac{p_2-1}{p_2+1}} = e^z,\ f_4(0) = 0,\ f_4'(0) = 1.$$

Then we have

$$f_4(z) = \int_0^z \exp\left(\int_0^x \frac{e^t - 1}{t}\, dt\right) dx = z + \frac{1}{2}z^2 + \frac{1}{4}z^3 + \cdots .$$

Clearly $a_2 = \frac{1}{2}$ and $a_3 = \frac{1}{4}$. Hence $T_{3,1}(f) = \frac{9}{16}$, i.e. the equality holds. □

Theorem 3. *Let $f \in \mathcal{R}_e$ and be given by (1). Then*

$$\frac{5}{9} \leq T_{3,1}(f) \leq 1. \tag{22}$$

The upper bound and lower bound are sharp.

Proof. Let $f \in \mathcal{C}_e$ and p be given by (6). There exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < |z|$ in $\mathbb{D}$ such that $f'(z) = e^{w(z)}$. From last equality and (9), by equating coefficients, we obtain

$$a_2 = \frac{1}{4}p_1, \qquad a_3 = -\frac{1}{24}p_1^2 + \frac{1}{6}p_2. \tag{23}$$

Prior to carrying out the estimates, we remark that both the class $\mathcal{P}$ of functions with positive real part and the class $\mathcal{R}_e$ are preserved under rotations. Consequently, without any loss of generality, and since $|p_n| \leq 2$, we may restrict ourselves to the case $0 \leq p_1 \leq 2$.

Applying Lemma 1, (23), for some $\xi \in \overline{\mathbb{D}}$ we get,

$$2\,\mathrm{Re}(a_2^2 \cdot \overline{a_3}) = 2\,\mathrm{Re}\left(\frac{p_1^2}{16}\right)\left(-\frac{1}{24}p_1^2 + \frac{1}{6}\overline{p}_2\right) = \frac{p_1^4}{192} + \frac{p_1^2(4-p_1^2)\ \mathrm{Re}\ \overline{\xi}}{96}, \tag{24}$$

$$-2|a_2|^2 = -\frac{1}{8}p_1^2, \tag{25}$$

$$-|a_3|^2 = -\left| -\frac{1}{24}p_1^2 + \frac{1}{6}p_2\right|^2 = -\left|\frac{1}{24}p_1^2 + \frac{1}{12}(4-p_1^2)\,\xi\right|^2 =$$
$$= -\frac{1}{576}p_1^4 - \frac{1}{144}p_1^2(4-p_1^2)\ \mathrm{Re}\ \xi - \frac{1}{144}(4-p_1^2)^2\,|\xi|^2. \tag{26}$$

By applying equations (24)–(26), equation (3) can be written as

$$T_{3,1}(f) = 1 + \frac{p_1^4}{192} + \frac{p_1^2(4-p_1^2)\text{ Re }\overline{\xi}}{96} - \frac{1}{8}p_1^2 - \frac{1}{576}p_1^4 - \frac{1}{144}p_1^2(4-p_1^2)\text{ Re }\xi - $$
$$-\frac{1}{144}(4-p_1^2)^2|\xi|^2 = \frac{1}{288}\left(288 + p_1^4 - 36p_1^2 + p_1^2(4-p_1^2)\text{ Re }\overline{\xi} - 2(4-p_1^2)^2|\xi|^2\right). \tag{27}$$

Next, we aim to maximize the right-hand side of (27). Since $\text{Re}\,\xi \leq |\xi|$, it follows from (27) that

$$T_{3,1}(f) \leq \frac{1}{288}\left(288 + p_1^4 - 36p_1^2 + p_1^2(4-p_1^2)|\xi| - 2(4-p_1^2)^2|\xi|^2\right) = \frac{1}{288}J(p_1^2, |\xi|).$$

Setting $p_1^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$, then $J(p_1^2, |\xi|)$ can be written as follows

$$J(x,y) = 288 + x^2 - 36x + x(4-x)y - 2(4-x)^2y^2. \tag{28}$$

Now, differentiating partially (28) with respect to x and y we obtain

$$\frac{\partial J(x,y)}{\partial x} = 2x - 36 + (4-2x)y + 4(4-x)y^2, \quad \frac{\partial J(x,y)}{\partial y} = x(4-x) - 4(4-x)^2y.$$

Solving the equations $\frac{\partial}{\partial x}J(x,y) = 0$ and $\frac{\partial}{\partial y}J(x,y) = 0$, we obtain that the only critical points are $(4,-7), (16, -\frac{1}{3}) \notin [0,4] \times [0,1]$. Therefore, the maximum value of $J(x,y)$ is attained on the boundary of $[0,4] \times [0,1]$.

