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SPECTRAL PROPERTIES OF THE SCHRÖDINGER OPERATOR WITH OPERATOR-VALUED POTENTIAL

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Let H be a separable Hilbert space, and let $\mathcal{H} := L_2(\mathbb{R}, H)$. In the space $\mathcal{H}$, we consider the self-adjoint Schrödinger operator of the form $T_q f = -f'' + qf$, where q is a reflectionless operator-valued potential. Let $\mathcal{P}_+$ and $\mathcal{P}_-$ be the spectral projectors of the operator T_q corresponding to the positive half-line $\mathbb{R}_+$ and the negative half-line $\mathbb{R}_-$, respectively. Define $\mathcal{H}_\pm := \mathcal{P}_\pm \mathcal{H}$, and let $T_q^\pm := T_q|_{\mathcal{H}_\pm}$.

In this paper, we show that the operator T_q has trivial kernel, and that the operator T_q^+ is unitarily equivalent to the unperturbed operator T_0. Next, let B be an arbitrary bounded negative operator in a separable Hilbert space H_1 ($\dim H_1 \le \dim H$ if $\dim H < \infty$). Then we prove that there exists a reflectionless potential q such that T_q^- is unitarily equivalent to the operator B.

A key role in this work is played by solutions of the operator Riccati equation of the form

$$S'(x) = KS(x) + S(x)K - 2S(x)KS(x), \qquad x \in \mathbb{R},$$

where $K \in \mathcal{B}_+(H) \setminus \{0\}$, and $S\colon \mathbb{R} \to \mathcal{B}(H)$. Here, $\mathcal{B}(H)$ is the Banach algebra of all bounded linear operators acting in H, and $\mathcal{B}_+(H) = \{A \in \mathcal{B}(H) \mid A \ge 0\}$.

1. Introduction. This paper is devoted to the study of the spectral properties of the Schrödinger operator with a reflectionless operator-valued potential. It continues the investigation started in [1] and [2].

1.1. Reflectionless potentials of the Schrödinger operator. Let H be a separable Hilbert space with the inner product $(\cdot \mid \cdot)$ that is linear in the first argument. Denote by $\mathcal{B}(H)$ the Banach algebra of all everywhere-defined linear continuous operators $A\colon H \to H$, and by $\mathcal{B}_{\mathrm{inv}}(H)$ the group of all invertible operators in $\mathcal{B}(H)$. Also, let $\mathcal{B}_+(H)$ be the cone of nonnegative operators, and let I be the identity operator in $\mathcal{B}(H)$. The domain, range, kernel, and the spectrum of a linear operator will be denoted by $\mathrm{dom}(\cdot)$, $\mathrm{ran}(\cdot)$, $\ker(\cdot)$, and $\sigma(\cdot)$, respectively. For arbitrary operators $A, B \in \mathcal{B}(H)$, we write $A < B$ if $A \le B$ and $\ker(B - A) = \{0\}$. If a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(H)$ converges to an operator A in the strong operator topology, we write $A = \operatorname*{s-lim}_{n\to\infty} A_n$.

Denote by $\mathcal{H} := L_2(\mathbb{R}, H)$ the Hilbert space of square integrable functions $f : \mathbb{R} \to H$ with the inner product

$$(f \mid g)_{\mathcal{H}} := \int_{\mathbb{R}} (f(x) \mid g(x))dx, \qquad f, g \in \mathcal{H},$$

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and let $\mathcal{I}$ be the identity operator in $\mathcal{H}$.

To simplify notation, we will use the following abbreviations for a function

$$z \mapsto F(z) \in \mathcal{B}(H): \ (F(z))^* = F^*(z), \ (F(z))^{-1} = F^{-1}(z).$$

Let $C(\mathbb{R}, \mathcal{B}(H))$ be the linear space of all continuous functions $f\colon \mathbb{R} \to \mathcal{B}(H)$, and set

$$C_b(\mathbb{R}, \mathcal{B}(H)) := \{f \in C(\mathbb{R}, \mathcal{B}(H)) \mid \|f\|_\infty < \infty\} \quad (\|f\|_\infty := \sup_{x\in\mathbb{R}} \|f(x)\|),$$
$$C_{b,s}(\mathbb{R}, \mathcal{B}(H)) := \{q \in C_b(\mathbb{R}, \mathcal{B}(H)) \mid \forall x \in \mathbb{R} \quad q^*(x) = q(x)\}.$$

We will associate every potential $q \in C_b(\mathbb{R}, \mathcal{B}(H))$ with the Schrödinger operator $T_q : \mathcal{H} \to \mathcal{H}$ defined by

$$T_q f = -f'' + qf \tag{1}$$

on the domain $\operatorname{dom} T_q := W_2^2(\mathbb{R}, H)$, where $W_2^2(\mathbb{R}, H)$ is the Sobolev space of functions $f\colon \mathbb{R} \to H$. If the potential $q \in C_{b,s}(\mathbb{R}, \mathcal{B}(H))$, then the operator T_q is self-adjoint.

Throughout the paper, we denote by $\mathcal{P}_+$ ($\mathcal{P}_-$) the spectral projector of the operator T_q corresponding to the positive half-line $\mathbb{R}_+$ (the negative half-line $\mathbb{R}_-$). We also set $\mathcal{H}_\pm := \mathcal{P}_\pm\mathcal{H}$, and define $T_q^\pm := T_q|_{\mathcal{H}_\pm}$.

Let $q \in C_{b,s}(\mathbb{R}, \mathcal{B}(H))$ and $z \in \mathbb{C}$. Consider the equation

$$-y'' + qy = zy. \tag{2}$$

As shown in [3], for every $z \in \mathbb{C} \setminus \mathbb{R}$, there exist the Weyl–Titchmarsh $\mathcal{B}(H)$-valued right $f_+(z,\cdot)$ and left $f_-(z,\cdot)$ normalized solutions of the equation (2), i.e., the solutions that satisfy the condition $f_+(z,0) = f_-(z,0) = I$ and for every $h \in H$

$$\int_{\mathbb{R}_\pm} \|f_\pm(z,x)h\|^2 dx < \infty.$$

The functions $m_\pm(z) := f'_\pm(z,0)$, $z \in \mathbb{C} \setminus \mathbb{R}$, are called the Weyl–Titchmarsh m-functions of the equation (2) on the half-lines $\mathbb{R}_\pm$. It is well-known (see [3]) that the equalities

$$m_\pm(\bar z) = m_\pm^*(z), \qquad z \in \mathbb{C} \setminus \mathbb{R}, \tag{3}$$

hold, and the functions m_+ and $-m_-$ are Herglotz functions in the upper half-plane, i.e.,

$$\pm \operatorname{Im} m_\pm(z) \geq 0, \qquad z \in \mathbb{C}_+. \tag{4}$$

Definition 1. Let $q \in C_{b,s}(\mathbb{R}, \mathcal{B}(H))$ and let $m_\pm$ be the Weyl–Titchmarsh functions of the equation (2). We call the potential q (the operator T_q) *reflectionless* if the $\mathcal{B}(H)$-valued function

$$n_q(\lambda) := \begin{cases} m_+(\lambda^2), & \operatorname{Im}\lambda > 0, \operatorname{Re}\lambda \neq 0; \\ m_-(\lambda^2), & \operatorname{Im}\lambda < 0, \operatorname{Re}\lambda \neq 0, \end{cases}$$

has an analytic continuation to the domain $\mathbb{C} \setminus i\mathbb{R}$. Denote by $\mathcal{Q}$ the set of all reflectionless potentials $q \in C_{b,s}(\mathbb{R}, \mathcal{B}(H))$.

