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ALGEBRAIC BASES OF SOME ALGEBRAS OF POLYNOMIALS ON BANACH SPACES

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The work is devoted to the study of algebraic bases of algebras of continuous polynomials on real and complex Banach spaces. A subset of an algebra is called an algebraic basis if every element of the algebra can be uniquely represented as a linear combination of products of powers of elements of the subset. Algebras of symmetric continuous polynomials on Banach spaces with some symmetric structure are typically equipped with finite or countable algebraic bases, which is important in the investigations of the respective algebras of symmetric analytic functions. Explicit constructions of such algebraic bases are often available when the Banach spaces are complex. In this work, we develop a method for extending these results to the case of real versions of such spaces. We also apply this method to the algebra of symmetric continuous polynomials on the Cartesian product of real Banach spaces of absolutely Lebesgue integrable in some powers functions on $[0, 1]$.

Introduction. Algebras of symmetric polynomials on Banach spaces have numerous applications in both theoretical and applied research, particularly in cryptography ([6]), machine learning ([22]) and statistical quantum physics ([4, 10]). More general algebras of block-symmetric ([9, 11, 17]), weakly symmetric ([2, 21]) and supersymmetric ([5]) polynomials also play an important role in applications. In many cases such algebras have finite or countable algebraic bases (see definition below). First, the existence of such an algebraic basis in an algebra allows one to explicitly determine the structure of its elements. Second, algebras of analytic functions that contain such an algebra as a dense subalgebra are finitely or countably generated, which significantly simplifies the description of their spectra ([14, 16]). Historically, the first algebraic bases to be described were those of the algebras of continuous symmetric polynomials on certain classical real separable Banach spaces ([8]). However, for more complicated cases, such as algebras of symmetric polynomials on some nonseparable Banach spaces ([7]) or on Cartesian products of Banach spaces ([1]), the problem of describing an algebraic basis in the complex setting is simpler than in the real one. A natural problem arises: to obtain a description of the algebraic basis of the algebra of symmetric polynomials on a real Banach space using the known description for its complex version. For some partial cases, this approach has been successfully implemented ([18, 19, 20]). In this paper, we realise this approach in the general case, even for algebras of polynomials that are not necessarily symmetric. We also apply this result to obtain the algebraic basis of the

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algebra of all symmetric continuous polynomials on the Cartesian product of real Banach spaces of absolutely Lebesgue integrable in some powers functions on $[0,1]$.

1. Preliminaries.

Polynomials. We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_+$ the set of all nonnegative integers. Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ resp. A mapping $A\colon X^m\to Y$, where $m\in\mathbb{N}$, is said to be *m-linear* if it is linear in each argument separately, when all other $m-1$ arguments are fixed. Let the norm of an m-linear mapping $A\colon X^m\to Y$ be defined by

$$\|A\| = \sup_{\|x_1\|_X\le 1,\ldots,\|x_m\|_X\le 1} \|A(x_1,\ldots,x_m)\|_Y. \tag{1}$$

It is known that A is continuous if and only if its norm is finite. An m-linear mapping $A\colon X^m\to Y$ is called *symmetric*, if $A(x_{\sigma(1)},\ldots,x_{\sigma(m)}) = A(x_1,\ldots,x_m)$ for every bijection $\sigma\colon\{1;\ldots;m\}\to\{1;\ldots;m\}$ and for every $x_1,\ldots,x_m\in X$.

A mapping $P\colon X\to Y$ is called an m-homogeneous polynomial with $m\in\mathbb{N}$, if there exists an m-linear mapping $A_P\colon X^m\to Y$ such that $P(x) = A_P\big(\underbrace{x,\ldots,x}_{m}\big)$ for every $x\in X$. Note that A_P is called an m-linear mapping *associated* with the polynomial P. One should observe that the existence of such an m-linear mapping guarantees the existence of the symmetric m-linear mapping associated with P (see [12, Proposition 1.6]). Note that for an m-homogeneous polynomial P the symmetric m-linear mapping associated with P can be recovered with the aid of the following formula (see [12, Theorem 1.10]):

$$A_P^s(x_1,\ldots,x_m) = \frac{1}{2^m m!}\sum_{j_k\in\{-1;1\}} j_1\cdot\ldots\cdot j_m P\Big(j_1x_1+\ldots+j_mx_m\Big), \tag{2}$$

where $k\in\{1;\ldots;m\}$.

A mapping $P\colon X\to Y$ is called a 0-homogeneous polynomial, if it is a constant mapping. A mapping $P\colon X\to Y$ is called a polynomial of degree at most N, if it can be represented in the form

$$P = P_0 + P_1 + \ldots + P_N, \tag{3}$$

where $P_j\colon X\to Y$ is a j-homogeneous polynomial for every $j\in\{0,1,\ldots,N\}$. It is known that a polynomial $P\colon X\to Y$ is continuous if and only if $\|P\|<+\infty$, where

$$\|P\| = \sup_{\|x\|_X\le 1}\|P(x)\|_Y.$$

By [12, Theorem 2.2], for a continuous m-homogeneous polynomial P and the symmetric m-linear form A_P associated with P,

$$\|P\| \le \|A_P\| \le \frac{m^m}{m!}\|P\|. \tag{4}$$

Lemma 1. *(See [19, Proposition 1]) Let $P\colon X\to Y$, where X and Y are Banach spaces over a field $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, be the polynomial of degree at most N. Let $\lambda_0,\ldots,\lambda_N$ be distinct nonzero real numbers. Then P_j defined by (3), is given by the equation $P_j(x)=\sum_{s=0}^N w_{js}P(\lambda_s x)$, for every $j\in\{0,\ldots,N\}$ and $x\in X$, where w_{js} are elements of the matrix $W=(w_{js})_{j,s=\overline{0,N}}$ which is the inverse of the Vandermonde matrix $V_{\lambda_0,\ldots,\lambda_N} = (\lambda_j^s)_{j,s=\overline{0,N}}$.*

Let $\mathcal{P}_a({}^kX;Y)$ be the vector space of k-homogeneous polynomials acting from X to Y and $\mathcal{P}({}^kX;Y)$ be the subspace of continuous k-homogeneous polynomials acting from X

to Y. Let $\mathcal{P}(X;Y)$ be the space of continuous polynomials acting from X to Y. Note that if Y is a unital commutative algebra, then $\mathcal{P}(X;Y)$ is also a unital commutative algebra.

