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AN ANALOG OF THE HILLE THEOREM FOR HYPERCOMPLEX FUNCTIONS IN A FINITE-DIMENSIONAL COMMUTATIVE ALGEBRA

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We prove that a locally bounded and differentiable in the sense of Gâteaux function given in a finite-dimensional commutative Banach algebra over the complex field is also differentiable in the sense of Lorc and it is a monogenic function. The algebra $\mathbb{A}_n^m$ has the Cartan basis for which the first m basic vectors $I_1, I_2, \ldots, I_m$ are idempotents, and next $n-m$ basis vectors $I_{m+1}, I_{m+2}, \ldots, I_n$ are nilpotent elements. Every locally bounded and differentiable in the sense of Gâteaux function $\Phi\colon \Omega \to \mathbb{A}_n^m$ can be represented in the form of linear combination of these idempotents, nilpotents and corresponding Cauchy-type integrals.

1. Introduction. For mappings of vector spaces as well as commutative Banach algebras, various concepts of differentiable mappings are used. Let us consider some of these concepts and the relationship between them (for more detailed information, see, for example, the monograph [1]). Let V_j, $j=1,2$, be normalized vector spaces with the norms $\|\cdot\|_{V_j}$, and let Ω be an open subset of V_1. A mapping $\Phi\colon \Omega \to V_2$ is *Fréchet differentiable* at a point $\zeta \in \Omega$ if there exists a bounded linear operator $A_\zeta\colon V_1 \to V_2$ such that

$$\frac{\|\Phi(\zeta+h)-\Phi(\zeta)-A_\zeta h\|_{V_2}}{\|h\|_{V_1}} \to 0, \quad \|h\|_{V_1} \to 0.$$

The operator A_ζ is called the *Fréchet derivative* of the mapping Φ at the point ζ (cf. M. Fréchet [2]). A mapping $\Phi\colon \Omega \to V_2$ is *Gâteaux differentiable* at a point $\zeta \in \Omega$ if the *Gâteaux differential*

$$\mathcal{D}_G\Phi(\zeta,h) := \lim_{\delta\to 0}\frac{\Phi(\zeta+\delta h)-\Phi(\zeta)}{\delta}, \tag{1}$$

exists for all $h \in V_1$, where δ is taken from the scalar field associated with the space V_1 (cf. R. Gâteaux [3]). In addition, if there exists a bounded linear operator $B_\zeta\colon V_1 \to V_2$ such that $\mathcal{D}_G\Phi(\zeta,h) \equiv B_\zeta h$, the operator B_ζ is called the *Gâteaux derivative* of the mapping Φ at the point ζ. It is evident that if a mapping $\Phi\colon \Omega \to V_2$ is Fréchet differentiable at a point $\zeta \in \Omega$, then the Gâteaux derivative of Φ exists at the point ζ and is equal to the Fréchet derivative of the mapping Φ at that point. The converse is not true in general.

The following theorem proved by E. Hille ([4], see also E. Hille and R. Phillips [5, p. 112]) indicates suffcient conditions of the coincidence of the Gâteaux derivative and the Fréchet derivative.

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Hille theorem. *Let V_1, V_2 be complex Banach spaces and Ω be an open subset of V_1. Suppose that a mapping $\Phi\colon \Omega \to V_2$ is locally bounded in Ω and has the Gâteaux derivative B_ζ at every point $\zeta \in \Omega$. Then B_ζ is the Fréchet derivative of the mapping Φ at every point $\zeta \in \Omega$.*

The considered Fréchet and Gâteaux derivatives are defined as bounded linear operators. Considering mappings given in commutative Banach algebras, it is possible to introduce the concepts of derivatives which are understood as functions defined in the same domain as the given function. Foremost, note that for functions $\Phi(\zeta)$ given in a domain of a finite-dimensional algebra, G. Scheffers [6] considered a derivative $\Phi'(\zeta)$ defined by $d\Phi = \Phi'(\zeta)d\zeta$, which is understood as a hypercomplex function. Generalizing such an approach to the case of functions given in a domain of an arbitrary commutative Banach algebra, E.R. Lorch [7] introduced a derivative, which is also understood as a function given in the same domain.

Let $\mathbb{A}$ be a commutative Banach algebra with unit 1 over either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, and let $\|\zeta\|$ denotes the norm of element $\zeta \in \mathbb{A}$. Let us fix a finite number k of the vectors $e_1 = 1, e_2, \ldots, e_k \in \mathbb{A}$ which are linearly independent over the field of real numbers $\mathbb{R}$. Let

$$E_k := \Big\{\zeta = \sum_{j=1}^{k} x_j e_j \colon x_j \in \mathbb{R}\Big\} \tag{2}$$

be the linear span of the vectors $e_1, e_2, \ldots, e_k$ over the field $\mathbb{R}$.