In the scalar case, Definition 1 is equivalent to the definitions given in [4] and [5].

1.2. Basic results from [1] and [2]. Hereafter, we assume that K is an arbitrary operator in $\mathcal{B}_+(H) \setminus \{0\}$, and let P be the orthogonal projector that projects H onto the subspace $H_1 := \overline{\operatorname{ran} K}$, and $P^\perp := I - P$.

Our construction is based on the operator Riccati equation of the form

$$S'(x) = KS(x) + S(x)K - 2S(x)KS(x), \qquad x \in \mathbb{R}, \tag{5}$$

where $S\colon \mathbb{R} \to \mathcal{B}(H)$.

Denote by $\mathscr{S}(K)$ the set of all solutions S of the equation (5) such that $0 < S(0) < I$, $S'(0) \geq 0$. A function $S \in \mathscr{S}(K)$ is called *regular* if the operators $S(0)$ and $I - S(0)$ belong to $\mathcal{B}_{\mathrm{inv}}(H)$. The set of all regular functions $S \in \mathscr{S}(K)$ is denoted by $\mathscr{S}_{\mathrm{reg}}(K)$.

It turns out that every solution $S \in \mathscr{S}(K)$ generates a reflectionless potential $q_S \in \mathcal{Q}$. Moreover, an explicit formula can be given for the mapping $S \mapsto q_S$.

Proposition 1 ([1])**.** *Let $S \in \mathscr{S}(K)$. Then for all $x \in \mathbb{R}$*

$$S'(x) \geq 0, \qquad 0 < S(x) < I, \tag{6}$$

$$S(x)P = PS(x), \qquad S'(x)P^{\perp} = P^{\perp}S'(x) = 0, \qquad S(x)P^{\perp} = S(0)P^{\perp}. \tag{7}$$

Moreover, the function S has an analytic continuation in the strip

$$\Pi_K := \{z = x + iy \mid x, y \in \mathbb{R}, \ |y| < \pi/(2\|K\|)\}.$$

This continuation is given by the formula

$$S(z) = e^{zK}(S^{-1}(0) - I + e^{2zK})^{-1}e^{zK}, \ z \in \Pi_K,$$

and the following estimate holds

$$\|S(z)\| \leq [\cos(y\|K\|)]^{-1}, \ z \in \Pi_K, \ y = \operatorname{Im} z.$$

With every function $S \in \mathscr{S}(K)$, we associate the operators

$$\Gamma := \Gamma_S := S^{-1}(0) - I, \ R := R_S := |S'(0)|^{1/2}S^{-1}(0),$$

and construct the following analytic operator-valued functions in Π_K:

$$L(z) := L_S(z) := e^{zK}(I - S(z)) + e^{-zK}S(z),$$

$$\Psi(z) := \Psi_S(z) := |S'(0)|^{1/2}L(z), \quad q(z) := q_S(z) := -4\Psi_S(z)K\Psi_S^*(\bar{z}). \tag{8}$$

It follows from (7) that

$$P\Gamma = \Gamma P, \ R = PR = RP, \ \Psi(z) = P\Psi(z) = \Psi(z)P, \ q(z) = Pq(z) = q(z)P \ (z \in \Pi_K). \tag{9}$$

Theorem 1 ([1])**.** *Let $S \in \mathscr{S}(K)$. Then q_S is a reflectionless potential and*

$$\|q_S(z)\| \leq \frac{2\|K\|^2}{\cos^2(y\|K\|)}, \qquad z \in \Pi_K, \qquad y = \operatorname{Im} z.$$

We define the following subsets of the set $\mathcal{Q}$ of all reflectionless potentials:

$$\mathcal{Q}(K) := \{q_S \mid S \in \mathscr{S}(K)\}, \qquad \mathcal{Q}_{\mathrm{reg}}(K) := \{q_S \mid S \in \mathscr{S}_{\mathrm{reg}}(K)\},$$

$$\mathcal{Q}_{\pi} := \bigcup_K \mathcal{Q}(K), \qquad \mathcal{Q}_{\mathrm{reg}} := \bigcup_K \mathcal{Q}_{\mathrm{reg}}(K).$$

Remark 1. The mapping $\mathscr{S}(K) \ni S \mapsto q_S \in \mathcal{Q}_{\pi}$ is surjective. However, it is not injective. Indeed, if $S, \widetilde{S} \in \mathscr{S}(K)$ and $S(0)P = \widetilde{S}(0)P$, then $\Psi_S = \Psi_{\widetilde{S}}$ and $q_S = q_{\widetilde{S}}$.

The case $q \in \mathcal{Q}_{\mathrm{reg}}$ is technically more convenient to study. For this reason, most of the results will first be established for $q \in \mathcal{Q}_{\mathrm{reg}}$. To extend them to the general case $q \in \mathcal{Q}_{\pi}$, we pass to the limit and apply the following proposition.

Proposition 2 ([1])**.** *Let $S \in \mathscr{S}(K)$ and*

$$S_{\varepsilon}(x) = e^{xK}(B_{\varepsilon}^{-1} - I + e^{2xK})^{-1}e^{xK}, \qquad x \in \mathbb{R}, \quad \varepsilon \in (0, 1/2),$$

where $B_{\varepsilon} := \varepsilon I + (1 - 2\varepsilon)S(0)$. Then $S_{\varepsilon} \in \mathscr{S}_{\mathrm{reg}}(K)$ for all $\varepsilon \in (0, 1/2)$. Moreover,

$$\|S(z) - S_{\varepsilon}(z)\| = o(1), \quad \|\Psi(z) - \Psi_{S_{\varepsilon}}(z)\| = o(1), \quad \|q(z) - q_{S_{\varepsilon}}(z)\| = o(1) \tag{10}$$

as $\varepsilon \to 0$ uniformly on compact sets in Π_K.