Complexification of a real Banach space. If E is a real Banach space, we can associate a complex vector space $\widetilde{E}$ with E (see [13, p. 1]). Let $\widetilde{E}$ be the Cartesian product $E \times E$, then $\widetilde{E}$ can be transformed into a complex space by defining the following two operations:

1. $(x;y)+(u;v)=(x+u;y+v)$ for every $x,y,u,v \in E$,
2. $(\alpha+i\beta)(x;y)=(\alpha x-\beta y;\beta x+\alpha y)$ for every $x,y\in E$ and $\alpha,\beta\in\mathbb{R}$.

From now on, let us refer to $\widetilde{E}$ as the *complexification* of E. If we identify $(x;0)\in\widetilde{E}$ with x, then E is the real kernel of $\widetilde{E}$. This identification allows us to use familiar notation $z=x+iy$ instead of $z=(x;y)$, and the elements $x=\operatorname{Re} z$, $y=\operatorname{Im} z$ are uniquely determined. If $z=x+iy\in\widetilde{E}$, then we define the conjugate of z by $\overline{z}=x-iy$.

According to [13, p. 4], if $\|\cdot\|_E$ is a norm on a real Banach space E and $\|\cdot\|_{\widetilde{E}}$ is any norm defined on the complexification $\widetilde{E}$ of the space E, then we say that $\|\cdot\|_{\widetilde{E}}$ is a *reasonable complexification norm* if the following two conditions are satisfied:

1. $\|\operatorname{Re} z\|_{\widetilde{E}}=\|\operatorname{Re} z\|_E$ for every $z\in\widetilde{E}$,
2. $\|z\|_{\widetilde{E}}=\|\overline{z}\|_{\widetilde{E}}$ for every $z\in\widetilde{E}$.

By [13, Proposition 3] such a norm always exists.

Theorem 1. *(See [3, Theorem 3]) Let $k\in\mathbb{N}$. Let E be a Banach space over $\mathbb{R}$. Given $P\in\mathcal{P}_a({}^kE;\mathbb{R})$, there exists exactly one element $\widehat{P}\in\mathcal{P}_a({}^k\widetilde{E};\mathbb{C})$ such that the restriction of $\widehat{f}$ to E is equal to P. The polynomial P is continuous if and only if the polynomial $\widehat{P}$ is continuous.*

We will call the polynomial $\widehat{P}$ the *complex extension* of the polynomial P.

Theorem 2. *(See [13, p. 16]) Let E be a Banach space over $\mathbb{R}$. Let $P\colon E\to\mathbb{R}$ be an m-homogeneous continuous polynomial. Then its complex extension $\widehat{P}$ is given by the formula $\widehat{P}(x)=A_{\widehat{P}}\big(\underbrace{x,\ldots,x}_{m}\big)$, where $x\in\widetilde{E}$ and $A_{\widehat{P}}$ is given by the formula*

$$A_{\widehat{P}}\big(x_1^{(0)}+ix_1^{(1)},\ldots,x_m^{(0)}+ix_m^{(1)}\big)=\sum_{j_1,\ldots,j_m\in\{0;1\}} i^{\sum_{q=1}^m j_q}A_P\big(x_1^{(j_1)},\ldots,x_m^{(j_m)}\big)$$

Moreover, by (1),

$$\|A_{\widehat{P}}\|\le 2^m\|A_P\|. \tag{5}$$

Let E be a Banach space over $\mathbb{R}$. For every polynomial $P\colon E\to\mathbb{R}$ such that $P=P_0+P_1+\ldots+P_m$, where P_j, $j\in\{0;\ldots;m\}$, is j-homogeneous, we will define its *complex extension* $\widehat{P}$ as

$$\widehat{P}=\widehat{P_0}+\widehat{P_1}+\ldots+\widehat{P_m}, \tag{6}$$

where $\widehat{P_j}$, $j\in\{1;\ldots;m\}$, are defined by Theorem 1 and $\widehat{P_0}=P_0$.

Proposition 1. *Let E be a Banach space over $\mathbb{R}$. For every polynomial $P\colon E\to\mathbb{R}$ there exists a unique polynomial $\widehat{P}\colon\widetilde{E}\to\mathbb{C}$, where $\widetilde{E}$ is the complexification of E, such that the restriction of $\widehat{P}$ to the space E is equal to P. This polynomial is given by* (6). *The polynomial P is continuous if and only if the polynomial $\widehat{P}$ is continuous.*

Proof. Suppose that there exists another polynomial $Q\colon\widetilde{E}\to\mathbb{C}$ such that the restriction of Q to the space E is equal to P. Let $G=Q-\widehat{P}$. Since G is a polynomial, there exists $n\in\mathbb{N}$ such that $G=G_0+G_1+\ldots+G_n$, where $G_j, j\in\{0;\ldots;n\}$, is j-homogeneous.

The restriction of G to E is the sum of restrictions of G_j, where every restriction of G_j is j-homogeneous. Note that the restriction of G to E is equal to 0. Therefore, taking into account Lemma 1, restrictions of all G_j are 0. Therefore, by Theorem 1, $G_j = 0$ for every $j \in \{0; \ldots; n\}$. Consequently, $G = 0$. Hence, $Q = \widehat{P}$. This contradicts the initial assumption. Thus, there exists only one complex-valued polynomial on $\widetilde{E}$ the restriction of which to E is equal to P.

