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DOMINATING POLYNOMIAL IN POWER SERIES EXPANSION FOR ANALYTIC FUNCTIONS IN A COMPLETE REINHARDT DOMAIN

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We generalized some criteria of boundedness of **L**-index in joint variables for analytic functions in a complete multiple circular domain, where $\mathbf{L}(z) = (l_1(z_1, z_2, \dots, z_n), l_2(z_1, z_2, \dots, z_n), \dots, l_n(z_1, z_2, \dots, z_n))$, $l_j\colon \mathbb{G} \to \mathbb{R}_+$ is a continuous function, $\mathbb{G}$ is the n-dimensional complete multiple circular domain in $\mathbb{C}^n$, i.e. for every point $(z_1, \dots, z_n)$ from this domain $\mathbb{G}$ and for each $r_j \in [0, 1]$, $\theta \in [0, 2\pi]$, $j \in \{1, 2, \dots, n\}$, the point-wise product $(r_1z_1, \dots, r_nz_n)$ belongs to the same domain $\mathbb{G}$ and the component-wise rotation $(z_1e^{i\theta_1}, \dots, z_ne^{i\theta_n})$ falls into this domain $\mathbb{G}$. The propositions describe a behavior of multiple power series expansion on a skeleton of a polydisc. There are presented estimation of power series expansion modulus by a dominating homogeneous polynomial with the degree that does not exceed some number depending only from radii of polydisc. Changing the center of the polydisc, we cover the whole domain $\mathbb{G}$. Replacing universal quantifier by existential quantifier for radii of bidisc, we also proved sufficient conditions of boundedness of **L**-index in joint variables for analytic functions which are weaker than necessary conditions.

1. Introduction. The complete Reinhardt domain is natural holomoprhy domain of multiple power series in n-dimensional complex space. Given this, it has attracted the attention of many investigators ([11–13]) in multidimensional complex analysis. This interest is generated by its geometric and analytical properties and its universality because it overlaps the balls ([1, 3, 9]), the polydiscs ([6, 7]), as the partial cases. Moreover, these cases are not biholomorphic equivalent. The study of the Reinhardt domain allows us to discover a deep connection between them. An increasing number of papers on various types of Reinhardt domains, on the Schwarz lemma ([32]), on Bergman kernels ([11]), and on the bounds of all the coefficients of homogeneous expansions ([30]) for the domain show the importance of this topic.

The present paper is addendum to papers [4, 8, 22]. There was introduced a notion of bounded **L**-index in joint variables for analytic functions in a complete Reinhardt domain. For this class of functions there were proved different characteristic local properties. In this paper, we present analog of results which were early obtained for analytic functions in bidisc ([6]), for entire multivariate functions ([5]) and for analytic vector-valued functions in two-dimensional unit ball ([2]). Interest to study these functions is generated by application of this notion in analytic theory of ordinary ([25, 26]) and directional differential equations

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([10]). It is quite often, the investigation of boundedness of l-index of analytic solutions is combined with the investigation of their geometric properties such as starlikeness, convexity ([27–29]).