The following proposition describes the properties of the function Ψ, which, along with S, plays a central role in this paper.

Proposition 3 ([1]). *Let $S \in \mathscr{S}(K)$. Then*

$$-\Psi''(x) + q(x)\Psi(x) = -\Psi(x)K^2, \quad x \in \mathbb{R}, \tag{11}$$

$$S'(x) = \Psi^*(x)\Psi(x), \quad x \in \mathbb{R}, \tag{12}$$

$$\|\Psi(z)\| \le \frac{\pi\|K\|^{1/2}}{2\cos(y\|K\|)}, \quad z \in \Pi_K, \quad y = \operatorname{Im} z.$$

If, in addition, $S \in \mathscr{S}_{\mathrm{reg}}(K)$, then $R, \Gamma \in \mathcal{B}(H)$ and

$$K\Gamma + \Gamma K = R^*R, \qquad \Psi(z) = Re^{-zK}S(z) \qquad (z \in \Pi_K). \tag{13}$$

1.3. Classes of operator-valued measures. Let $B(\mathbb{R})$ be the σ-algebra of all Borel subsets of the real line $\mathbb{R}$, and let $B_b(\mathbb{R})$ denote the ring of all bounded subsets in $B(\mathbb{R})$.

Definition 2. A mapping $\mu\colon B_b(\mathbb{R}) \to \mathcal{B}(H)$ is called an *operator-valued measure* on $\mathbb{R}$ if it satisfies the following conditions

1) $\mu(\varnothing) = 0$ and $\mu(A) \ge 0$ for all $A \in B_b(\mathbb{R})$;
2) the function μ is strongly countably additive, i.e., if $A = \bigsqcup_{j\in\mathbb{N}} A_j$ is a disjoint decomposition of a set $A \in B_b(\mathbb{R})$ into subsets $A_j \in B_b(\mathbb{R})$, then

$$\mu\Big(\bigsqcup_{j\in\mathbb{N}} A_j\Big) = \operatorname*{s-lim}_{n\to\infty} \sum_{j=1}^{n} \mu(A_j).$$

We denote by $\mathcal{M}_l$ ($\mathcal{M}_b$) the set of all operator-valued measures $\mu\colon B_b(\mathbb{R}) \to \mathcal{B}(H)$ for which the integral

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+|t|} \qquad \Big(\int_{\mathbb{R}} d\mu(t)\Big)$$

converges in the strong operator topology. Denote by $\mathcal{M}$ the set of all measures $\mu \in \mathcal{M}_b$ with compact support.

We will prove the following theorem.

Theorem 2. *Let $q \in \mathcal{Q}$ and let $r := (\|q\|_\infty)^{1/2}$. Then there exists a unique measure $\nu_q \in \mathcal{M}$ such that*

$$n_q(\lambda) = i\lambda I + \int \frac{d\nu_q(t)}{t - i\lambda}, \qquad \lambda \in \mathbb{C} \setminus i\mathbb{R}.$$

Moreover,

$$\operatorname{supp}\nu_q \subset [-r, r], \qquad \int d\nu_q(t) = -\frac{1}{2}q(0) \text{ and } m_\pm(\lambda^2) = n_q(\pm\lambda), \quad \lambda \in \mathbb{C}_+ \setminus i\mathbb{R}. \tag{14}$$

The mapping

$$\mathcal{Q} \ni q \mapsto \nu_q \in \mathcal{M} \tag{15}$$

is an analogue of the mapping constructed by V.A. Marchenko in the scalar case (see [4]). This mapping, which we call the Marchenko parametrization, plays an important role in the spectral theory of reflectionless potentials. The authors plan to devote a separate publication to its detailed study. In particular, we expect that the mapping (15) will be used to prove that $\mathcal{Q}_\pi = \mathcal{Q}$.

1.4. The formulation of the main result. The main result of this paper is

Theorem 3. *Let $S \in \mathscr{S}(K)$ and $q = q_S$. Then*

(I) $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$;

(II) the operator T_q^- is unitarily equivalent to the operator $-K_1^2$, where $K_1 := K|_{H_1}$;

(III) the operator T_q^+ is unitarily equivalent to the unperturbed operator T_0.

Interestingly, the results of this work imply the following fact: the operator T_q with potential $q \in \mathcal{Q}_\pi$ is uniquely determined by its negative part, i.e., the operator T_q^-. In particular, the following proposition holds.

Proposition 4. *Let $q \in C_{b,s}(\mathbb{R}, \mathcal{B}(H))$. Then*

(I) the operator $(-T_q^-)^{1/2}$ is a continuous positive integral operator on $\mathcal{H}_-$, acting according to the formula

$$[(-T_q^-)^{1/2}\varphi](x) = \underset{a\to+\infty}{\text{s-lim}} \int_{-a}^{a} \mathscr{K}_q(x,t)\varphi(t)dt, \qquad x \in \mathbb{R}, \quad \varphi \in \mathcal{H}_-,$$

where $\mathscr{K}_q \colon \mathbb{R}^2 \to \mathcal{B}(H)$ is a continuous bounded function uniquely determined by the operator T_q^- (by the potential q);

(II) if $q \in \mathcal{Q}_\pi$, then $q(x) = -4\mathscr{K}_q(x,x)$, $\quad x \in \mathbb{R}$.

Proposition 4 leads to the following question.

Question 1. *Does the following equality hold*

$$\mathcal{Q} = \{q \in C_{b,s}(\mathbb{R}, \mathcal{B}(H)) \mid \forall x \in \mathbb{R} \quad q(x) = -4\mathscr{K}_q(x,x)\}?$$

The structure of the paper is as follows. In Section 2, we prove Theorem 2. In Section 3, we show that $\ker T_q = \{0\}$ for any $q \in \mathcal{Q}$. In Section 4, we study an isometric operator V that is closely related to the operator T_q^-. In Section 5, we construct an analogue of the classical transformation operator for T_q with $q \in \mathcal{Q}_{\text{reg}}(K)$. In Section 6, we construct an isometric operator $\mathfrak{A}$ that realizes a unitary equivalence between the operators T_q^+ and T_0. Finally, in Section 7, we complete the proof of Theorem 3 and prove Proposition 4. In the Appendix, we prove a result concerning a certain special operator equation.