Note that Lemma 1 guarantees that homogeneous components of a continuous polynomial are continuous, and Theorem 2 guarantees that complex extensions of a homogeneous continuous polynomials are continuous. Therefore, the complex extension of a continuous polynomial is continuous.

Let $P\colon E \to \mathbb{R}$ be such that the complex extension $\widehat{P}$ is continuous. Then $\|\widehat{P}\| < \infty$. Thus, $\|P\| < \infty$. Therefore P is a continuous polynomial. Thus, the polynomial P is continuous if and only if the polynomial $\widehat{P}$ is continuous. □

Algebraic basis. Let $\mathbb{K} \in \{\mathbb{R}; \mathbb{C}\}$. Let A be a unital commutative algebra over the field $\mathbb{K}$. For every polynomial $Q\colon \mathbb{K}^n \to \mathbb{K}$ of the form

$$Q(z_1, \ldots, z_n) = \sum_{(k_1,\ldots,k_n)\in\Omega} \alpha_{(k_1,\ldots,k_n)} z_1^{k_1} \cdots z_n^{k_n},$$

where $\alpha_{(k_1,\ldots,k_n)} \in \mathbb{K}$ and Ω is some nonempty finite subset of $\mathbb{Z}_+^n$, let us define the mapping $Q_A\colon A^n \to A$ by

$$Q_A(a_1, \ldots, a_n) = \sum_{(k_1,\ldots,k_n)\in\Omega} \alpha_{(k_1,\ldots,k_n)} a_1^{k_1} \cdots a_n^{k_n}, \tag{7}$$

where $a_1, \ldots, a_n \in A$ (we consider the zero power a_j^0 of an element a_j to be the unit element of A).

Let $a, a_1, \ldots, a_n \in A$. The element a is called an *algebraic combination* of $a_1, \ldots, a_n$, if there exists a polynomial $Q\colon \mathbb{K}^n \to \mathbb{K}$ such that $a = Q_A(a_1, \ldots, a_n)$.

A nonempty set $B \subset A$ is called a *generating system* of A, if every element of A can be represented as an algebraic combination of some elements of B.

A nonempty set $B \subset A$ is called an *algebraic basis* of A if every element of A can be uniquely represented as an algebraic combination of some elements of B.

A finite nonempty set $\{a_1; \ldots; a_n\} \subset A$ is called algebraically independent, if the equality $Q_A(a_1, \ldots, a_n) = 0$ is possible only if the polynomial Q is identically equal to 0. An infinite set $A_0 \subset A$ is called algebraically independent, if its each finite nonempty subset is algebraically independent.

Evidently, every algebraic basis is algebraically independent. Furthermore, every algebraically independent generating system is an algebraic basis.

Lemma 2. *(See [18, Lemma 3.1]) Let X be a vector space over a field $\mathbb{K} \in \{\mathbb{R}; \mathbb{C}\}$. Let $n \in \mathbb{N}$ and a set of polynomials $\{P_1; \ldots; P_n\} \subset \mathcal{P}(X; \mathbb{K})$ is such that following conditions are satisfied:*

1) for every $j \in \{1; \ldots; n\}$ there exists $m_j \in \mathbb{Z}_+$ such that the polynomial P_j is m_j-homogeneous;

2) for every polynomial $Q\colon \mathbb{K}^n \to \mathbb{K}$ of the form

$$Q(z_1, \ldots, z_n) = \sum_{\substack{k_1;\ldots;k_n\in\mathbb{Z}_+\\ k_1m_1+\ldots+k_nm_n=l}} \alpha_{(k_1;\ldots;k_n)} z_1^{k_1} \cdots z_n^{k_n},$$

with $l \in \mathbb{Z}_+$ and $\alpha_{(k_1;\ldots;k_n)} \in \mathbb{K}$, the equality $Q_{\mathcal{P}(X;\mathbb{K})}(P_1, \ldots, P_n) \equiv 0$ holds if and only if

$Q \equiv 0$, *where the polynomial* $Q_{\mathcal{P}(X;\mathbb{K})}$ *is defined by* (7).

Then the set $\{P_1, \ldots, P_n\}$ *is algebraically independent.*

Symmetric mappings. Let A, B be arbitrary nonempty sets. Let S be an arbitrary fixed set of mappings that act from A to itself. A mapping $f\colon A \to B$ is called *S-symmetric* if $f(s(a)) = f(a)$ for every $a \in A$ and $s \in S$.

Spaces of Lebesgue-measurable functions. Let $\mathbb{K} \in \{\mathbb{R};\mathbb{C}\}$. Let $L_p^{(\mathbb{K})}$, where $p \in [1;+\infty)$, be the Banach space of measurable functions $x\colon [0;1] \to \mathbb{K}$ for which the pth power of the absolute value is Lebesgue integrable, i.e. the integral $\int_0^1 |x(t)|^p\,dt$ is finite, with norm

$$\|x\|_{L_p^{(\mathbb{K})}} = \Big(\int_0^1 |x(t)|^p\,dt\Big)^{\frac{1}{p}}. \tag{8}$$

Let us define the norm on the Cartesian product $L_{p_1}^{(\mathbb{K})} \times L_{p_2}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}$, where $p_1, p_2, \ldots, p_n \in [1;+\infty)$ as

$$\|x\|_{L_{p_1}^{(\mathbb{K})} \times L_{p_2}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}} = \max_{j\in\{1;\ldots;n\}} \|x_j\|_{L_j^{(\mathbb{K})}} \tag{9}$$

for every $x = (x_1;\ldots;x_n) \in L_{p_1}^{(\mathbb{K})} \times L_{p_2}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}$.