We will need the following standard notations from the theory of holomorphic multivariate functions (see, for example, [4, 19, 20]). In particular, $\mathbb{R}_+ = [0, +\infty)$, $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$. For $A = (a_1, \dots, a_n) \in \mathbb{R}^n$, $B = (b_1, \dots, b_n) \in \mathbb{R}^n$, we define $AB = (a_1b_1, \cdots, a_nb_n)$, $A/B = (a_1/b_1, \dots, a_n/b_n)$, $A^B = a_1^{b_1}a_2^{b_2}\cdot\ldots\cdot a_n^{b_n}$, $A+B = (a_1+b_1, \dots, a_n+b_n)$, $kA = (ka_1, \dots, ka_n)$, $rA = (ra_1, \dots, ra_n)$, $\|A\| = \sum_{j=1}^n a_j$ and $A! = a_1! \dots a_n!$, if each $a_j \in \mathbb{Z}_+$. And all vector inequalities are understood as coordinate inequalities. This concerns the inequalities $A < B$, $A \le B$, and so on. Denote $\mathbb{D}^n(z^0, R) = \{z \in \mathbb{C}^n \colon |z_j - z_j^0| < r_j \ \ j \in \{1, \dots, n\}\}$, $\mathbb{T}^n(z^0, R) = \{z \in \mathbb{C}^n \colon \ |z_j - z_j^0| = r_j, \ j \in \{1, \dots, n\}\}$, $\mathbb{D}^n[z^0, R] = \{z \in \mathbb{C}^n \colon \ |z_j - z_j^0| \le r_j, \ j = 1, \dots, n\}$.The domain $\mathbb{G} \subset \mathbb{C}^n$ is called the complete Reinhardt domain ([17, 18]), if: $\forall z \in \mathbb{G}$ $\forall R \in [0, 1]^n$ one has $Rz \in \mathbb{G}$ and for all angles $(\theta_1, \dots, \theta_n) \in [0, 2\pi]^n)$ $(z_1e^{i\theta_1}, \dots, z_ne^{i\theta_n}) \in \mathbb{G}$. Denote by $\partial\mathbb{G}$ the boundary of the domain $\mathbb{G}$. For $J \in \mathbb{Z}_+^n$ we will denote $H^{(J)}(z) = \frac{\partial^{\|J\|}H}{\partial z^J}(z) = \frac{\partial^{j_1+j_2+\cdots+j_n}H}{\partial z_1^{j_1}\partial z_2^{j_2}\ldots\partial z_n^{j_n}}(z_1, z_2, \dots, z_n)$. Suppose that the function $\mathbf{L} \colon \mathbb{G} \to \mathbb{R}_+^n$ is continuous and for any $j \in \{1, 2, \dots, n\}$ and $\beta > 1$ one has

$$l_j(z) > \frac{\beta}{\inf_{\substack{\widehat{R}_j z \in \partial\mathbb{G},\\ r>1}} (r|z_j|) - |z_j|}, \tag{1}$$

where $\widehat{R}_j = (1, \dots, 1, \underbrace{r}_{j\text{-th item}}, 1 \dots, 1)$. Denote $\mathcal{B} = (0, \beta]$, $\mathcal{B}^n = (0, \beta]^n$. Below we suppose everywhere that $\mathbb{G} \subset \mathbb{C}^n$ is the complete Reinhardt domain.

An analytic function $H \colon \mathbb{G} \to \mathbb{C}$ is called a function with *bounded* $\mathbf{L}$*-index (in joint variables)* if $\exists n_0 \in \mathbb{Z}_+$ $\forall J \in \mathbb{Z}_+^n$ $\forall z \in \mathbb{G}$:

$$\frac{|H^{(J)}(z)|}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{|H^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \ K \in \mathbb{Z}_+^n, \ \|K\| \le n_0\right\}. \tag{2}$$

The least corresponding number $N(H, \mathbf{L}, \mathbb{G}) = n_0$ is the $\mathbf{L}$*-index in joint variables.* If $\mathbb{G} = \mathbb{C}^n$, $\mathbf{L} = \mathbf{1}$, then it is a definition of an entire multivariate function of a bounded index ([20, 21]). One should note that the notion of bounded $l - M$-index for entire functions ([23, 24]) is not yet considered in multidimensional case. Denote

$$\lambda_j(R) = \sup_{z, w \in \mathbb{G}}\left\{\frac{l_j(z)}{l_j(w)} \colon |z_k - w_k| \le \frac{r_k}{\min\{l_k(z), l_k(w)\}}, k \in \{1, \dots, n\}\right\}. \tag{3}$$

The class of these mapping $\mathbf{L} \colon \mathbb{G} \to \mathbb{R}_+^n$ for which (1) is true and $\lambda_j(R)$ is finite for any $R \in \mathcal{B}^n$ and $j \in \{1, 2, \dots, n\}$ is denoted by $Q(\mathbb{G})$. Although all elementary functions and their compositions belong to the class $Q(\mathbb{C})$, it is known ([14]) that if $l \colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $rl(r) \to +\infty$ as $r = |z| \to +\infty$, then there exists an entire function of single variable having bounded l-index. We will need the recent result, which is analog of Hayman's theorem ([15]) for this class of functions.