2. The proof of Theorem 2. Let $q \in C_{b,s}(\mathbb{R}, \mathcal{B}(H))$, and let $\alpha = \alpha^* \in \mathcal{B}(H)$. We denote by $\mathcal{T}_{q,\alpha}$ the self-adjoint Schrödinger operator acting in the space $L_2(\mathbb{R}_+, H)$ by the formula

$$\mathcal{T}_{q,\alpha} f = -f'' + qf$$

on the domain

$$\operatorname{dom} \mathcal{T}_{q,\alpha} := \left\{g \in W_2^2(\mathbb{R}_+, H) \middle| (\cos\alpha)g(0) + (\sin\alpha)g'(0) = 0\right\}.$$

According to the results of [3], the Weyl–Titchmarsh function m_α of the operator $\mathcal{T}_{q,\alpha}$ admits the representation

$$m_\alpha(\lambda) = C_\alpha + \int_{\mathbb{R}} \left[\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right] d\rho_\alpha(t), \qquad \lambda \in \mathbb{C}_+, \tag{16}$$

where $C_\alpha = C_\alpha^* \in \mathcal{B}(H)$ and $\rho_\alpha \in \mathcal{M}_l$. Moreover, the following theorem holds.

Theorem 4 ([3])**.** *The operator $\mathcal{T}_{q,\alpha}$ is unitarily equivalent to the multiplication operator Υ by the independent variable in the space $L_2(\mathbb{R}, d\rho_\alpha, H)$.*

In particular, Theorem 4 implies the identity $\operatorname{supp} \rho_\alpha = \sigma(\mathcal{T}_{q,\alpha})$.

Lemma 1. *Let $q \in C_{b,s}(\mathbb{R}, \mathcal{B}(H))$ and let $r := (\|q\|_\infty)^{1/2}$. Then the functions $m_\pm$ admit analytic continuation to the domain $\mathbb{C} \setminus [-r^2, \infty)$.*

Proof. Note that m_+ is the Weyl–Titchmarsh m-function of $\mathcal{T}_{q,0}$, i.e., $m_+ = m_0$. Hence,

$$m_+(z) = C + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\rho(t), \qquad z \in \mathbb{C} \setminus \sigma(\mathcal{T}_{q,0}), \tag{17}$$

where $\rho \in \mathcal{M}_l$, $C \in \mathcal{B}(H)$ with $C = C^*$, and $\operatorname{supp} \rho = \sigma(\mathcal{T}_{q,0})$. It is clear that $\sigma(\mathcal{T}_{q,0}) \subset [-r^2, +\infty)$. Therefore, $\operatorname{supp} \rho \subset [-r^2, +\infty)$. It follows from (17) that m_+ admits an analytic continuation to the domain $\mathbb{C} \setminus [-r^2, \infty)$.

It is easy to verify that the function $-m_-$ is the Weyl–Titchmarsh m-function for the operator $\mathcal{T}_{\widetilde{q},0}$ with the potential $\widetilde{q}(x) := q(-x),\ x \in \mathbb{R}$. Thus, from the above proof, it follows that m_- also admits an analytic continuation to the domain $\mathbb{C} \setminus [-r^2, \infty)$. □

Lemma 2. *Let $q \in \mathcal{Q}$ and let $r := (\|q\|_\infty)^{1/2}$. Then the function n_q has a unique representation of the form*

$$n_q(\lambda) = C + i\lambda D + \int_{\mathbb{R}} \frac{d\rho(t)}{t - i\lambda}, \qquad \lambda \in \mathbb{C} \setminus i\mathbb{R}, \quad \operatorname{Re} \lambda \neq 0, \tag{18}$$

where $\rho \in \mathcal{M}$, $C, D \in \mathcal{B}(H)$ with $C^* = C$, $D \geq 0$, and $\operatorname{supp} \rho \subset [-r, r]$.

Proof. It follows from Definition 1 and the equalities (14) that for all $\lambda \in \mathbb{C}_+ \setminus i\mathbb{R}$,

$$m_\pm(\lambda^2) = n_q(\pm\lambda), \qquad n_q^*(\bar{\lambda}) = m_\mp^*(\bar{\lambda}^2) = n_q(-\lambda). \tag{19}$$

Moreover, from (4) and (3), we have

$$\begin{aligned} \operatorname{Im} n_q(\lambda) &= \operatorname{Im} m_+(\lambda^2) \geq 0, \qquad \text{if } 0 < \arg \lambda < \pi/2, \\ \operatorname{Im} n_q(\lambda) &= \operatorname{Im} m_-(\lambda^2) \geq 0, \qquad \text{if } 3\pi/2 < \arg \lambda < 2\pi. \end{aligned}$$

Hence,

$$\operatorname{Im} n_q(\lambda) \geq 0, \quad \text{if } \operatorname{Re} \lambda > 0, \quad \operatorname{Im} \lambda \neq 0. \tag{20}$$

In view of Definition 1 and Lemma 1, we have that the function n_q admits an analytic continuation to the domain $\mathbb{C} \setminus [-ir, ir]$. Therefore, the function $L(\lambda) := n_q(-i\lambda)$ is analytic in the domain $\mathbb{C} \setminus [-r, r]$. From (19) and (20), we obtain

$$L(\lambda) = L^*(\bar{\lambda}), \qquad \operatorname{Im} L(\lambda) \geq 0, \qquad \lambda \in \overline{\mathbb{C}}_+ \setminus [-r, r]. \tag{21}$$

Thus, L is a Herglotz function. Therefore, it has a unique representation (see [3]) of the form

$$L(\lambda) = B + \lambda D + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\rho(t), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{22}$$

where $B, D \in \mathcal{B}(H)$, $\rho \in \mathcal{M}_l$, with $B = B^*$ and $D \geq 0$. From (21), we have $L(\xi) = L^*(\xi),\ \xi \in \mathbb{R} \setminus [-r, r]$. According to the Stieltjes inversion formula (see [3]), for any interval $[a, b]$ disjoint from $[-r, r]$, we find

$$\rho((a, b]) = \pi^{-1} \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \int_{a+\delta}^{b+\delta} \operatorname{Im} L(\xi + i\varepsilon) d\xi = \pi^{-1} \int_a^b \operatorname{Im} L(\xi) d\xi = 0.$$

Therefore, $\rho \in \mathcal{M}$ and $\operatorname{supp}\rho \subset [-r, r]$. Consequently, the representation (22) can be rewritten as

$$L(\lambda) = C + \lambda D + \int_{\mathbb{R}} \frac{d\rho(t)}{t-\lambda}, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

where $C = B - \int_{\mathbb{R}} \frac{t}{1+t^2} d\rho(t)$. □

Proof of Theorem 2. In view of Lemma 2, it remains to prove that $C = 0$ and $D = I$ in the formula (18), and $\int d\rho(t) = -\frac{1}{2}q(0)$. Fix a function $\varphi \in C(\mathbb{R})$ with support in $(-1, 0)$ such that $\varphi \geq 0$, $\int_{\mathbb{R}} \varphi(t)dt = 1$. Denote by A_ε the operator from H to $\mathcal{H}$ acting according to the formula

$$[A_\varepsilon h](x) := \varepsilon^{-1}\varphi(x/\varepsilon)h, \ x \in \mathbb{R}, \ h \in H.$$