Let us refer to $L_p^{(\mathbb{C})}$ and $L_{p_1}^{(\mathbb{C})} \times L_{p_2}^{(\mathbb{C})} \times \ldots \times L_{p_n}^{(\mathbb{C})}$ as L_p and $L_{p_1} \times L_{p_2} \times \ldots \times L_{p_n}$ respectively.

Symmetric functions on $L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}$. Let $p_1, \ldots, p_n \in [1;+\infty)$. Let $\Xi_{[0;1]}$ be the set of all bijections $\sigma\colon [0;1] \to [0;1]$ such that both σ and σ^{-1} are measurable and preserve Lebesgue measure, i.e. $\mu(\sigma(E)) = \mu(\sigma^{-1}(E)) = \mu(E)$ for every Lebesgue measurable set $E \subset [0;1]$, where μ is Lebesgue measure. For $\sigma \in \Xi_{[0;1]}$ and $x = (x_1;\ldots;x_n) \in L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}$ let $x \circ \sigma = (x_1 \circ \sigma;\ldots;x_n \circ \sigma)$. For $\sigma \in \Xi_{[0;1]}$, let the operator $g_\sigma^{(\mathbb{K})}$ be defined by

$$g_\sigma^{(\mathbb{K})}\colon x \in L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})} \mapsto x \circ \sigma \in L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}.$$

Let $G_{p_1;\ldots;p_n}^{(\mathbb{K})} = \{g_\sigma^{(\mathbb{K})}\colon\ \sigma \in \Xi_{[0;1]}\}$. It can be verified that $G_{p_1;\ldots;p_n}^{(\mathbb{K})}$ is a group of continuous operators on $L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}$. In the following work, let us refer to a $G_{p_1;\ldots;p_n}^{(\mathbb{K})}$-symmetric function $f\colon L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})} \to \mathbb{K}$ as *symmetric*, i.e. a function $f\colon L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})} \to \mathbb{K}$ is called *symmetric* if $f(x) = f(x \circ \sigma)$ for every $x \in L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}$ and $\sigma \in \Xi_{[0;1]}$.

Symmetric polynomials on $L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}$. Let $p_1, \ldots, p_n \in [1;+\infty)$. Let

$$\aleph_{p_1,\ldots,p_n} = \big\{\alpha = (k_1;\ldots;k_n) \in \mathbb{Z}_+^n \setminus \{(0;\ldots;0)\}\colon\ k_1/p_1 + \ldots + k_n/p_n \le 1\big\}. \tag{10}$$

For every $\alpha = (k_1;k_2;\ldots;k_n) \in \aleph_{p_1,\ldots,p_n}$, let $R_\alpha^{\left(L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}\right)}\colon L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})} \to \mathbb{K}$ be defined by

$$R_\alpha^{\left(L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}\right)}((x_1;\ldots;x_n)) = \int_0^1 x_1^{k_1}(t) \cdot \ldots \cdot x_n^{k_n}(t)\,dt, \tag{11}$$

where $x_j \in L_{p_j}^{(\mathbb{K})}$ for $j \in \{1;\ldots;n\}$. In the case $k_j = 0$ and $x_j \equiv 0$ we will consider $x_j^{k_j}$ to be equal to 1. Note that $R_\alpha^{\left(L_{p_1}^{(\mathbb{K})} \times \ldots \times L_{p_n}^{(\mathbb{K})}\right)}$ is a $(k_1 + \ldots + k_n)$-homogeneous polynomial.

Lemma 3. *(See [15, Theorems 3 and 4]) The polynomial* $R_\alpha^{(L_{p_1} \times \ldots \times L_{p_n})}$ *defined by* (11) *is continuous and symmetric. Moreover* $\|R_\alpha^{(L_{p_1} \times \ldots \times L_{p_n})}\| = 1$.

Theorem 3. *(See [15, Theorem 11]) Let* $p_1, \ldots, p_n \in [1;+\infty)$. *The set* $\big\{R_\alpha^{(L_{p_1} \times \ldots \times L_{p_n})}\colon \alpha \in \aleph_{p_1,\ldots,p_n}\big\}$, *where* $R_\alpha^{(L_{p_1} \times \ldots \times L_{p_n})}$ *is defined by* (11) *and* $\aleph_{p_1,\ldots,p_n}$ *is defined by* (10), *is an algebraic basis of the algebra of all symmetric continuous complex-valued polynomials on* $L_{p_1} \times \ldots \times L_{p_n}$.

2. The main result.

Lemma 4. *Let E be a real Banach space. Let $P, Q\colon E \to \mathbb{R}$ be polynomials and let $\widehat{P}, \widehat{Q}, \widehat{PQ}$ be defined by* (6)*. Then $\widehat{P}\widehat{Q} = \widehat{PQ}$.*

Proof. The restrictions of both polynomials $\widehat{P}\widehat{Q}$ and $\widehat{PQ}$ to the space E are equal to PQ. By Proposition 1, every real-valued polynomial on E including PQ has the unique polynomial complex extension. Consequently, the equality $\widehat{P}\widehat{Q} = \widehat{PQ}$ holds. □

Theorem 4. *Let E be a real Banach space. Let Γ be a set of indices. For every $\gamma \in \Gamma$ let $P_\gamma\colon E \to \mathbb{R}$ be some m_γ-homogeneous continuous polynomial, where $m_\gamma \in \mathbb{Z}_+$. Suppose the set $\{\widehat{P_\gamma}\colon \gamma \in \Gamma\}$ is algebraically independent, where $\widehat{P_\gamma}$ are the complex extensions of polynomials P_γ. Then the set $\{P_\gamma\colon \gamma \in \Gamma\}$ is algebraically independent.*