Definition 1. A function $\Phi\colon \Omega \to \mathbb{A}$ given in a domain $\Omega \subset E_k$ is called *differentiable in the sense of Lorch at a point* $\zeta \in \Omega$ if there exists an element $\Phi'_L(\zeta) \in \mathbb{A}$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in E_k$ with $\|h\| < \delta$ the following inequality is fulfilled $\|\Phi(\zeta + h) - \Phi(\zeta) - h\Phi'_L(\zeta)\| \le \|h\|\varepsilon$. Here, $\Phi'_L(\zeta)$ is called the *Lorch derivative* of the function Φ at the point ζ (cf. E.R. Lorch [7]).

Clearly, a function Φ, which is differentiable in the sense of Lorch at a point $\zeta \in \Omega$, is also the Fréchet differentiable at the same point. The converse is not true. Using the Gâteaux differential ((1)), I.P. Mel'nichenko [8] defined a derivative which is also a hypercomplex function.

Definition 2. A function $\Phi\colon \Omega \to \mathbb{A}$ given in a domain $\Omega \subset E_k$ is called *differentiable in the sense of Gâteaux at a point* $\zeta \in \Omega$, if there exists an element $\Phi'_G(\zeta) \in \mathbb{A}$ such that

$$\lim_{\delta \to 0^+} \frac{\Phi(\zeta + \delta h) - \Phi(\zeta)}{\delta} = h\Phi'_G(\zeta) \qquad \forall h \in E_k.$$

We call $\Phi'_G(\zeta)$ the *Gâteaux–Mel'nichenko derivative* of the function Φ at the point ζ (cf. I.P. Mel'nichenko [8]). Clearly, if a function Φ is differentiable in the sense of Lorch in a domain $\Omega \subset E_k$, then it is also differentiable in the sense of Gâteaux, and $\Phi'_L(\zeta) = \Phi'_G(\zeta)$ for all $\zeta \in \Omega$. Obviously, the converse statement is not true because the existence of all directional derivatives at a point does not guarantee a strong differentiability (or even continuity) of function at that point. Moreover, the Fréchet differentiability of a function $\Phi\colon \Omega \to \mathbb{A}$ does not imply the differentiability of this function in the sense of Gâteaux (see Examples 3.1 and 3.2 in [1]). In the paper [9], an analog of the Hille theorem was proved for hypercomplex functions in some three-dimensional commutative algebra over the complex field (see also Theorem 6.17 in [1]). Namely, it is proved that the locally bounded and differentiable in the sense of Gâteaux functions defined in a three-dimensional commutative harmonic algebra with two-dimensional radical are also differentiable in the sense of Lorch. Note that it is impossible to deduce this result by the Hille theorem because the Fréchet differentiability does not imply the existence of Lorch derivative. The purpose of this paper is to prove a similar theorem in an arbitrary finite-dimensional commutative Banach algebra.

2. A finite-dimensional commutative associative algebra and the Cartan basis. E. Cartan [10] proved that for an arbitrary n-dimensional commutative associative algebra $\mathbb{A}$ with unit over the field of complex number $\mathbb{C}$, there exists a basis $\{I_k\}_{k=1}^n$ and there exist structural constants $\Upsilon_{r,k}^s$ such that the following multiplication rules hold:

$$1.\forall r,s\in[1,m]\cap\mathbb{N}:\quad I_rI_s=\begin{cases}0, & \text{for}\quad r\neq s,\\ I_r, & \text{for}\quad r=s;\end{cases}$$

$$2.\forall r,s\in[m+1,n]\cap\mathbb{N}:\quad I_rI_s=\sum_{k=\max\{r,s\}+1}^{n}\Upsilon_{r,k}^sI_k;$$

$$3.\forall s\in[m+1,n]\cap\mathbb{N}\quad \exists!\ u_s\in[1,m]\cap\mathbb{N}\quad \forall r\in[1,m]\cap\mathbb{N}:$$

$$I_rI_s=\begin{cases}0, & \text{for}\quad r\neq u_s,\\ I_s, & \text{for}\quad r=u_s,\end{cases}$$

where $\mathbb{N}$ is the set of natural numbers. Obviously, the first m basic vectors $I_1, I_2, \ldots, I_m$ are idempotents, and the vectors $I_{m+1}, I_{m+2}, \ldots, I_n$ are nilpotent elements. The algebra $\mathbb{A}$ with the Cartan basis is denoted as $\mathbb{A}_n^m$. The element $1 = I_1 + I_2 + \cdots + I_m$ is the unit in the algebra $\mathbb{A}_n^m$ that is a commutative Banach algebra with Euclidean norm defined by the equality $\|v\| := (\sum_{j=1}^n |v_j|^2)^{1/2}$ for $v = \sum_{j=1}^n v_jI_j,\ v_i \in \mathbb{C}$. The algebra $\mathbb{A}_n^m$ contains m maximal ideals

$$\mathcal{I}_u := \left\{\sum_{r=1,r\neq u}^{n}\lambda_rI_r : \lambda_r\in\mathbb{C}\right\},\quad u\in\{1,2,\ldots,m\},$$

and their intersection is the radical $\mathcal{R} := \{\sum_{r=m+1}^n \lambda_rI_r\colon \lambda_r \in \mathbb{C}\}$. We define m linear functionals $f_u\colon \mathbb{A}_n^m \to \mathbb{C}$ by the equalities $f_u(I_u) = 1,\ f_u(\omega) = 0\ \forall\omega \in \mathcal{I}_u,\ u \in \{1, 2, \ldots, m\}$. Since the kernel of the functional f_u is the maximal ideal $\mathcal{I}_u$, this functional is also continuous and multiplicative (see [5, p. 147]).