As shown in [3], the resolvent of the operator T_q has the form

$$[(T_q - z\mathcal{I})^{-1}g](x) =$$
$$= \int_{-\infty}^{x} f_+(z,x)W^{-1}(z)[f_-(\bar{z},t)]^* g(t)dt + \int_{x}^{\infty} f_-(z,x)W^{-1}(z)[f_+(\bar{z},t)]^* g(t)dt, \tag{23}$$

where $z \in \mathbb{C}\setminus\mathbb{R}$, $x \in \mathbb{R}$, $g \in \mathcal{H}$, and the operator

$$W(z) := m_-(z) - m_+(z), \quad z \in \mathbb{C}\setminus\mathbb{R}, \tag{24}$$

is invertible in $\mathcal{B}(H)$. Set

$$T_{q,\lambda} := (T_q - \lambda^2\mathcal{I})^{-1}, \ \lambda \in \mathbb{C}_+ \setminus i\mathbb{R}.$$

Using (23), we obtain that for all $\lambda \in \mathbb{C}_+ \setminus i\mathbb{R}$, $h \in H$ and $x \geq 0$ the following limit exists

$$\lim_{\varepsilon\to+0}[T_{q,\lambda}A_\varepsilon h](x) = f_+(\lambda^2, x)W^{-1}(\lambda^2)h. \tag{25}$$

From (23), we also have

$$(T_{0,\lambda}f)(x) = \frac{i}{2\lambda}\int_{\mathbb{R}} e^{i\lambda|x-t|}f(t)dt, \qquad x \in \mathbb{R}, \quad f \in \mathcal{H}, \quad \lambda \in \mathbb{C}_+. \tag{26}$$

Denote by $\mathfrak{Q}$ the multiplication operator by the function q, i.e., $\mathfrak{Q}f := qf$, $f \in \mathcal{H}$. Put

$$\xi(q) := 1 + 2\|\mathfrak{Q}\| \text{ and } \Omega(q) := \{z \in \mathbb{C}_+ \mid \operatorname{Re} z < \operatorname{Im} z, \ \operatorname{Im} z > \xi(q)\}.$$

It is easy to check the identity

$$T_{q,\lambda} = T_{0,\lambda} - T_{0,\lambda}X_q(\lambda)T_{0,\lambda}, \qquad \lambda \in \Omega(q), \tag{27}$$

where $X_q(\lambda) := \mathfrak{Q}(I + T_{0,\lambda}\mathfrak{Q})^{-1}$. Since $\|T_{0,\lambda}\| \leq |\operatorname{Im}\lambda|^{-1}$, we have

$$\|X_q(\lambda)\| \leq \xi(q) \qquad \text{if} \quad \lambda \in \Omega(q). \tag{28}$$

Define the operators $B_j(\lambda)\colon \mathcal{H} \to H$ for $\lambda \in \Omega(q)$ $(j \in \{0,1\})$ by

$$B_0(\lambda)g := (T_{0,\lambda}g)(0), \ B_1(\lambda)g := (T_{0,\lambda}g)'(0), \ g \in \mathcal{H}.$$

From (26), we obtain

$$B_0(\lambda)g = \frac{i}{2\lambda}\int_{\mathbb{R}} e^{i\lambda|t|}g(t)dt, \quad B_1(\lambda)g = \frac{1}{2}\int_{\mathbb{R}} (\operatorname{sign} t)e^{i\lambda|t|}g(t)dt, \quad g \in \mathcal{H}.$$

Using the Cauchy-Schwarz inequality, we get

$$\|B_0(\lambda)\| = O(\lambda^{-3/2}), \quad \|B_1(\lambda)\| = O(\lambda^{-1/2}), \quad \Omega(q) \ni \lambda \to \infty. \tag{29}$$

It follows from (26) that for all $\lambda \in \Omega(q)$ and $h \in H$

$$\lim_{\varepsilon\to+0}[T_{0,\lambda}A_\varepsilon h](x) = \frac{i}{2\lambda}\lim_{\varepsilon\to+0}\int_{\mathbb{R}} \varepsilon^{-1}\varphi(t/\varepsilon)e^{i\lambda|x-t|}h dt = \frac{i}{2\lambda}e^{i\lambda|x|}h = [B_0(-\bar{\lambda})^* h](x).$$

Hence, using (27), we obtain

$$\lim_{\varepsilon\to+0}[T_{q,\lambda}A_\varepsilon h](x) = \frac{i}{2\lambda}e^{i\lambda|x|}h - [T_{0,\lambda}X_q(\lambda)[B_0(-\bar\lambda)]^*h](x), \qquad \lambda\in\Omega(q).$$

Taking into account (25), we conclude that for all $\lambda\in\Omega(q)$, $x\ge 0$, $h\in H$

$$f_+(\lambda^2,x)W^{-1}(\lambda^2)h = \frac{i}{2\lambda}e^{i\lambda x}h - [T_{0,\lambda}X_q(\lambda)[B_0(-\bar\lambda)]^*h](x).$$

Since $f_+(\lambda^2,0)=I$ and $f_+'(\lambda^2,0)=m_+(\lambda^2)$, we get

$$W^{-1}(\lambda^2) = \frac{i}{2\lambda}I - B_0(\lambda)X_q(\lambda)[B_0(-\bar\lambda)]^*, \tag{30}$$

$$m_+(\lambda^2)W^{-1}(\lambda^2) = -\frac{1}{2}I - B_1(\lambda)X_q(\lambda)[B_0(-\bar\lambda)]^*.$$

The estimates (28) and (29) imply that

$$\|2i\lambda W^{-1}(\lambda^2)+I\| = O(\lambda^{-2}), \quad \Omega(q)\ni\lambda\to\infty,$$

$$\|2i\lambda m_+(\lambda^2)W^{-1}(\lambda^2)+i\lambda I\| = O(\lambda^{-1}), \quad \Omega(q)\ni\lambda\to\infty.$$

From these estimates, it follows that

$$\|W(\lambda^2)+2i\lambda I\| = O(\lambda^{-1}), \quad \Omega(q)\ni\lambda\to\infty, \tag{31}$$

$$\|m_+(\lambda^2)-i\lambda I\| = O(\lambda^{-1}), \quad \Omega(q)\ni\lambda\to\infty. \tag{32}$$

Applying (18) and (19), we find

$$W(\lambda^2) = m_-(\lambda^2)-m_+(\lambda^2) = -n_q(\lambda)-n_q(-\lambda) = -2i\lambda\left(D+\int_{\mathbb{R}}\frac{d\rho(t)}{t^2+\lambda^2}\right). \tag{33}$$

In view of (31), we conclude that $D=I$. Then, it follows from (18) and (32) that $C=0$.