Theorem 1 ([22]). *Let* $\mathbf{L} \in Q(\mathbb{G})$*. An analytic function* F *in* $\mathbb{G}$ *has bounded* $\mathbf{L}$*-index in joint variables if and only if there exist* $p \in \mathbb{Z}_+$ *and* $c \in \mathbb{R}_+$ *such that for each* $z \in \mathbb{G}$

$$\max\left\{\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} \colon \ \|J\| = p + 1\right\} \le c \cdot \max\left\{\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} \colon \ \|K\| \le p\right\}. \tag{4}$$

2. Main results. Let $z^0 \in \mathbb{G}$. We develop an analytic function $F\colon \mathbb{G} \to \mathbb{C}$ in the power series written in a diagonal form

$$F(z) = \sum_{k=0}^{\infty} p_k(z - z^0) = \sum_{k=0}^{\infty} \sum_{\|J\|=k} b_J (z - z^0)^J, \tag{5}$$

where p_k are homogeneous polynomials of k-th degree, $b_J = \frac{F^{(J)}(z^0)}{J!}$. A polynomial $p_{k_0}, k_0 \in \mathbb{Z}_+$, is called a dominating polynomial in the power series expansion (5) on $\mathbb{T}^n(z^0, R)$ if for every $z \in \mathbb{T}^n(z^0, R)$ the next inequality holds

$$\Big| \sum_{k \neq k^0} p_k(z - z^0) \Big| \leq \frac{1}{2} \max\{|b_J| R^J \colon \|J\| = k^0\}.$$

Recently, there was obtained another estimate in [16] decribing behavior of maximum modulus of homogeneous polynomial in a bounded complete multiple-circular domain. These polynomials are contained in some gap power series representations for entire functions (see also [31]).

Theorem 2. *Let $\mathbf{L} \in Q(\mathbb{G})$. If an analytic function F in $\mathbb{G}$ has bounded $\mathbf{L}$-index in joint variables then there exists $p \in \mathbb{Z}_+$ that for all $d \in (0, \beta]$ there exists $\eta(d) \in (0, d)$ such that for each $z^0 \in \mathbb{G}$ and some $r = r(d, z^0) \in (\eta(d), d)$, $k^0 = k^0(d, z^0) \leq p$ the polynomial p_{k^0} is a dominating polynomial in the series* (5) *on $\mathbb{T}^n(z^0, \frac{r\mathbf{1}}{\mathbf{L}(z^0)})$.*

Proof. Let F be an analytic function of bounded $\mathbf{L}$-index in joint variables with $N = N(F, \mathbf{L}, \mathbb{G}) < +\infty$ and n_0 be the $\mathbf{L}$-index in joint variables at a point $z^0 \in \mathbb{D}^2$, i.e. n_0 is the least number, for which inequality (2) holds at the point z^0. Then for each $z^0 \in \mathbb{G}$ $n_0 \leq N$. We put $a_J^* = \frac{|b_J|}{\mathbf{L}^J(z^0)} = \frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^J(z^0)}$, $a_k = \max\{a_J^*\colon \|J\| = k\}$, $c = 2\{(N+n+1)!(n+1)! + (N+1)C_{n+N-1}^N\}$. Let $d \in (0, \beta]$ be an arbitrary number. We also denote $r_m = \frac{d}{(d+1)c^m}$, $\mu_m = \max\{a_k r_m^k \colon k \in \mathbb{Z}_+\}$, $s_m = \min\{k\colon a_k r_m^k = \mu_m\}$ for $m \in \mathbb{Z}_+$.

Since $z^0 \in \mathbb{G}$ is a fixed point the inequality $a_K^* \leq \max\{a_J^*\colon \|J\| \leq n_0\}$ is valid for all $K \in \mathbb{Z}_+^n$. Then $a_k \leq a_{n_0}$ for all $k \in \mathbb{Z}_+$. Hence, for all $k > n_0$, in view of $r_0 < 1$, we have $a_k r_0^k < a_{n_0} r_0^{n_0}$. This implies $s_0 \leq n_0$. Since $c r_m = r_{m-1}$, we obtain that for each $k > s_{m-1}$ $(r_{m-1} < 1)$

$$a_{s_{m-1}} r_m^{s_{m-1}} = a_{s_{m-1}} r_{m-1}^{s_{m-1}} c^{-s_{m-1}} \geq a_k r_{m-1}^k c^{-s_{m-1}} = a_k r_m^k c^{k - s_{m-1}} \geq c a_k r_m^k. \tag{6}$$