On the boundary of the rectangular region $[0,4] \times [0,1]$, the function $J(x,y)$ takes the following forms: $J(0,y) = 288 - 32y^2 \leq 288,\ I(4,y) = 160$ for all $y \in [0,1]$ and $J(x,0) = 288 + x^2 - 36x,\ I(x,1) = -2x^2 - 16x + 256$ for all $x \in [0,4]$. We see that $J'(x,0) < 0$ and $J'(x,1) < 0$ for all $x \in [0,4]$. Therefore $J(x,0) \leq 288$ and $I(x,1) \leq 288$ for all $x \in [0,4]$. From above discussion, we deduce that $T_{3,1}(f) \leq \max\{160, 288\}/288 = 1$.

Next, we aim to minimize the right-hand side of (27). Now from (27) we have

$$T_{3,1}(f) \geq \frac{1}{288}\left(288 + p_1^4 - 36p_1^2 - p_1^2(4-p_1^2)|\xi| - 2(4-p_1^2)^2|\xi|^2\right) \geq \frac{1}{288}K(p_1^2, 1).$$

Setting $p_1^2 =: x \in [0,4]$. then $I(p_1^2, |\xi|)$ can be written as follows

$$K(x,1) = 288 + x^2 - 36x - x(4-x) - 2(4-x)^2 = 256 - 24x.$$

Here, it is easy to clear that $K'(x,1) < 0$ for $x \in [0,4]$. Moreover, it follows that $K(x,1)$ attains its minimum at $x = 4$. Therefore $K(x,1) \geq 160$ for all $x \in [0,4]$.

From above discussion, we deduce that

$$T_{3,1}(f) \geq \frac{160}{288} = \frac{5}{9}.$$

The equalities $T_{3,1}(f_5) = 1$ and $T_{3,1}(f_6) = \frac{5}{9}$ is attained for the functions

$$f_5(z) = \int_0^z e^{t^3}\,dt = z + \tfrac{1}{4}z^4 + \tfrac{1}{14}z^7 + \ldots, \quad f_6(z) = \int_0^z e^t\,dt = z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3 + \ldots,$$

respectively. □

REFERENCES

1. M.F. Ali, D.K. Thomas, A. Vasudevarao, *Toeplitz determinants whose elements are the coefficients of analytic and univalent functions*, Bull. Aust. Math. Soc., **97** (2018), №2, 253–264.
2. K. Cudna, O.S. Kwon, A. Lecko, Y.J. Sim, B. Smiarowska, *The second and third-order Hermitian Toeplitz determinants for starlike and convex functions of order* α, Bol. Soc. Mat. Mex. (3), **26** (2020), №2, 361–375.
3. P. Hartman, A. Wintner, *The spectra of Toeplitz's matrices*, Amer. J. Math., **76** (1954), 867–882.
4. A.W. Goodman, Univalent functions, Mariner Publishing Company, Inc., Tampa, Florida, 1983.
5. W. Ma, D. Minda, *A unified treatment of some special classes of univalent functions*, In: Proceeding of the Conference on Complex Analysis, Z. Li, F. Ren, L. Yang and S. Zhang (Eds), Int. Press, 1994, 157–169.
6. R. Mendiratta, S. Nagpal, V. Ravichandran, *On a subclass of strongly starlike functions associated with exponential function*, Bull. Malays. Math. Sci. Soc., **38** (2015), №1, 365–386.
7. D. Kucerovsky, K. Mousavand, A. Sarraf, *On some properties of Toeplitz matrices*, Cogent Mathematics, **3** (2016), 1154705.
8. V. Kumar, R. Srivastava, N.E. Cho, *Sharp estimation of Hermitian-Toeplitz determinants for Janowski type starlike and convex functions*, Miskolc Math. Notes, **21** (2020), №2, 939–952.
9. S. Kumar, R.K. Pandey, P. Rai, *Sharp bounds on Hankel and Hermitian-Toeplitz determinants of associated Sakaguchi functions*, Lith. Math. J., **64** (2024), 491–506.
10. R.J. Libera, E.J. Zlotkiewicz, *Coefficient bounds for the inverse of a function with derivative in* $\mathcal{P}$, Proc. Amer. Math. Soc., **87** (1983), №2, 251–257.

Department of Mathematics, Raiganj University
West Bengal, Raiganj, India
naba.iitbmath@gmail.com

Received 22.09.2025
Revised 03.12.2025