3. The main result. Let us consider vectors $e_1 = 1, e_2, \ldots, e_k \in \mathbb{A}_n^m$, $2 \le k \le 2n$, which are linearly independent over the field $\mathbb{R}$. Thus, the equality $\sum_{j=1}^k \alpha_je_j = 0$ holds with $\alpha_j \in \mathbb{R}$ if and only if $\alpha_j = 0$ for all $j \in \{1, 2, \ldots, k\}$. The following decompositions with respect to the Cartan basis $\{I_r\}_{r=1}^n$ hold $e_1 = \sum_{r=1}^m I_r,\ e_j = \sum_{r=1}^n a_{jr}I_r,\ a_{jr} \in \mathbb{C},\ j \in \{2, 3, \ldots, k\}$.

The linear span E_k is defined by equality (2). Next, we impose the following restriction on the choice of the linear span E_k:

$$\{f_u(\zeta)\colon \zeta\in E_k\}=\mathbb{C},\qquad u\in\{1,2,\ldots,m\},\tag{3}$$

i.e., the images of the set E_k under all mappings f_u must be the whole complex plane (cf. [11]). Obviously, it holds if and only if for every fixed $u \in \{1, 2, \ldots, m\}$ at least one of the numbers $a_{2u}, a_{3u}, \ldots, a_{ku}$ belongs to $\mathbb{C} \setminus \mathbb{R}$.

Definition 3. We say that a function $\Phi\colon \Omega \longrightarrow \mathbb{A}_n^m$ is *monogenic* in a domain $\Omega \subset E_k$ if Φ is continuous and differentiable in the sense of Gâteaux at every point of Ω.

In [1, 12] for the case $k = 3$, we developed a theory of monogenic functions $\Phi\colon \Omega \longrightarrow \mathbb{A}_n^m$, which includes analogs of classical theorems of complex analysis (the Cauchy integral theorems for a curvilinear and for a surface integral, the Cauchy integral formula, the Morera theorem, the Taylor theorem). These results are generalized in the papers [13, 14] for the case $2 \le k \le 2n$. Note that we use the notion of monogenic function in the sense of existence of derived numbers for this function (cf. the monographs by E. Goursat [15] and Ju.Ju. Trokhimchuk [16]) in a combination with its continuity. In the scientific literature, the concept of monogenic function is used else for functions given in non-commutative algebras and satisfying certain conditions similar to the classical Cauchy–Riemann conditions (see, for example, F. Sommen [17] and J. Ryan [18]). The main result of this paper is the following analogue of the Hille theorem.

Theorem 1. *Suppose that condition* (3) *is satisfied. For a function* $\Phi\colon \Omega \to \mathbb{A}_n^m$ *given in an arbitrary domain* $\Omega \subset E_k$ *the following properties are equivalent:*

(I) Φ *is a function locally bounded and differentiable in the sense of Gâteaux in* Ω*;*

(II) Φ *is a monogenic function in* Ω*;*

(III) Φ *is a function differentiable in the sense of Lorch in* Ω*.*

It follows from Theorem 1 that under condition (3) the property of continuity in Definition 3 of a monogenic function can be replaced by its local boundedness. Note that in Theorem 2 in [19], which is an analog of Menchov–Trokhimchuk theorem, the conditions of monogeneity are weakened in another way for continuous functions taking values in one of three-dimensional commutative algebras over the complex field (cf. also Theorem 6.18 in [1]).

Certainly, Theorem 1 yields that the property of function to be locally bounded and differentiable in the sense of Gâteaux in Ω is also equivalent to the various definitions of monogenic function, which are stated in Theorem 9.9 in [1].