It yields that $s_m \leq s_{m-1}$ for all $m \in \mathbb{N}$. Thus, we can rewrite

$$\mu_0 = \max\{a_k r_0^k \colon k \leq n_0\},\ \mu_m = \max\{a_k r_m^k \colon k \leq s_{m-1}\},\ m \in \mathbb{N}.$$

Let us introduce additional notations for $m \in \mathbb{N}$

$$\mu_0^* = \max\left\{a_k r_0^k \colon s_0 \neq k \leq n_0\right\},\ s_0^* = \min\{k\colon k \neq s_0, a_k r_0^k = \mu_0^*\},$$
$$\mu_m^* = \max\{a_k r_m^k \colon s_m \neq k \leq s_{m-1}\},\ s_m^* = \min\{k\colon k \neq s_m, a_k r_m^k = \mu_m^*\}.$$

We will show that there exists $m_0 \in \mathbb{Z}_+$ such that

$$\frac{\mu_{m_0}^*}{\mu_{m_0}} \leq \frac{1}{c}. \tag{7}$$

Suppose that for all $m \in \mathbb{Z}_+$ the next inequality holds

$$\frac{\mu_m^*}{\mu_m} > \frac{1}{c}. \tag{8}$$

If $s_m^* < s_m$ ($s_m^* \neq s_m$ in view of definition) then we have

$$a_{s_m^*} r_{m+1}^{s_m^*} = \frac{a_{s_m^*} r_m^{s_m^*}}{c^{s_m^*}} = \frac{\mu_m^*}{c^{s_m^*}} > \frac{\mu_m}{c^{s_m^*+1}} = \frac{a_{s_m} r_m^{s_m}}{c^{s_m^*+1}} = \frac{a_{s_m} r_{m+1}^{s_m}}{c^{s_m^*+1-s_m}} \geq a_{s_m} r_{m+1}^{s_m}.$$

Besides, for every $k > s_m^*$, $k \neq s_m$, (i. e., $k-1 \geq s_m^*$) it can be deduced similarly that

$$a_{s_m^*} r_{m+1}^{s_m^*} = \frac{a_{s_m^*} r_m^{s_m^*}}{c^{s_m^*}} \geq \frac{a_k r_m^k}{c^{s_m^*}} \geq \frac{a_k r_m^k}{c^{k-1}} = c a_k r_{m+1}^k.$$

Hence, $a_{s_m^*} r_{m+1}^{s_m^*} > a_k r_{m+1}^k$ for all $k > s_m^*$. Then

$$s_{m+1} \leq s_m^* \leq s_m - 1. \tag{9}$$

On the contrary, if $s_m < s_m^* \leq s_{m-1}$, then the equality $s_{m+1} = s_m$ may holds. Indeed, by definition $s_{m+1} \leq s_m$. It means that the specified equality is possible. But if $s_{m+1} < s_m$ then $s_{m+1} \leq s_m - 1$ (they are natural numbers!). Hence, we obtain (9).

Thus, the inequalities $s_{m+1}^* \leq s_m$ and $s_m^* \neq s_{m+1}$ imply that $s_{m+1}^* < s_{m+1}$. As above instead of (9) we have

$$s_{m+2} \leq s_{m+1}^* \leq s_{m+1} - 1 = s_m - 1.$$

Therefore, if for all $m \in \mathbb{Z}_+$ (8) holds, then for every $m \in \mathbb{Z}_+$ either $s_{m+2} \leq s_{m+1} \leq s_m - 1$ or $s_{m+2} \leq s_m - 1$ holds, that is $s_{m+2} \leq s_m - 1$, because $s_{m+2} \leq s_{m+1}$. It follows that

$$s_m \leq s_{m-2} - 1 \leq \ldots \leq s_{m-2[m/2]} - [m/2] \leq s_0 - [m/2] \leq n_0 - [m/2] \leq N - [m/2].$$