Proof. Let Γ_1 be an arbitrary finite subset of Γ. Let n be the cardinality of Γ_1, i.e. $\Gamma_1 = \{\gamma_1, \ldots, \gamma_n\}$ for some $\gamma_1, \ldots, \gamma_n \in \Gamma$. Let us check the conditions of Lemma 2 for the space E and the set $\{P_\gamma\colon \gamma \in \Gamma_1\}$. The first condition holds by the definition of the polynomials P_{γ_j}. Let us show that the second condition holds. Let $Q\colon \mathbb{R}^n \to \mathbb{R}$, defined by

$$Q(z_1, \ldots, z_n) = \sum_{\substack{k_1,\ldots,k_n \in \mathbb{Z}_+ \\ k_1 m_1 + \ldots + k_n m_n = l}} \alpha_{(k_1,\ldots,k_n)} z_1^{k_1} \cdots z_n^{k_n},$$

with $l \in \mathbb{Z}_+$ and $\alpha_{(k_1,\ldots,k_n)} \in \mathbb{R}$, be such that $R \equiv 0$, where $R = Q_{\mathcal{P}(E;\mathbb{R})}(P_{\gamma_1}, \ldots, P_{\gamma_n})$ (see (7) for the definition of $Q_{\mathcal{P}(E;\mathbb{R})}$). Note that R is a continuous l-homogeneous polynomial on the space E and, by Lemma 4 and the fact that the complex extension of some sum of polynomials is equal to the sum of their complex extensions, $\widehat{R} = Q_{\mathcal{P}(\widetilde{E};\mathbb{C})}(\widehat{P_{\gamma_1}}, \ldots, \widehat{P_{\gamma_n}})$. Let A_R be the multilinear symmetric mapping associated with R and $A_{\widehat{R}}$ be the multilinear symmetric mapping associated with $\widehat{R}$ (see Theorem 2). By (4) and (2), $\|\widehat{R}\| \leq \|A_{\widehat{R}}\| \leq 2^l \|A_R\| \leq \frac{(2l)^l}{l!}\|R\|$. Therefore, since $R \equiv 0$, it follows that $\widehat{R} \equiv 0$. Consequently, since the set $\{\widehat{P_{\gamma_1}}, \ldots, \widehat{P_{\gamma_n}}\}$ is algebraically independent, $Q \equiv 0$. So, the second condition of Lemma 2 is satisfied. Consequently, the set $\{P_\gamma\colon \gamma \in \Gamma_1\}$ is algebraically independent for every finite set $\Gamma_1 \subset \Gamma$. Thus, the set $\{P_\gamma\colon \gamma \in \Gamma\}$ is algebraically independent. □

Theorem 5. *Let E be a real Banach space and $\widetilde{E}$ be the complexification of E. Let $\widehat{A}$ be a subalgebra of $\mathcal{P}(\widetilde{E};\mathbb{C})$ and A be a subalgebra of $\mathcal{P}(E;\mathbb{R})$. Let Γ be some set of indices. Let $H = \{R_\gamma\colon \gamma \in \Gamma\} \subset A$ be some set of homogeneous (potentially, with different degrees of homogeneity) polynomials. If two following conditions are satisfied:*

1. *the algebra $\widehat{A}$ contains complex extensions of all elements of the algebra A,*
2. *the set $\widehat{H} = \{\widehat{R_\gamma}\colon \gamma \in \Gamma\}$, where $\widehat{R_\gamma}$ are the complex extensions of R_γ, is an algebraic basis of $\widehat{A}$,*

then H is an algebraic basis of A.

Proof. It is enough to show that H is an algebraically independent generating system. In view of algebraic independence of $\widehat{H}$, by Theorem 4 the set H is also algebraically independent. Let us show that H is a generating system. It is enough to show that an arbitrary $P \in A$ is an algebraic combination of elements of H with real coefficients. Let $P \in A$. Since $\widehat{H}$ is an algebraic basis of $\widehat{A}$ and $\widehat{P} \in \widehat{A}$, it follows that $\widehat{P}$ is an algebraic combination of some elements of $\widehat{H}$. Consequently, there exist $n \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $Q\colon \mathbb{C}^n \to \mathbb{C}$ such that

$$\widehat{P}(z) = Q(\widehat{R}_{\gamma_1}(z), \ldots, \widehat{R}_{\gamma_n}(z)) \tag{12}$$

for every $z \in \widetilde{E}$. Taking into account that $Q\colon \mathbb{C}^n \to \mathbb{C}$ is a polynomial, there exist $N \in \mathbb{N}$, $c_1, \ldots, c_N \in \mathbb{C}$ and $\{k_{s;j}\} \subset \mathbb{Z}_+$, where $j \in \{1; \ldots; N\}$ and $s \in \{1; \ldots; n\}$, such that

$$\widehat{P}(z) = \sum_{j=1}^{N} c_j \widehat{R}_{\gamma_1}^{k_{1;j}}(z) \cdot \ldots \cdot \widehat{R}_{\gamma_n}^{k_{n;j}}(z) \tag{13}$$

for every $z \in \widetilde{E}$.