4. Proof of the main result.

4.1. Auxiliary statements. For

$$\zeta := \sum_{j=1}^{k} x_j e_j, \qquad x_j \in \mathbb{R}. \tag{4}$$

one has

$$\xi_u := f_u(\zeta) = x_1 + \sum_{j=2}^{k} x_j a_{ju}, \qquad u \in \{1, 2, \ldots, m\}. \tag{5}$$

The following expansion of resolvent is proved in Lemma 1 from [13]:

$$(te_1 - \zeta)^{-1} = \sum_{u=1}^{m} \frac{1}{t - \xi_u} I_u + \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \frac{Q_{r,s}}{(t - \xi_{u_s})^r} I_s \tag{6}$$

$(\forall t \in \mathbb{C},\ t \neq \xi_u)$, $u \in \{1, 2, \ldots, m\}$, where the coefficients $Q_{r,s}$ are determined by the following recurrence relations

$$Q_{2,s} = T_s, \quad Q_{r,s} = \sum_{q=r+m-2}^{s-1} Q_{r-1,q} B_{q,s} \text{ for } r \in \{3, 4, \ldots, s - m + 1\},$$

and $T_s := \sum_{j=2}^{k} x_j a_{js}$ for $s \in \{m+1, m+3, 2 \ldots, n\}$, and

$$B_{q,s} := \sum_{p=m+1}^{s-1} T_p \Upsilon_{q,s}^{p} \text{ for } s \in \{m+2, m+3, \ldots, n\},$$

and the natural numbers u_s are defined in the rule 3 of the multiplication table of algebra $\mathbb{A}_n^m$. Expansion (6) generalizes the corresponding expansion obtained in [12] (see also Section 8.3 in [1]) for the case $k = 3$.

It follows from expansion (6) that

$$\zeta^{-1} = \sum_{u=1}^{m} \frac{1}{\xi_u} I_u - \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} (-1)^r \frac{Q_{r,s}}{(\xi_{u_s})^r} I_s,$$

and the union $\bigcup_{1 \le u \le m} M_u$ of the sets $M_u := \{\zeta \in E_k \colon \xi_u = 0\}$ is the set of all noninvertible elements in E_k. Since the equality (4) establishes a one-to-one correspondence between the points $\zeta \in E_k$ and $(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, the linear span (2) is a k-dimensional real vector space. Therefore, we shall use some geometric concepts (parallelism, convexity, $(k-2)$-plane etc.) of real vector spaces for the objects from E_k.

Taking into account the equality (5), we can state that the set M_u of noninvertible elements (4) is $(k-2)$-dimensional linear subspace of E_k, which is defined by the equalities

$$x_1 + \sum_{j=2}^{k} x_j \mathrm{Re} a_{ju} = 0, \qquad \sum_{j=2}^{k} x_j \mathrm{Im} a_{ju} = 0$$

for $u \in \{1, 2, \ldots, m\}$. We shall say that $(k-2)$-plane $L \subset E_k$ is parallel to M_u if L can be obtained by translation of M_u. Denote the line segment with the start point $\zeta_1 \in E_k$ and the end point $\zeta_2 \in E_k$ as $s[\zeta_1, \zeta_2]$. We shall use the same denotation of the line segment $s[\xi_1, \xi_2]$ in the case where $\xi_1, \xi_2 \in \mathbb{C}$.

Consider the expansion

$$\Phi(\zeta) = \sum_{r=1}^{n} V_r(\zeta) I_r \tag{7}$$

of a function $\Phi \colon \Omega \to \mathbb{A}_n^m$ with respect to the basis $\{I_r\}_{r=1}^{n}$, where $V_r \colon \Omega \to \mathbb{C}$.

Lemma 1. *Let a function $\Phi \colon \Omega \to \mathbb{A}_n^m$ be differentiable in the sense of Gâteaux at a point ζ_0 of a domain $\Omega \subset E_k$.*

(a) For any $u \in \{1, 2, \ldots, m\}$, let $\Sigma_u \subset \Omega$ be a two-dimensional surface containing four segments $s[\zeta_0, \zeta_1]$, $s[\zeta_0, \zeta_2]$, $s[\zeta_0, \zeta_3]$, $s[\zeta_0, \zeta_4]$ such that their images under the mapping f_u are subsets of the segments $s[\xi_0, \xi_0 + 1]$, $s[\xi_0, \xi_0 + i]$, $s[\xi_0, \xi_0 - 1]$, $s[\xi_0, \xi_0 - i]$, respectively, where $\xi_0 = f_u(\zeta_0)$, and, in addition, let the restriction of the functional f_u to the surface Σ_u defines a one-to-one mapping of this surface onto the domain $Q_u := \{\xi = f_u(\zeta) \colon \zeta \in \Sigma_u\}$. Then the function $H_u \colon Q_u \to \mathbb{C}$ defined as $H_u(\xi) := V_u(\zeta)$ for all $\xi = \tau + i\eta = f_u(\zeta)$ with $\zeta \in \Sigma_u$, $\tau, \eta \in \mathbb{R}$, and V_u, given by (7), has the partial derivatives $\partial H_u / \partial \tau$, $\partial H_u / \partial \eta$ at the point ξ_0, and the following equality holds:

$$\frac{\partial H_u}{\partial \eta} = i \frac{\partial H_u}{\partial \tau}. \tag{8}$$