In other words, $s_m < 0$ for $m > 2N+1$, which is impossible. Therefore, there exists $m_0 \leq 2N+1$ such that (7) holds. We put $r = r_{m_0}$, $\eta(d) = \frac{d}{(d+1)c^{2(N+1)}}$, $p = N$ and $k_0 = s_{m_0}$. Then for all $\|J\| \neq k_0 = s_{m_0}$ in $\mathbb{T}^n(z^0, \frac{r_1}{\mathbf{L}(z^0)})$, in view (6) and (7) we obtain

$$|b_J||(z-z^0)^J| = a_J^* r^{\|J\|} \leq a_{\|J\|} r^{\|J\|} \leq \frac{1}{c} a_{s_{m_0}} r_{m_0}^{s_{m_0}} = \frac{1}{c} a_{k_0} r^{k_0}.$$

Thus, for $z \in \mathbb{T}^n(z^0, \frac{r\mathbf{1}}{\mathbf{L}(z^0)})$

$$\left| \sum_{\|J\| \neq k_0} b_J (z-z^0)^J \right| \leq \sum_{\|J\| \neq k_0} a_j^* r^{\|J\|} \leq \sum_{\substack{k=0,\\ k \neq k_0}}^{\infty} a_k C_{n+k-1}^k r^k =$$

$$= \sum_{\substack{k=0,\\ k \neq s_{m_0}}}^{s_{m_0-1}} a_k C_{n+k-1}^k r^k + \sum_{k=s_{m_0-1}+1}^{\infty} a_k C_{n+k-1}^k r^k. \tag{10}$$

We will estimate two sums in (10). From (7) it follows that $\mu_{m_0}^* \leq \frac{1}{c}\mu_{m_0}$ or $\max\{a_k r_{m_0}^k \colon k \neq s_{m_0},\ k \leq s_{m_0-1}\} \leq \frac{1}{c} \max\{a_k r_{m_0}^k \colon k \neq s_{m_0}, k \leq s_{m_0-1}\}$, i. e. $a_k r^k \leq \frac{1}{c} a_{k_0} r^{k_0}$. Taking into account (9), it can be deduced that

$$\sum_{\substack{k=0,\\ k \neq s_{m_0}}}^{s_{m_0-1}} a_k C_{n+k-1}^k r^k \leq \frac{a_{k_0} r^{k_0}}{c} \sum_{k=0}^{N} C_{n+k-1}^k \leq \frac{a_{k_0} r^{k_0}}{c} (N+1) C_{n+N-1}^N. \tag{11}$$

For all $k \geq s_{m_0-1}+1$ $a_k r^k_{m_0-1} \leq \mu_{m_0-1}$ holds. Then $a_k r^k_{m_0} = \frac{a_k r^k_{m_0-1}}{c^k} \leq \frac{\mu_{m_0-1}}{c^k}$. In view of (7) we deduce

$$\sum_{k=s_{m_0-1}+1}^{\infty} a_k C^k_{n+k-1} r^k \leq \mu_{m_0-1} \sum_{k=s_{m_0-1}+1}^{\infty} C^k_{n+k-1}\frac{1}{c^k} \leq$$

$$\leq a_{s_{m_0-1}} r_{m_0}^{s_{m_0-1}} c^{s_{m_0-1}} \sum_{k=s_{m_0-1}+1} (k+1)(k+2)\dots(k+n)\frac{1}{c^k} \leq$$

$$\leq \frac{a_{s_{m_0}} r^{s_{m_0}}}{c} c^{s_{m_0-1}} \Bigg(\sum_{k=s_{m_0-1}+1}^{\infty} x^{k+n}\Bigg)^{(n)}\Bigg|_{x=\frac{1}{c}} = \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0-1}} \Bigg\{\frac{x^{s_{m_0-1}+n+1}}{1-x}\Bigg\}^{(n)}\Bigg|_{x=\frac{1}{c}} =$$

$$= \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0-1}} \sum_{j=0}^{n} C^j_n (n-j)!(s_{m_0-1}+n+1)\dots(s_{m_0-1}+n-j+2)\times$$

$$\times \frac{x^{s_{m_0-1}+1+n-j}}{(1-x)^{n-j+1}}\Bigg|_{x=\frac{1}{c}} \leq \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0-1}} n!(N+n+1)! \sum_{j=0}^{n} \frac{(1/c)^{s_{m_0-1}+1+n-j}}{(1-1/c)^{n-j+1}} =$$

$$= n!(N+n+1)!\frac{a_{k_0} r^{k_0}}{c} \sum_{j=0}^{n} \frac{1}{(c-1)^{n-j+1}} \leq (n+1)!(N+n+1)!\frac{a_{k_0} r^{k_0}}{c}, \tag{12}$$

because $c \geq 2$. Hence, from (10)–(12) it follows that

$$\Bigg|\sum_{\|J\|\neq k_0} b_J (z-z^0)^J\Bigg| \leq \frac{((N+1)C^N_{n+N-1} + (n+1)!(N+n+1)!)a_{k_0} r^{k_0}}{c} \leq \frac{1}{2} a_{k_0} r^{k_0}.$$

