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ON NONLINEAR EQUATIONS RELATED TO THE DISTRIBUTIONS OF COMPOSITION OF PROCESSES

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The relationship between non-trivial linear partial differential equations and the probability densities of compositions of relevant processes has been pointed out in the recent literature. In this note, we construct exact solutions for nonlinear partial differential equations starting from these linear equations by using simple transformations. In this way, we have an interesting bridge between the fundamental solutions of linear equations with a clear probabilistic meaning and the construction of exact interesting solutions for the corresponding nonlinear equations. This method can be obviously generalized to many other cases.

1. Introduction. The obtaining of exact solutions for nonlinear partial differential equations is relevant both for the applications and for pure mathematical research, in order to better understand the evolution of nonlinear phenomena. There are many different methods to construct exact solutions for nonlinear equations, we refer for example to the Handbook by Polyanin and Zaitsev [4], since there is a wide literature based on different tecniques. It is well-known that in some simple, but relevant, cases, it is possible to introduce transformations in order to reduce a nonlinear equation to a linear one, obtaining quite simply the exact solutions. One of the most relevant cases is the Hopf-Cole transform that allows to reduce the nonlinear Burgers equation into the linear heat equation. On the other hand, it is possible to develope the inverse idea: starting from a linear equation with known solutions, by using a suitable transformation we can construct an exact solution for a corresponding nonlinear equation. This idea has been recently developed by Vitanov in [5], in which the author has considered the classical linear wave, Laplace and heat equations with known solutions and then the exact solutions for corresponding nonlinear equations have been obtained by means of simple transforms.

In this note we follow this idea, by considering particular linear equations whose known solutions correspond to the probability density of the compositions of relevant processes. These equations have been obtained in [2]. Here we construct exact solutions for the corresponding nonlinear equations by using simple transforms. In this way, we have an interesting bridge between solutions with a clear probabilistic meaning and exact solutions for non-trivial nonlinear partial differential equations.

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2. Exact solutions for nonlinear PDEs obtained from the linear equations governing composition of processes. In the recent preprint [5], Vitanov has shown the utility of applying different simple transformations in order to construct exact solutions for nonlinear equations starting from kwown solutions of the classical linear equations of mathematical physics. This is a simple and concrete method to construct explicit solutions for non-trivial nonlinear complicated equations.
In this note we use the same idea in order to construct exact solutions for complicated nonlinear equations starting from kwnown solutions with probabilistic meaning for linear partial differential equations governing the distributions of composition of processes. Even if this is technically a simple idea, in our view it gives and interesting bridge between different fields of research. Moreover, we can obtain new exact solutions for non-trivial nonlinear equations starting from equations emerging from a probabilistic motivation.

We recall that in the recent papers [1] and [2], interesting relations between the distributions of the compositions of Brownian, fractional Brownian and Cauchy processes with linear partial differential equations of mathematical physics have been pointed out. Here we apply simple transformations in order to construct exact solutions for nonlinear equations starting from the linear equations obtained in [2]. This is essentially a construction, starting from the knowledge of the form of the marginal law of the process. However, it can be useful to show a new link and to identify new classes of exactly solvable nonlinear PDEs. We provide some relevant examples but this scheme can be clearly applied in more cases.

The first case that we consider is given by the iterated fractional Brownian motion, namely $B^1_{H_1}\left(|B^2_{H_2}|\right)$, with Hurst index respetively H_1 and H_2. According to [2], Theorem 2.5, the distribution of this process solves the linear partial differential equation

$$(1+H_1H_2)t\frac{\partial p}{\partial t}+t^2\frac{\partial^2 p}{\partial t^2}=H_1^2H_2^2\left\{2x\frac{\partial p}{\partial x}+x^2\frac{\partial^2 p}{\partial x^2}\right\}, \tag{1}$$

We have the following result.

Proposition 1. *The nonlinear equation*

$$(1+H_1H_2)t\frac{\partial \psi}{\partial t}+t^2\left[\frac{\partial^2 \psi}{\partial t^2}+\left(\frac{\partial \psi}{\partial t}\right)^2\right]=H_1^2H_2^2\left\{2x\frac{\partial \psi}{\partial x}+x^2\left[\frac{\partial^2 \psi}{\partial x^2}+\left(\frac{\partial \psi}{\partial x}\right)^2\right]\right\} \tag{2}$$

admits a solution of the form $\psi(x,t)=\ln p(x,t)$, *where*

$$p(x,t)=2\int_0^\infty \frac{e^{-\frac{x^2}{2s^{2H_1}}}}{\sqrt{2\pi s^{2H_1}}}\frac{e^{-\frac{s^2}{2t^{2H_2}}}}{\sqrt{2\pi t^{2H_2}}}ds \tag{3}$$

is the density of the iterated fractional Brownian motion $B^1_{H_1}(|B^2_{H_2}(t)|)$.