(b) Suggest that the function Φ takes values in the radical $\mathcal{R}$, i.e., expansion (7) takes the form

$$\Phi(\zeta)=\sum_{r=s}^{n}V_r(\zeta)I_r, \qquad s\geq m, \tag{9}$$

and the function $H_s\colon Q_{u_s}\to\mathbb{C}$ is defined as $H_s(\xi):=V_s(\zeta)$ for all $\xi=\tau+i\eta=f_{u_s}(\zeta)$ with $\zeta\in\Sigma_{u_s}$, $\tau,\eta\in\mathbb{R}$ and u_s is the number given by the multiplication rule 3 for the basis $\{I_r\}_{r=1}^n$. Then the function H_s has the partial derivatives $\partial H_s/\partial\tau$, $\partial H_s/\partial\eta$ at the point $\xi_0=f_{u_s}(\zeta_0)$, and equality (8) holds for $u=s$.

Proof. (a) Let a vector h_l be collinear to the respective vector $\zeta_l-\zeta_0$ for $l\in\{1,2,3,4\}$, and let $\theta_l:=f_u(h_l)\in\{1,i,-1,-i\}$. Since the function Φ is differentiable in the sense of Gâteaux at the point ζ_0, we have

$$\lim_{\delta\to0^+}\frac{\Phi(\zeta_0+\delta h_l)-\Phi(\zeta_0)}{\delta}=h_l\Phi'_G(\zeta_0). \tag{10}$$

Applying the continuous multiplicative functional f_u to the both parts of equality (10), we get $\lim\limits_{\delta\to0^+}\frac{V_u(\zeta_0+\delta h_l)-V_u(\zeta_0)}{\delta}=\theta_l f_u\big(\Phi'_G(\zeta_0)\big)$ that can be rewritten as

$$\lim_{\delta\to0^+}\frac{H_u(\xi_0+\delta\theta_l)-H_u(\xi_0)}{\delta}=\theta_l f_u\big(\Phi'_G(\zeta_0)\big). \tag{11}$$

Equalities (11) for $l\in\{1,2,3,4\}$ yield the following equalities:

$$\frac{\partial H_u}{\partial\tau}=\lim_{\delta\to0^+}\frac{H_u(\xi_0+\delta)-H_u(\xi_0)}{\delta}=f_u\big(\Phi'_G(\zeta_0)\big)=\lim_{\delta\to0^+}\frac{H_u(\xi_0-\delta)-H_u(\xi_0)}{-\delta},$$

$$\frac{\partial H_u}{\partial\eta}=\lim_{\delta\to0^+}\frac{H_u(\xi_0+\delta i)-H_u(\xi_0)}{\delta}=if_u\big(\Phi'_G(\zeta_0)\big)=\lim_{\delta\to0^+}\frac{H_u(\xi_0-\delta i)-H_u(\xi_0)}{-\delta},$$

which also imply the equality (8).

(b) Let $\Sigma_{u_s}\subset\Omega$ be a two-dimensional surface containing four segments $s[\zeta_0,\zeta_1]$, $s[\zeta_0,\zeta_2]$, $s[\zeta_0,\zeta_3]$, $s[\zeta_0,\zeta_4]$ such that their images under the mapping f_{u_s} are subsets of the segments $s[\xi_0,\xi_0+1]$, $s[\xi_0,\xi_0+i]$, $s[\xi_0,\xi_0-1]$, $s[\xi_0,\xi_0-i]$, respectively, where $\xi_0=f_{u_s}(\zeta_0)$, and, in addition, let the restriction of the functional f_{u_s} to the surface Σ_u defines a one-to-one mapping of this surface onto the domain $Q_{u_s}:=\{\xi=f_{u_s}(\zeta)\colon\zeta\in\Sigma_{u_s}\}$. Let a vector h_l be collinear to the respective vector $\zeta_l-\zeta_0$ for $l\in\{1,2,3,4\}$, and let $\theta_l:=f_{u_s}(h_l)\in\{1,i,-1,-i\}$. Since the function Φ is differentiable in the sense of Gâteaux at a point $\zeta_0\in E_k$, we have equality (10) and, moreover, $\Phi'_G(\zeta_0)=\frac{\partial\Phi}{\partial x_1}(\zeta_0)=\sum_{r=s}^{n}\frac{\partial V_r}{\partial x_1}(\zeta_0)I_r$. Let us substitute the expression $h_l=\theta_l I_{u_s}+\rho_l$ into equality (10), where ρ_l is some vector belonging to the ideal $\mathcal{I}_{u_s}$. Taking into account the multiplication rules for the Cartan basis and the uniqueness of decomposition of elements of the algebra $\mathbb{A}_n^m$ with respect to the basis, we equate complex-valued functions for I_s at the both sides of equality (10) and get $\lim\limits_{\delta\to0^+}\frac{V_s(\zeta_0+\delta h_l)-V_s(\zeta_0)}{\delta}=\theta_l\frac{\partial V_r}{\partial x_1}(\zeta_0)$. It can be rewritten as $\lim\limits_{\delta\to0^+}\frac{H_s(\xi_0+\delta\theta_l)-H_s(\xi_0)}{\delta}=\theta_l\frac{\partial V_r}{\partial x_1}(\zeta_0)$.