It remains to prove the equality $\int d\rho(t) = -\frac{1}{2}q(0)$. Taking into account (33) and the fact that $D=I$, we obtain

$$W^{-1}((i\xi)^2) = \frac{1}{2\xi}\left(I+\int_{\mathbb{R}}\frac{d\rho(t)}{t^2-\xi^2}\right)^{-1} = \frac{1}{2\xi}\left(I+\xi^{-2}\int_{\mathbb{R}}d\rho(t)+Y(\xi)\right),$$

where $\|Y(\xi)\| = O(\xi^{-4})$, $\xi\to+\infty$. Hence, in view of (30), we have

$$\int_{\mathbb{R}}d\rho(t) = -\lim_{\xi\to+\infty}2\xi B_0(i\xi)X_q(i\xi)[B_0(i\xi)]^*.$$

Since $\|X_q(i\xi)-\mathfrak{Q}\| = O(\xi^{-1})$, $\xi\to+\infty$, we get

$$\int_{\mathbb{R}}d\rho(t) = -\lim_{\xi\to+\infty}\frac{\xi}{2}\int_{\mathbb{R}}e^{-2\xi|t|}q(t)dt = -\frac{1}{2}\lim_{\xi\to+\infty}\int_{\mathbb{R}}e^{-2|y|}q(y/\xi)dy = -\frac{1}{2}q(0).$$

□

Theorem 2 and the equalities (33) imply the following corollary.

Corollary 1. *The equalities*

$$W(\lambda^2) = m_-(\lambda^2)-m_+(\lambda^2) = -2i\lambda\left(I+\int_{\mathbb{R}}\frac{d\nu_q(t)}{t^2+\lambda^2}\right), \qquad \lambda\in\mathbb{C}_+\setminus i\mathbb{R}, \tag{34}$$

hold.

3. The absence of the zero eigenvalue. The main result of this section is the following theorem.

Theorem 5. *Let* $q \in \mathcal{Q}$. *Then* $\ker T_q = \{0\}$.

Before starting the proof, we make some preparatory steps.

It is well-known ([3]) that for any pair of self-adjoint operators $\alpha, \beta \in \mathcal{B}(H)$ the functions m_α and m_β are connected by the relation

$$m_\alpha(\lambda) = (A_{21} + A_{22} m_\beta(\lambda))(A_{11} + A_{12} m_\beta(\lambda))^{-1}, \ \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where the operators $A_{ij} \in \mathcal{B}(H)$ are determined by the equality

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

In particular, if $\beta = 0$, then

$$m_\alpha(\lambda) = (-\sin\alpha + (\cos\alpha) m_0(\lambda))(\cos\alpha + (\sin\alpha) m_0(\lambda))^{-1}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{35}$$

Lemma 3. *Let* $\alpha = \alpha^* \in \mathcal{B}(H)$. *Then* $\ker \mathcal{T}_{q,\alpha} = \{0\}$.

Proof. Note that m_0 coincides with the function m_+. Thus, in view of Theorem 2,

$$m_0(\lambda^2) = n_q(\lambda) = i\lambda I + \int \frac{d\nu_q(t)}{t - i\lambda}, \qquad \lambda \in \mathbb{C}_+ \setminus i\mathbb{R}. \tag{36}$$

Obviously, the function

$$r(\lambda) := n_q(-i\lambda) = \lambda I + \int \frac{d\nu_q(t)}{t - \lambda}, \qquad \lambda \in \mathbb{C}_+,$$

is a Herglotz function. Let us consider the function

$$r_\alpha(\lambda) := (-\sin\alpha + (\cos\alpha) r(\lambda))(\cos\alpha + (\sin\alpha) r(\lambda))^{-1}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{37}$$

Note that for the operator $A \in \mathcal{B}(H \oplus H)$ defined by

$$A := \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},$$

the equality $A^* J A = J$ holds with $J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Therefore, according to results of Krein and Shmuljan ([7]), r_α is also a Herglotz function. In particular, we have

$$\|r_\alpha(z)\| = O(1/\operatorname{Im} z), \qquad z \in \mathbb{C}_+, \quad z \to 0. \tag{38}$$

On the other hand, taking into account the equalities (35), (36) and (37), we obtain

$$m_\alpha(\lambda^2) = r_\alpha(i\lambda), \qquad \lambda \in \mathbb{C}_+ \setminus i\mathbb{R}. \tag{39}$$

It follows from (38) and (39) that

$$\|m_\alpha(2i\xi^2)\| = \|r_\alpha(\xi + i\xi)\| = O(\xi^{-1}), \ \xi \in \mathbb{R}_+, \ \xi \to 0,$$

thus, $\|m_\alpha(i\xi)\| = O(\xi^{-1/2})$, $\xi \in \mathbb{R}_+$, $\xi \to 0$. From this, in view of (17), we have that $\rho_\alpha(\{0\}) = 0$. This means that the point $\lambda = 0$ cannot be an eigenvalue of the multiplication operator Υ by the independent variable in the space $L_2(\mathbb{R}, d\rho_\alpha, H)$. According to Theorem 4, the operators Υ and $\mathcal{T}_{q,\alpha}$ are unitarily equivalent, and hence $\lambda = 0$ cannot be an eigenvalue of the operator $\mathcal{T}_{q,\alpha}$. □

Proof of Theorem 5. Assume that $\varphi \in \ker T_q \setminus \{0\}$. It suffices to show that there exists a self-adjoint operator $\alpha \in \mathcal{B}(H)$ such that $\ker \mathcal{T}_{q,\alpha} \neq \{0\}$. Indeed, in this case, we get a contradiction with Lemma 3, and hence $\ker T_q = \{0\}$.

If $\varphi(0) = 0$, then obviously the restriction of φ to $[0, \infty)$ is a nonzero function belonging to $\ker \mathcal{T}_{q,0}$.

Let $\varphi(0) \neq 0$. From the equality $-\varphi'' + q\varphi = 0$ we have

$$\int_0^\infty (q(x)\varphi(x) \mid \varphi(x))dx = \int_0^\infty (\varphi''(x) \mid \varphi(x))dx =$$

$$= \int_0^\infty [(\varphi'(x) \mid \varphi(x))]'dx - \int_0^\infty \|\varphi'(x)\|^2 dx = -(\varphi'(0) \mid \varphi(0)) - \int_0^\infty \|\varphi'(x)\|^2 dx.$$

From this, it follows that the number $(\varphi'(0) \mid \varphi(0))$ is real. Denote by P_0 the orthoprojector onto the subspace $\{c\varphi(0) \mid c \in \mathbb{C}\}$ and set $\alpha := aP_0$, where

$$a := \operatorname{arcctg}\left(-\frac{(\varphi'(0) \mid \varphi(0))}{\|\varphi(0)\|^2}\right).$$

Since

$$\cos\alpha = I - P_0 + (\cos a)P_0, \quad \sin\alpha = (\sin a)P_0, \quad P_0 = \frac{(\cdot \mid \varphi(0))}{\|\varphi(0)\|^2}\varphi(0),$$

we have

$$(\cos\alpha)\varphi(0) + (\sin\alpha)\varphi'(0) = (\cos a)\varphi(0) + (\sin a)P_0\varphi'(0) = 0.$$

Hence, the restriction of φ to $[0, \infty)$ is a nonzero function belonging to $\ker \mathcal{T}_{q,\alpha}$. □

From Theorem 5, we get the following corollary.