So, P_{k_0} is the dominating polynomial in the series (5) on skeleton $\mathbb{T}^n(z^0, \frac{r\mathbf{1}}{\mathbf{L}(z^0)})$. □

Theorem 3. *Let $\mathbf{L} \in Q(\mathbb{G})$. If there exist $p \in \mathbb{Z}_+$, $d \in (0,1]$, $\eta \in (0,d)$ such that for each $z^0 \in \mathbb{G}$ and some $R = (r_1, \dots, r_n)$ with $r_j = r_j(d, z^0) \in (\eta, d)$, $j \in \{1, \dots, n\}$, and certain $k^0 = k^0(d, z^0) \leq p$ the polynomial p_{k^0} is the dominating polynomial in the series (5) on $\mathbb{T}^2(z^0, R/\mathbf{L}(z^0))$ then the analytic in $\mathbb{G}$ function F has bounded $\mathbf{L}$-index in joint variables.*

Proof. Suppose that there exist $p \in \mathbb{Z}_+$, $d \leq 1$ and $\eta \in (0,d)$ such that for each $z^0 \in \mathbb{G}$ and some $R = (r_1, \dots, r_n)$ with $r_j = r_j(d, z^0) \in (\eta, d)$, $j \in \{1, \dots, n\}$, and $k_0 = k_0(1, z^0) \leq p$ the polynomial P_{k_0} is a dominating polynomial in the series (5) on $\mathbb{T}^n(z^0, \frac{R}{\mathbf{L}(z^0)})$. Let us denote $r_0 = \max_{1\leq j\leq n} r_j$. Then

$$\Bigg|\sum_{\|J\|\neq k_0} b_J (z-z^0)^J\Bigg| = \Bigg|F(z) - \sum_{\|J\|=k_0} b_J (z-z^0)^J\Bigg| \leq \frac{a_{k_0} r_0^{k_0}}{2}.$$

Using Cauchy's inequality we have $|b_J (z-z^0)^J| = a^*_j R^J \leq \frac{a_{k_0} r_0^{k_0}}{2}$ for all $J \in \mathbb{Z}^n_+$, $\|J\| \neq k_0$, that is for all $\|J\| = k \neq k_0$

$$a_k R^J \leq \frac{a_{k_0} r_0^{k_0}}{2}. \tag{13}$$

Suppose that F is not a function of bounded $\mathbf{L}$-index in joint variables. Then in view of Theorem 1 for all $p_1 \in \mathbb{Z}_+$ and $c \geq 1$ there exists $z^0 \in \mathbb{G}$ such that the next inequality

$$\max\left\{\frac{|F^{(J)}(z^0)|}{\mathbf{L}^J(z^0)} \colon \|J\| = p_1 + 1\right\} > c \max\left\{\frac{|F^{(K)}(z^0)|}{\mathbf{L}^K(z^0)} \colon \|K\| \leq p_1\right\}$$

holds. We put $p_1 = p$ and $c = \left(\frac{(p+1)!}{\eta^{p+1}}\right)^n$. Then for this $z^0(p_1, c)$

$$\max\left\{\frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^J(|z^0|)} : \|J\| = p+1\right\} > \frac{1}{\eta^{p+1}} \max\left\{\frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^K(|z^0|)} : \|K\| \le p\right\},$$

that is $a_{p+1} > \frac{a_{k_0}}{\eta^{p+1}}$. Hence, $a_{p+1} r_0^{p+1} > \frac{a_{k_0} r_0^{p+1}}{\eta^{p+1}} \ge a_{k_0} r^{k_0}$. The last inequality contradicts (13). Therefore, F is of bounded $\mathbf{L}$-index in joint variables. □

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