Now, similarly to the part (a) of the proof, we can conclude that the partial derivatives $\partial H_s/\partial\eta$, $\partial H_s/\partial\tau$ exist at the point $\xi_0=f_{u_s}(\zeta_0)$ and equality (8) holds for $u=s$. □

Lemma 2. *Suppose that the condition* (3) *is satisfied and all intersections of a domain* $\Omega \subset E_k$ *with parallel* $(k-2)$*-planes to* M_u *are linearly connected. Suppose also that* $\Phi\colon \Omega \to \mathbb{A}_n^m$ *is a locally bounded and Gâteaux-differentiable function in* Ω*. If* $\zeta_1, \zeta_2 \in \Omega$ *and* $\zeta_2 - \zeta_1 \in M_u$*, then* $\Phi(\zeta_2) - \Phi(\zeta_1) \in \mathcal{I}_u$*. Moreover, if the function* Φ *is of the form* (9)*, then for* $\zeta_1, \zeta_2 \in \Omega$ *such that* $\zeta_2 - \zeta_1 \in M_{u_s}$*, the following inclusion holds:*

$$\Phi(\zeta_2) - \Phi(\zeta_1) \in \left\{ \sum_{r=s+1}^{n} \lambda_r I_r \colon \lambda_r \in \mathbb{C} \right\}. \tag{12}$$

Proof. Let $\zeta_1, \zeta_2 \in \Omega$ and $\zeta_2 - \zeta_1 \in M_u$. Consider the $(k-2)$-plane L containing ζ_1 and parallel to M_u. It is evident that $\zeta_2 \in L$.

Since $L \cap \Omega$ is linearly connected, we can choose a curve γ that lies in $L \cap \Omega$ and connects the points ζ_1 and ζ_2. In view of the compactness of the curve γ, there exists a finite set of open balls in Ω with centers lying on γ, which cover this curve.

Let us add the points ζ_1 and ζ_2 to the centers of the mentioned balls and number them as $\psi_1, \psi_2, \ldots, \psi_m$ in the order in which they are encountered when moving along the curve γ, starting from the point $\psi_1 = \zeta_1$ and ending at the point $\psi_m = \zeta_2$.

Consider the polygonal chain consisting of the line segments $s[\psi_j, \psi_{j+1}]$, $j \in \{1, 2, \ldots, m-1\}$, each of which is parallel to some vector belonging to M_u, because $\psi_j, \psi_{j+1} \in \gamma \subset L$ and L is parallel to M_u.

Clearly, there exists $\rho > 0$ such that for $j \in \{1, 2, \ldots, m-1\}$ the convex neighborhood

$$\Omega_j := \{\xi \in \Omega \colon \|\xi - \zeta\| < \rho, \zeta \in s[\psi_j, \psi_{j+1}]\}$$

of the line segment $s[\psi_j, \psi_{j+1}]$ is contained in the mentioned cover of the curve γ by a finite set of open balls in Ω.

Now, let us prove the relations $\Phi(\psi_{j+1}) - \Phi(\psi_j) \in \mathcal{I}_u$ equivalent to the equalities $f_u(\Phi(\psi_{j+1})) = f_u(\Phi(\psi_j))$ for $j \in \{1, 2, \ldots, m-1\}$.

Since the condition (3) is satisfied, there exists an element $e_2^* \in E_k$ such that $f_u(e_2^*) = i$. Consider the lineal span $E_3^* := \{\zeta = xe_1^* + ye_2^* + ze_3^* \colon x, y, z \in \mathbb{R}\}$ of the vectors $e_1^* = 1$, e_2^* and $e_3^* = \psi_{j+1} - \psi_j$. Introduce the convex neighborhood $\Omega_j^* := \Omega_j \cap E_3^*$ of the line segment $s[\psi_j, \psi_{j+1}]$ in the three-dimensional real vector space E_3^*.

Let us construct in Ω_j^* two surfaces Υ_j and Σ_j satisfying the following conditions (cf. [1, p. 143]):

- Υ_j and Σ_j have the same edge;
- the surface Υ_j contains the point ψ_j and the surface Σ_j contains the point ψ_{j+1};
- restrictions of the functional f_u onto the sets Υ_j and Σ_j are one-to-one mappings of the mentioned sets onto the same domain Q_j of the complex plane.
- any line segment $s[\xi_0, \xi] \subset Q_j$ contains a line subsegment $s[\xi_0, \xi_1]$ such that its preimages on the surfaces Υ_j and Σ_j over the mapping f_u are the line segments as well.

As the surface Υ_j, we can take an equilateral triangle having the center ψ_j and, in addition, the plane of this triangle is perpendicular to the line segment $s[\psi_j, \psi_{j+1}]$ in E_3^*. As the surface Σ_j, we can take the lateral surface of the pyramid with the base Υ_j and the apex ψ_{j+1}.