Corollary 2. *The equalities $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{P}_+ + \mathcal{P}_- = \mathcal{I}$ hold.*

4. The operator V. The main result of this section is

Theorem 6. *Let $S \in \mathscr{S}(K)$ and let $q := q_S$, $\Psi := \Psi_S$. Then the formula*

$$(Vh)(x) := \Psi(x)h, \qquad x \in \mathbb{R}, \quad h \in H, \tag{40}$$

defines a partial isometry. Moreover, the operator V maps H into the Sobolev space $W_2^2(\mathbb{R}, H)$ and satisfies

$$T_q V = -VK^2, \qquad V^*V = P. \tag{41}$$

Here, P is the orthogonal projector from H onto the subspace H_1.

First, we prove the following lemma.

Lemma 4. *Let $S \in \mathscr{S}(K)$. Then the limits $S(\pm\infty) := \operatorname*{s-lim}_{x\to\pm\infty} S(x)$ exist, moreover, $S(-\infty)P = 0$ and $S(+\infty)P = P$.*

Proof. It follows from (6) that the limits $S(+\infty)$ and $S(-\infty)$ exist and

$$0 \le S(-\infty) \le S(0) \le S(+\infty) \le I. \tag{42}$$

Taking into account that (see (5))

$$S'(x) = KS(x) + S(x)K - 2S(x)KS(x), \qquad x \in \mathbb{R}, \tag{43}$$

we obtain the existence of the limits

$$S'(+\infty) = \operatorname*{s-lim}_{x\to+\infty} S'(x), \qquad S'(-\infty) = \operatorname*{s-lim}_{x\to-\infty} S'(x).$$

Let $E_\pm := S(\pm\infty)$. For an arbitrary $h \in H$,

$$\int_{\mathbb{R}} (S'(x)h \mid h)dx = (E_+h \mid h) - (E_-h \mid h) \leq \|h\|^2. \tag{44}$$

Using the equalities $\lim_{x\to\pm\infty}(S'(x)h \mid h) = (S'(\pm\infty)h \mid h)$ and the convergence of the integral in (44), we conclude $(S'(\pm\infty)h \mid h) = 0,\ h \in H$. Thus $S'(\pm\infty) = 0$. Passing to the limit in (43) as $x \to \pm\infty$, we get

$$KE_\pm + E_\pm K - 2E_\pm KE_\pm = 0.$$

Multiplying the last equality on both sides by P, we obtain

$$KPE_\pm P + PE_\pm PK - 2PE_\pm PKPE_\pm P = 0.$$

Let $\widetilde{E}_\pm$ be the restrictions of the operators $PE_\pm P$ on the subspace $H_1 = PH$, and let K_1 be the restriction of the operator K on H_1. Then

$$K_1\widetilde{E}_\pm + \widetilde{E}_\pm K_1 - 2\widetilde{E}_\pm K_1\widetilde{E}_\pm = 0.$$

From this, in view of Lemma 10, we conclude that the operators $\widetilde{E}_+$ and $\widetilde{E}_-$ are orthogonal projections.

From (42), it follows that for all $h \in H_1$

$$(\widetilde{E}_-h \mid h) \leq (S(0)h \mid h) \leq (\widetilde{E}_+h \mid h).$$

Since $0 < S(0) < I$, we get $\widetilde{E}_- = 0$ and $\widetilde{E}_+ = P$, thus $S(-\infty)P = 0, S(+\infty)P = P$. □

Proof of Theorem 6. Using (12) and (7), we get

$$\|\Psi(x)h\|^2 = (S'(x)h \mid h) = (S'(x)Ph \mid Ph),\ x \in \mathbb{R},\ h \in H.$$

Therefore, for an arbitrary $a > 0$,

$$\int_{-a}^{a} \|\Psi(x)h\|^2 dx = (S(a)Ph \mid Ph) - (S(-a)Ph \mid Ph).$$

In view of Lemma 4, we obtain that the operator V is bounded and

$$\|Vh\|^2 = \int_{\mathbb{R}} \|\Psi(x)h\|^2 dx = \|Ph\|^2,$$

thus, $(V^*Vh \mid h) = (Ph \mid h),\ h \in H$. Therefore, the operator V is a partial isometry and $V^*V = P$.

Next, we show that $\operatorname{ran} V \subset W_2^2(\mathbb{R}, H)$ and that (41) holds. Let $h \in H$. By the analyticity of the function Ψ, it follows that Vh belongs locally to the Sobolev space $W_2^2(\mathbb{R}, H)$. Using the equality (11), we obtain

$$(Vh)''(x) = q(x)(Vh)(x) + (VK^2h)(x), \qquad x \in \mathbb{R}. \tag{45}$$

Since the function q is bounded on the real axis, the right-hand side of (45) belongs to the space $\mathcal{H}$. Hence, we conclude that Vh belongs to the Sobolev space $W_2^2(\mathbb{R}, H)$ and the equality (41) holds. □

From Theorem 6, we obtain the following corollary.

Corollary 3. *The space $\mathfrak{H}_- := \operatorname{ran} V$ is an invariant subspace of the operator T_q, and the operator $T_q|_{\mathfrak{H}_-}$ is unitarily equivalent to the operator $-K_1^2$. Moreover, $\mathfrak{H}_- \subset \mathcal{H}_-$.*

Proof. Consider the operator $V_1\colon H_1 \to \mathfrak{H}_-$ defined by the formula $V_1 h := VPh,\ h \in H_1$. It follows from (41) that $T_q\mathfrak{H}_- \subset \mathfrak{H}_-$ and the operator V_1 maps H_1 unitarily onto $\mathfrak{H}_-$, moreover, $V_1^* T_q V_1 = -K_1^2$, i.e., the operator $T_q|_{\mathfrak{H}_-}$ is unitarily equivalent to the operator $-K_1^2$. From this, in particular, it follows that $T_q|_{\mathfrak{H}_-} < 0$, and thus $\mathfrak{H}_- \subset \mathcal{H}_-$. □

5. Classical transformation operators. In this subsection, we construct analogs of the classical transformation operators for the operator T_q with $q \in \mathcal{Q}_{\mathrm{reg}}$. The main result of this subsection is

Theorem 7. *Let $S \in \mathscr{S}_{\mathrm{reg}}(K)$, and let $q := q_S$, $\Psi := \Psi_S$. Then the formula*

$$(Uf)(x) := f(x) - \operatorname*{s-lim}_{a\to+\infty} \int_x^a \Psi(x)S^{-1}(t)\Psi^*(t)f(t)dt, \quad x \in \mathbb{R}, \tag{46}$$

defines an isometric operator in the space $\mathcal{H}$. Moreover, the operator U maps the Sobolev space $W_2^2(\mathbb{R}, H)$ into itself and satisfies

$$T_q U = UT_0, \tag{47}$$

$$UU^* + VV^* = \mathcal{I}. \tag{48}$$

First, we prove two auxiliary lemmas.