For every $x \in E$ and $G\colon E \to \mathbb{R}$ such that $G \in A$,

$$G(x) = \widehat{G}(x + 0 \cdot i). \tag{14}$$

Let us substitute $z = x + 0 \cdot i$ into the formula (13). Then, taking into account (14),

$$P(x) = \sum_{j=1}^{N} c_j R_{\gamma_1}^{k_{1;j}}(x) \cdot \ldots \cdot R_{\gamma_n}^{k_{n;j}}(x) \tag{15}$$

for every $x \in E$. To show that the polynomial P is an algebraic combination of $R_{\gamma_1}, \ldots, R_{\gamma_n}$, it is enough to show that $c_j \in \mathbb{R}$ for every $j \in \{1; \ldots; N\}$. Since $P(x), R_{\gamma_j}(x) \in \mathbb{R}$ for every $x \in E$, it follows that $P(x) = \overline{P(x)}$ and $R_{\gamma_j}(x) = \overline{R_{\gamma_j}(x)}$ for every $x \in E$. Consequently, by (15),

$$P(x) = \sum_{j=1}^{N} \overline{c_j} R_{\gamma_1}^{k_{1;j}}(x) \cdot \ldots \cdot R_{\gamma_n}^{k_{n;j}}(x). \tag{16}$$

By subtracting (16) from (15) we get $2i(\sum_{j=1}^{N} (\mathrm{Im}\ c_j) R_{\gamma_1}^{k_{1;j}}(x) \cdot \ldots \cdot R_{\gamma_n}^{k_{n;j}}(x)) = 0$ for every $x \in E$. Taking into account that the set H is algebraically independent, it follows that $\mathrm{Im}\, c_j = 0$ for every $j \in \{1; \ldots; N\}$. Thus, $c_1, \ldots, c_N \in \mathbb{R}$. Consequently, P is an algebraic combination of $R_{\gamma_1}, \ldots, R_{\gamma_n}$. Therefore, H is a generating system in A.

Thus, H is an algebraic basis in A. □

Theorem 5 implies the following corollary.

Corollary 1. *Let E be a real Banach space and $\widetilde{E}$ be the complexification of E. Let S be some set of operators acting from E to E and $\widehat{S}$ be some set of operators acting from $\widetilde{E}$ to $\widetilde{E}$. Let $\widehat{A}$ be the algebra of all $\widehat{S}$-symmetric continuous complex-valued polynomials on $\widetilde{E}$ and A be the algebra of all S-symmetric continuous real-valued polynomials on E. Let Γ be some set of indices. Let $H = \{P_\gamma \colon \gamma \in \Gamma\} \subset A$ be some set of homogeneous (potentially, with different degrees of homogeneity) polynomials. If two following conditions are satisfied:*

1. *sets S and $\widehat{S}$ are such that the complex extension of every S-symmetric continuous real-valued polynomial is a $\widehat{S}$-symmetric continuous complex-valued polynomial,*
2. *the set $\widehat{H} = \{\widehat{P_\gamma} \colon \gamma \in \Gamma\}$, where $\widehat{P_\gamma}$ are the complex extensions of polynomials P_γ, is an algebraic basis of $\widehat{A}$,*

then H is an algebraic basis of A.

Proof. We apply Theorem 5 to the space E, the algebras A, $\widehat{A}$ and the set H. Both conditions of the corollary match with the conditions of Theorem 5. □

Let $p_1, \ldots, p_n \in [1; +\infty)$. Let us apply Corollary 1 to the algebra of all symmetric continuous real-valued polynomials on $L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}$. First we establish some auxiliary results.

Lemma 5. *The complexification of the space $L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}$ is equal to $L_{p_1} \times \ldots \times L_{p_n}$ and the norm $\|\cdot\|_{L_{p_1} \times \ldots \times L_{p_n}}$ is a reasonable complexification norm with respect to the norm $\|\cdot\|_{L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}}$, where $\|\cdot\|_{L_{p_1} \times \ldots \times L_{p_n}}$ and $\|\cdot\|_{L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}}$ are defined by (9).*

Proof. Let us show that there exists a linear bijection between the complexification $(L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}) \times (L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})})$ of the space $L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}$ and $L_{p_1} \times \ldots \times L_{p_n}$. Let $f((x;y)) = x + iy$, where $x, y \in L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}$. Then f is an injective linear function acting from $(L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}) \times (L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})})$ to $L_{p_1} \times \ldots \times L_{p_n}$. Let us show that f is surjective. Let $z = (z_1; \ldots; z_n) \in L_{p_1} \times \ldots \times L_{p_n}$. Let us construct $x, y \in L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}$ such that $f((x;y)) = z$. Let $x = (\operatorname{Re} z_1; \ldots; \operatorname{Re} z_n)$ and $y = (\operatorname{Im} z_1; \ldots; \operatorname{Im} z_n)$. Then $x, y \in L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}$. Note that $f((x;y)) = z$. Thus, f is a surjection. Therefore, f is a linear bijection between $(L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}) \times (L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})})$ and $L_{p_1} \times \ldots \times L_{p_n}$. One can verify that two conditions required by the definition of complexification are satisfied. Then, $L_{p_1} \times \ldots \times L_{p_n}$ is the complexification of $L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}$.

Let us show that the norm $\|\cdot\|_{L_{p_1} \times \ldots \times L_{p_n}}$ is a reasonable complexification norm with respect to the norm $\|\cdot\|_{L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}}$. For every $z = (z_1; \ldots; z_n) \in L_{p_1} \times \ldots \times L_{p_n}$

$$\|\operatorname{Re} z\|_{L_{p_1} \times \ldots \times L_{p_n}} = \max_{j \in \{1; \ldots; n\}} \|\operatorname{Re} z_j\|_{L_{p_j}} = \max_{j \in \{1; \ldots; n\}} \|\operatorname{Re} z_j\|_{L_{p_j}^{(\mathbb{R})}} = \|\operatorname{Re} z\|_{L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}}.$$

Therefore, the first condition required by the definition of a reasonable norm has been established. For every $z = (z_1; \ldots; z_n) \in L_{p_1} \times \ldots \times L_{p_n}$

$$\|z\|_{L_{p_1} \times \ldots \times L_{p_n}} = \max_{j \in \{1; \ldots; n\}} \Big(\int_0^1 |z_j(t)|^{p_j} \Big)^{\frac{1}{p_j}} = \max_{j \in \{1; \ldots; n\}} \Big(\int_0^1 |\overline{z}_j(t)|^{p_j} \Big)^{\frac{1}{p_j}} = \|\overline{z}\|_{L_{p_1} \times \ldots \times L_{p_n}}.$$