Consider the function $V_u \colon \Omega \to \mathbb{C}$ defined by the equality $V_u(\zeta) := f_u\big(\Phi(\zeta)\big)$ for all $\zeta \in \Omega$. Now, the relation $V_u(\psi_{j+1}) = V_u(\psi_j)$ can be proved in a similar way as Lemma 5.3 [12] or Lemma 8.5 [1].

For each $\xi \in Q_j$, we define two complex-valued functions H_1 and H_2 as follows

$$H_1(\xi) := V_u(\zeta) \text{ for} \zeta \in \Upsilon_j, \quad H_2(\xi) := V_u(\zeta) \text{ for} \zeta \in \Sigma_j,$$

where the correspondence between the points ζ and $\xi \in Q_j$ is determined by the relation $f_u(\zeta) = \xi$.

Each of the functions H_1, H_2 satisfies the classical Cauchy–Riemann condition of the form (8) at all points $\xi \in Q_j$ due to the part (a) of Lemma 1. Therefore, by virtue of Tolstoff's result [20], the functions H_1 and H_2 are holomorphic in the domain Q_j.

Since, in addition, H_1, H_2 are continuous in the closure of domain Q_j and $H_1(\xi) \equiv H_2(\xi)$ on the boundary of Q_j, this identity is also fulfilled everywhere in Q_j. Therefore, $V_u(\psi_{j+1}) = V_u(\psi_j)$ and $\Phi(\psi_{j+1}) - \Phi(\psi_j) \in \mathcal{I}_u$. Hence, taking into account the equality $\Phi(\zeta_2) - \Phi(\zeta_1) = \sum_{j=1}^{m-1} \big(\Phi(\psi_{j+1}) - \Phi(\psi_j)\big)$, we get $\Phi(\zeta_2) - \Phi(\zeta_1) \in \mathcal{I}_u$. If the function Φ has the form (9), then in the above reasoning we can consider the function V_s instead of V_u, the set M_{u_s} instead of M_u and use the part (b) of Lemma 1. In such a way, for $\zeta_1, \zeta_2 \in \Omega$ such that $\zeta_2 - \zeta_1 \in M_{u_s}$, we get the equality $V_s(\zeta_2) - V_s(\zeta_1) = 0$ that yields inclusion (12). □

Note that the condition of Lemma 2 on the linear connectivity of all intersections of the domain $\Omega \subset E_k$ with parallel $(k-2)$-planes to M_u is a weakening of the convexity condition with respect to the set of directions M_u in the case $k > 3$, which is accepted in the papers [11, 13]. At the same time, these conditions are equivalent in the case $k = 3$.

4.2. A constructive description of locally bounded and Gâteaux-differentiable functions. A constructive description of monogenic functions given in a domain $\Omega \subset E_3$ is obtained in the paper [12] (see also Theorem 8.2 in [1]) by means of holomorphic functions of complex variables. This result is generalized in the paper [13] to the case of a domain $\Omega \subset E_k$, where $2 \le k \le 2n$. As a consequence of such descriptions, it can be established that every function monogenic in Ω is differentiable in the sense of Lorch in the domain Ω. Now, we shall obtain the same constructive description of Gâteaux-differentiable functions in a domain $\Omega \subset E_k$ provided that these functions are locally bounded, i.e., in comparison with the mentioned results of the papers [12, 13], the condition of continuity of functions will be weakened.

We introduce the linear operators A_u , $u \in \{1, 2, \ldots, m\}$, which assign holomorphic functions $F_u \colon D_u \to \mathbb{C}$ to every monogenic function $\Phi \colon \Omega \to \mathbb{A}_n^m$ by the formula

$$F_u(\xi_u) := f_u(\Phi(\zeta)), \tag{13}$$

where $\xi_u = f_u(\zeta)$ and $\zeta \in \Omega$. It follows from Lemma 2 that the value $F_u(\xi_u)$ does not depend on a choice of a point ζ for which $f_u(\zeta) = \xi_u$. The following theorem will be proved below similarly as Theorem 1 [13] (see also Theorem 8.2 in [1]).