Proof. Let us consider the governing equation (1), by using the transformation $p(x,t)=\exp\left(\psi(x,t)\right)$, we obtain the equation (2). By using the known probabilistic solution (3), we directly construct an exact solution for (2). □

This quite cumbersome equation can be in someway interpretated as a sort of nonlinear Euler-Poisson-Darboux equation.

Another interesting transformation considered by Vitanov can be useful in order to obtain exact solutions for another other class of nonlinear equations starting from the linear equation (1).

Proposition 2. *The nonlinear equation*

$$(1+H_1H_2)t\psi\frac{\partial\psi}{\partial t}+t^2\Big[\psi\frac{\partial^2\psi}{\partial t^2}+(\alpha-1)\left(\frac{\partial\psi}{\partial t}\right)^2\Big]=$$
$$=H_1^2H_2^2\Big\{2x\psi\frac{\partial\psi}{\partial x}+x^2\Big[\psi\frac{\partial^2\psi}{\partial x^2}+(\alpha-1)\left(\frac{\partial\psi}{\partial x}\right)^2\Big]\Big\}$$

admits a solution of the form $\psi(x,t)=(p(x,t))^{1/\alpha}$, $\alpha\neq 0$, *where* $p(x,t)$ *is given in* (3).

A second relevant case is given by the well-known Laplace equation

$$\frac{\partial^2 p}{\partial t^2}+\frac{\partial^2 p}{\partial x^2}=0, \tag{4}$$

whose solution (under suitably intial and boundary conditions) is given by the Cauchy distribution

$$p(x,t)=\frac{t}{\pi(x^2+t^2)}.$$

In this case we have the following result.

Proposition 3. *The nonlinear equation*

$$\frac{\partial^2\psi}{\partial t^2}+\left(\frac{\partial\psi}{\partial t}\right)^2+\frac{\partial^2\psi}{\partial x^2}+\left(\frac{\partial\psi}{\partial x}\right)^2=0$$

admits a solution of the form

$$\psi(x,t)=\ln\left(\frac{t}{\pi(x^2+t^2)}\right).$$

We underline that the Cauchy distribution plays a peculariar role in this framework as a bridge between linear, nonlinear and fractional PDEs. Moreover, a similar method can be applied also for higher order more complex equations related to higher order Cauchy random variables. For example (see [3] for details), the probability density

$$p(x,t)=\left(\frac{2\sin(\pi/4)}{\pi}\right)\frac{t^3}{t^4+x^4},$$

is a solution for the equation

$$\left(\frac{\partial^4}{\partial t^4}+\frac{\partial^4}{\partial x^4}\right)p=0.$$

The corresponding nonlinear equation can be directly obtained by the same method and it is quite cumbersome, involving many terms of higher order.

In analogy with Proposition 2, in relation to the Laplace equation (4), we can state the following result

Proposition 4. *The nonlinear equation*

$$\psi\frac{\partial^2\psi}{\partial t^2}+(\alpha-1)\left(\frac{\partial\psi}{\partial t}\right)^2+\psi\frac{\partial^2\psi}{\partial x^2}+(\alpha-1)\left(\frac{\partial\psi}{\partial x}\right)^2=0$$

admits a solution of the form

$$\psi(x,t)=\left(\frac{t}{\pi(x^2+t^2)}\right)^{1/\alpha},\quad \alpha\neq 0.$$

In [2], other interesting connections between partial differential equations and compositions of Cauchy processes have been pointed out. In particular, it was proved in Theorem 4.1

that the probability law $q(x,t)$ of the composition of the iterated Cauchy process $C^1(|C^2(t)|)$, $t>0$, has the following explicit form

$$q(x,t)=\frac{2t}{\pi^2(t^2-x^2)}\ln\frac{t}{|x|}$$

and satisfies the non-homogeneous wave equation

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)q(x,t)=-\frac{2}{\pi^2x^2t}.$$

Following the same line of the previous results, it is simple to prove the following result.

Proposition 5. *The nonlinear equation*

$$\frac{\partial^2\psi}{\partial t^2}+\left(\frac{\partial\psi}{\partial t}\right)^2-\frac{\partial^2\psi}{\partial x^2}-\left(\frac{\partial\psi}{\partial x}\right)^2=-\frac{2e^{-\psi}}{\pi^2x^2t}$$

admits a solution of the form

$$\psi(x,t)=\ln\left(\frac{t^2-x^2}{x^2t}\frac{1}{\ln(t/|x|)}\right).$$

Similar nonlinear equations with exponential nonlinearity can be obtained also by using transformations like $q(x,t)=\psi\partial_x\psi$ as suggested in [5]. We leave to the reader the analysis of further interesting examples. The main aim of this note is to give a useful way to construct exact solutions for nonlinear equations starting from known probabilistic solutions of particular linear PDE's studied in the framework of the iterated processes.

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