Thus, the second condition required by the definition of a reasonable norm has been established. So, the norm $\|\cdot\|_{L_{p_1} \times \ldots \times L_{p_n}}$ is a reasonable complexification norm with respect to the norm $\|\cdot\|_{L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}}$. □

By Theorem 2, for the complex extension $\widehat{P} \colon L_{p_1} \times \ldots \times L_{p_n} \to \mathbb{C}$ of an m-homogeneous polynomial $P \colon L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})} \to \mathbb{R}$, its associated m-linear symmetric mapping $A_{\widehat{P}} \colon (L_{p_1} \times \ldots \times L_{p_n})^m \to \mathbb{C}$ is given by the following formula:

$$A_{\widehat{P}}(x_1, \ldots, x_m) = \sum_{j_1=0}^{1} \ldots \sum_{j_m=0}^{1} i^{j_1 + \ldots + j_m} A_P \Big(w_{j_1}(x_1), \ldots, w_{j_m}(x_m) \Big) \tag{17}$$

for $x_1, \ldots, x_m \in L_{p_1} \times \ldots \times L_{p_n}$, where A_P is the symmetric m-linear mapping associated with P,

$$w_0(x_k) = (\operatorname{Re} x_{k;1}; \ldots; \operatorname{Re} x_{k;n}) \quad \text{and} \quad w_1(x_k) = (\operatorname{Im} x_{k;1}; \ldots; \operatorname{Im} x_{k;n}) \tag{18}$$

for $x_k = (x_{k,1}; \ldots; x_{k;n})$, $k \in \{1; \ldots; m\}$. Then

$$\widehat{P}(z) = A_{\widehat{P}}(z, \ldots, z) \tag{19}$$

for every $z \in L_{p_1} \times \ldots \times L_{p_n}$.

Lemma 6. *Let $P \colon L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})} \to \mathbb{R}$ be a symmetric m-homogeneous polynomial. Then for the symmetric associated with P m-linear mapping $A_P \colon \big(L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}\big)^m \to \mathbb{R}$ the formula $A_P(x_1, \ldots, x_m) = A_P(x_1 \circ \sigma, \ldots, x_m \circ \sigma)$ holds for every $\sigma \in \Xi_{[0;1]}$ and $x_j \in L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}$, where $j \in \{1; \ldots; m\}$.*

Proof. Let $\sigma \in \Xi_{[0;1]}$ and $x_1, \ldots, x_m \in L_{p_1}^{(\mathbb{R})} \times \ldots \times L_{p_n}^{(\mathbb{R})}$. For every $j_1, \ldots, j_m \in \{-1; 1\}$,

$$(j_1 x_1 + \ldots + j_m x_m) \circ \sigma = j_1 (x_1 \circ \sigma) + \ldots + j_m (x_m \circ \sigma).$$

Then, taking into account that P is symmetric,

$$P(j_1 x_1 + \ldots + j_m x_m) = P(j_1 (x_1 \circ \sigma) + \ldots + j_m (x_m \circ \sigma)) \tag{20}$$

for every $j_1, \ldots, j_m \in \{-1; 1\}$. By (2) and (20), $A_P(x_1, \ldots, x_m) = A_P(x_1 \circ \sigma, \ldots, x_m \circ \sigma)$. □

Lemma 7. *For every continuous polynomial* $R\colon L_{p_1}\times\ldots\times L_{p_n}\to\mathbb{C}$ *its restriction to* $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ *is continuous.*

Proof. Let $R\colon L_{p_1}\times\ldots\times L_{p_n}\to\mathbb{C}$ be a continuous polynomial. Then $\|R\|<\infty$. Then, the norm of the restriction of R to $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ is also finite. Therefore the restriction of R to $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ is continuous. □

Lemma 8. *For every symmetric mapping* $Q\colon L_{p_1}\times\ldots\times L_{p_n}\to\mathbb{C}$ *its restriction to* $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ *is symmetric.*

Proof. Let $Q\colon L_{p_1}\times\ldots\times L_{p_n}\to\mathbb{C}$ be symmetric. Let us show that the restriction of Q to $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ is symmetric. Let $\sigma\in\Xi_{[0;1]}$. For every $z\in L_{p_1}\times\ldots\times L_{p_n}$, since Q is symmetric, $Q(z)=Q(z\circ\sigma)$, i.e.

$$Q(\operatorname{Re}z+i\operatorname{Im}z)=Q((\operatorname{Re}z+i\operatorname{Im}z)\circ\sigma). \tag{21}$$

Let $x\in L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$. Let $z_1=x+i\cdot 0$. By (21),

$$Q(x)=Q(\operatorname{Re}z_1+i\operatorname{Im}z_1)=Q((\operatorname{Re}(z_1)+i\operatorname{Im}z_1)\circ\sigma)=Q(x\circ\sigma).$$

Thus, the restriction of Q to $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ is symmetric. □

Lemma 9. *For every symmetric continuous polynomial* $P\colon L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}\to\mathbb{R}$, *its complex extension* $\widehat{P}\colon L_{p_1}\times\ldots\times L_{p_n}\to\mathbb{C}$ *is symmetric and continuous.*