Theorem 2. *Suppose that condition* (3) *is satisfied and all intersections of a domain* $\Omega \subset E_k$ *with the parallel planes to* M_u *are linearly connected. Then every locally bounded and Gâteaux differentiable function* $\Phi \colon \Omega \to \mathbb{A}_n^m$ *can be expressed in the form*

$$\Phi(\zeta) = \sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt + \sum_{s=m+1}^{n} I_s \frac{1}{2\pi i} \int_{\Gamma_{u_s}} G_s(t)(te_1 - \zeta)^{-1} dt, \tag{14}$$

where F_u *and* G_s *are certain holomorphic functions in the domains* D_u *and* D_{u_s}*, respectively, and* Γ_u *is a closed rectifiable Jordan curve in* D_u*, which embraces the point* ξ_u *and contains no points* ξ_q *for* $q \in \{1, 2, \ldots, m\}$*,* $q \ne u$*.*

Proof. We define the holomorphic functions F_u, $u \in \{1, 2, \ldots, m\}$, by equality (13). Let us show that the values of the function

$$\Phi_0(\zeta) := \Phi(\zeta) - \sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int\limits_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt \tag{15}$$

belong to the radical $\mathcal{R}$, i.e., $\Phi_0(\zeta) \in \mathcal{R}$ for all $\zeta \in \Omega$. As a consequence of equality (6), we obtain the equality

$$I_u \frac{1}{2\pi i} \int\limits_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt = I_u \frac{1}{2\pi i} \int\limits_{\Gamma_u} \frac{F_u(t)}{t - \xi_u} dt + \frac{1}{2\pi i} \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \int\limits_{\Gamma_u} \frac{F_u(t) Q_{r,s}}{(t - \xi_{u_s})^r} dt I_s I_u,$$

which implies the equality

$$f_u \left(\sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int\limits_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt \right) = F_u(\xi_u). \tag{16}$$

Acting on equality (15) by the functional f_u and taking into account relations (13) and (16), we get the equality $f_u(\Phi_0(\zeta)) = F_u(\xi_u) - F_u(\xi_u) = 0$ for all $u \in \{1, 2, \ldots, m\}$, i.e., $\Phi_0(\zeta) \in \mathcal{R}$. Therefore, the function Φ_0 is of the form $\Phi_0(\zeta) = \sum_{s=m+1}^{n} V_s(\zeta) I_s$, $V_s \colon \Omega \to \mathbb{C}$.

It follows from Lemma 2 that $V_{m+1}(\zeta) \equiv G_{m+1}(\xi_{u_{m+1}})$ for all $\zeta \in \Omega$, where $\xi_{u_{m+1}} = f_{u_{m+1}}(\zeta)$ and $G_{m+1} \colon D_{u_{m+1}} \to \mathbb{C}$. In addition, the function G_{m+1} satisfies the classical Cauchy–Riemann condition of the form (8) at all points $\xi_{u_{m+1}} \in D_{u_{m+1}}$ due to the part (b) of Lemma 1. Hence, by virtue of Tolstoff's result [20], the function G_{m+1} is holomorphic in the domain $D_{u_{m+1}}$. Therefore,

$$\Phi_0(\zeta) = G_{m+1}(\xi_{u_{m+1}}) I_{m+1} + \sum_{s=m+2}^{n} V_s(\zeta) I_s. \tag{17}$$

In view of expansion (6), we have the representation

$$I_{m+1} \frac{1}{2\pi i} \int_{\Gamma_{u_{m+1}}} G_{m+1}(t)(te_1 - \zeta)^{-1} dt = G_{m+1}(\xi_{u_{m+1}}) I_{m+1} + \Psi(\zeta), \tag{18}$$

where $\Psi(\zeta)$ is a function taking values in the set $\{\sum_{s=m+2}^{n} \lambda_s I_s \colon \lambda_s \in \mathbb{C}\}$. Now, consider the function $\Phi_1(\zeta) := \Phi_0(\zeta) - I_{m+1} \frac{1}{2\pi i} \int_{\Gamma_{u_{m+1}}} G_{m+1}(t)(te_1 - \zeta)^{-1} dt$. In view of the relations (17), (18), the function Φ_1 can be represented as $\Phi_1(\zeta) = \sum_{s=m+2}^{n} \widetilde{V}_s(\zeta) I_s$, with $\widetilde{V}_s \colon \Omega \to \mathbb{C}$. Since the function Φ_1 is locally bounded and differentiable in the sense of Gâteaux in Ω, the function $\widetilde{V}_{m+2}$ is similar to the function V_{m+1}, and we can state that $\widetilde{V}_{m+2}(\zeta) \equiv G_{m+2}(\xi_{u_{m+2}})$ for all $\zeta \in \Omega$, where $\xi_{u_{m+2}} = f_{u_{m+2}}(\zeta)$ and $G_{m+2} \colon D_{u_{m+2}} \to \mathbb{C}$ is a function holomorphic in the domain $D_{u_{m+2}}$. In such a way, step by step, considering the functions $\Phi_j(\zeta) := \Phi_{j-1}(\zeta) - I_{m+j} \frac{1}{2\pi i} \int_{\Gamma_{u_{m+j}}} G_{m+j}(t)(te_1 - \zeta)^{-1} dt$ for $j \in \{2, 3, \ldots, n - m - 1\}$, we get representation (14) of the function Φ. □

As a result of Theorem 2, we get the statement of Theorem 1. Indeed, it follows from representation (14) that the function Φ is differentiable in the sense of Lorch in the domain Ω, which, in turn, implies that the function Φ is monogenic in Ω.

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