Proof. By Proposition 1, the complex extension of every continuous polynomial is continuous, thus, $\widehat{P}$ is continuous. Let us show that the complex extension of every symmetric continuous polynomial P that acts from $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ to $\mathbb{R}$ is symmetric. By the definition of the complex extension of a polynomial (see (6)), it is enough to show this fact for every its homogeneous component. By Lemma 1, every homogeneous component of P is symmetric. Let $m\in\mathbb{Z}_+$ and P_m be an arbitrary m-homogeneous symmetric continuous polynomial acting from $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ to $\mathbb{R}$. Let $\sigma\in\Xi_{[0;1]}$. For every $x_j\in L_{p_j}$, $\operatorname{Re}(x_j(\sigma(t)))=(\operatorname{Re}x_j)(\sigma(t))$ and $\operatorname{Im}(x_j(\sigma(t)))=(\operatorname{Im}x_j)(\sigma(t))$ for every $t\in[0;1]$. Thus, for every $x\in L_{p_1}\times\ldots\times L_{p_n}$,

$$w_0(x\circ\sigma)=w_0(x)\circ\sigma \text{ and } w_1(x\circ\sigma)=w_1(x)\circ\sigma, \tag{22}$$

where w_0 and w_1 are defined by (18). Let $z\in L_{p_1}\times\ldots\times L_{p_n}$. By (22) and Lemma 6, $A_{P_m}\big(w_{j_1}(z\circ\sigma),\ldots,w_{j_m}(z\circ\sigma)\big)=A_{P_m}\big(w_{j_1}(z),\ldots,w_{j_m}(z)\big)$ for every $j_1,\ldots,j_m\in\{0,1\}$. Consequently, by (19) and (17), $\widehat{P_m}(z\circ\sigma)=\widehat{P_m}(z)$, i.e. $\widehat{P}_m$ is symmetric. Therefore the complex extension of every symmetric continuous polynomial acting from $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ to $\mathbb{R}$ is symmetric. □

Lemma 10. *Let* $\alpha\in\aleph_{p_1,\ldots,p_n}$, *where* $\aleph_{p_1,\ldots,p_n}$ *is defined by (10). Let* $R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}$ *and* $R_\alpha^{(L_{p_1}\times\ldots\times L_{p_n})}$ *be defined by (11). Then*

1. *the complex extension of the polynomial* $R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}$ *is equal to* $R_\alpha^{(L_{p_1}\times\ldots\times L_{p_n})}$;
2. *the polynomial* $R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}$ *is symmetric and continuous;*
3. $\big\|R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}\big\|=1.$

Proof. Let us prove the first part. The restriction of the polynomial $R_\alpha^{(L_{p_1}\times\ldots\times L_{p_n})}$ to the space $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ is equal to $R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}$. By Theorem 1, the homogeneous polynomial $R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}$ can be extended uniquely to a polynomial acting from the space $L_{p_1}\times\ldots\times L_{p_n}$ to $\mathbb{C}$. So, its complex extension is equal to $R_\alpha^{(L_{p_1}\times\ldots\times L_{p_n})}$.

Let us prove the second part. The polynomial $R_\alpha^{(L_{p_1}\times\ldots\times L_{p_n})}$ is continuous and symmetric by Lemma 3. So, the second part of the lemma immediately follows from Lemmas 7 and 8 for $R_\alpha^{(L_{p_1}\times\ldots\times L_{p_n})}$.

Let us prove the third part. By Lemma 3, $\|R_\alpha^{(L_{p_1}\times\ldots\times L_{p_n})}\| = 1$. Since $\|R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}\| \le \|R_\alpha^{(L_{p_1}\times\ldots\times L_{p_n})}\|$, it follows that $\|R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}\| \le 1$. Since $\|(1;1;\ldots;1)\|_{L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}} = 1$ and $R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}((1;1;\ldots;1)) = 1$, it follows that $\|R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}\| \ge 1$. So, $\|R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}\| = 1$. □

Theorem 6. *Let* $p_1;\ldots;p_n \in [1;+\infty)$. *The set* $\{R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})} : \alpha \in \aleph_{p_1,\ldots,p_n}\}$, *where* $R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}$ *are defined by* (11) *and* $\aleph_{p_1,\ldots,p_n}$ *is defined by* (10), *is an algebraic basis of the algebra of all symmetric continuous real-valued polynomials on* $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$.

Proof. Let us check the conditions of Corollary 1, where $E = L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$, $S = G_{p_1,\ldots,p_n}^{(\mathbb{R})}$, $\widehat{S} = G_{p_1,\ldots,p_n}^{(\mathbb{C})}$, A is the algebra of all symmetric continuous real-valued polynomials on $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$, $\widehat{A}$ is the algebra of all symmetric continuous complex-valued polynomials on the complexification of $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ and $H = \{R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})} : \alpha \in \aleph_{p_1,\ldots,p_n}\}$. By Lemma 5, the complexification of $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ is equal to $L_{p_1}\times\ldots\times L_{p_n}$.

Let us show that H is a subset of A. It is enough to show that for an arbitrary $\alpha \in \aleph_{p_1,\ldots,p_n}$ the polynomial $R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})}$ is symmetric and continuous. It immediately follows from Lemma 10, part 2.

By Lemma 9, the complex extensions of symmetric continuous real-valued polynomials on $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$ are symmetric continuous complex-valued polynomials on $L_{p_1}\times\ldots\times L_{p_n}$. Thus, the first condition of Corollary 1 is satisfied. By Lemma 10, part 1, the set $\widehat{H}$ of complex extensions of elements of the set H is equal to $\{R_\alpha^{(L_{p_1}\times\ldots\times L_{p_n})} : \alpha \in \aleph_{p_1,\ldots,p_n}\}$, which, by Theorem 3, is an algebraic basis of $\widehat{A}$. Thus, the second condition of Corollary 1 is satisfied. Therefore, by Corollary 1, the set $\{R_\alpha^{(L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})})} : \alpha \in \aleph_{p_1,\ldots,p_n}\}$ is an algebraic basis of the algebra of all symmetric continuous real-valued polynomials on $L_{p_1}^{(\mathbb{R})}\times\ldots\times L_{p_n}^{(\mathbb{R})}$. □

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