Lemma 5. *Let $S \in \mathscr{S}_{\mathrm{reg}}(K)$ and $\Psi := \Psi_S$. Then the formula*

$$(\mathfrak{N}f)(x) := \int_x^\infty \Psi(x)S^{-1}(t)\Psi^*(t)f(t)dt, \quad x \in \mathbb{R}, \quad f \in \operatorname{dom}\mathfrak{N} := C_0(\mathbb{R}, H), \tag{49}$$

defines a bounded operator $\mathfrak{N}\colon \mathcal{H} \to \mathcal{H}$, and

$$\|\mathfrak{N}f\|_{\mathcal{H}}^2 = (\mathfrak{N}f \mid f)_{\mathcal{H}} + (f \mid \mathfrak{N}f)_{\mathcal{H}}, \qquad f \in \operatorname{dom}\mathfrak{N}. \tag{50}$$

Here, $C_0(\mathbb{R}, H)$ denotes the linear space of all continuous functions $f\colon \mathbb{R} \to H$ with compact support in $\mathbb{R}$.

Proof. In view of (7), (9), and (12), we have

$$S'(x) = \Psi^*(x)\Psi(x), \quad S(x)P = PS(x), \quad P\Psi(x) = \Psi(x)P = \Psi(x), \qquad x \in \mathbb{R}. \tag{51}$$

Therefore, for arbitrary $h_1, h_2 \in H$,

$$\int_a^b (\Psi(x)h_1 \mid \Psi(x)h_2)dx = \int_a^b (\Psi(x)Ph_1 \mid \Psi(x)Ph_2)dx =$$
$$= \int_a^b (S'(x)Ph_1 \mid Ph_2)dx = (S(b)Ph_1 \mid Ph_2) - (S(a)Ph_1 \mid Ph_2) \qquad (a < b).$$

Hence, by Lemma 4, it follows that

$$\int_{-\infty}^t (\Psi(x)h_1 \mid \Psi(x)h_2)dx = (S(t)Ph_1 \mid Ph_2), \qquad h_1, h_2 \in H, \quad t \in \mathbb{R}. \tag{52}$$

Let $f \in C_0(\mathbb{R}, H)$ and set $\widetilde{f}(t) := S^{-1}(t)\Psi^*(t)f(t)$. From the definition of the operator $\mathfrak{N}$, we have

$$\|\mathfrak{N}f\|^2_{\mathcal{H}} = \iiint\limits_{x\le t\le\tau} (\Psi(x)\widetilde{f}(t) \mid \Psi(x)\widetilde{f}(\tau))dxdtd\tau + \iiint\limits_{x\le\tau\le t} (\Psi(x)\widetilde{f}(t) \mid \Psi(x)\widetilde{f}(\tau))dxdtd\tau.$$

Thus, taking into account (52) and (51), we get

$$\|\mathfrak{N}f\|^2_{\mathcal{H}} = \iint\limits_{t\le\tau} (S(t)P\widetilde{f}(t) \mid P\widetilde{f}(\tau))dtd\tau + \iint\limits_{\tau\le t} (S(\tau)P\widetilde{f}(t) \mid P\widetilde{f}(\tau))dtd\tau =$$

$$= \iint\limits_{t\le\tau} (\Psi^*(t)f(t) \mid S^{-1}(\tau)\Psi^*(\tau)f(\tau))dtd\tau + \iint\limits_{\tau\le t} (S^{-1}(t)\Psi^*(t)f(t) \mid \Psi^*(\tau)f(\tau))dtd\tau =$$

$$= \int_{\mathbb{R}} (f(t) \mid (\mathfrak{N}f)(t))dt + \int_{\mathbb{R}} ((\mathfrak{N}f)(\tau) \mid f(\tau))d\tau.$$

The function $\mathbb{R} \ni t \mapsto ((\mathfrak{N}f)(t) \mid f(t)) \in \mathbb{C}$ is continuous with compact support, and hence $\|\mathfrak{N}f\|^2_{\mathcal{H}} < \infty$ and the identity (50) holds. It follows from (50) that $\|\mathfrak{N}f\|^2_{\mathcal{H}} \le 2\|f\|_{\mathcal{H}}\|\mathfrak{N}f\|_{\mathcal{H}}$. Therefore, the operator $\mathfrak{N}$ is bounded, and $\|\mathfrak{N}\|_{\mathcal{H}\to\mathcal{H}} \le 2$. □

Corollary 4. *The formula* (46) *defines an isometric operator* $U\colon \mathcal{H} \to \mathcal{H}$.

Proof. It follows from (50) that for all $f \in C_0(\mathbb{R}, H)$

$$\|Uf\|^2_{\mathcal{H}} = \|(\mathcal{I} - \mathfrak{N})f\|^2_{\mathcal{H}} = \|f\|^2_{\mathcal{H}} + \|\mathfrak{N}f\|^2_{\mathcal{H}} - (\mathfrak{N}f \mid f)_{\mathcal{H}} - (f \mid \mathfrak{N}f)_{\mathcal{H}} = \|f\|^2_{\mathcal{H}}.$$

Since $C_0(\mathbb{R}, H)$ is everywhere dense in $\mathcal{H}$, the formula (46) defines an isometric operator $U\colon \mathcal{H} \to \mathcal{H}$. □

Lemma 6. *For all* $f \in C_0(\mathbb{R}, H)$ *the following equality holds*

$$\|\mathfrak{N}^*f\|^2_{\mathcal{H}} = (\mathfrak{N}f \mid f)_{\mathcal{H}} + (f \mid \mathfrak{N}f)_{\mathcal{H}} - \|V^*f\|^2_{\mathcal{H}}. \tag{53}$$

Proof. Using (51), we obtain that for arbitrary $h_1, h_2 \in H$

$$\int_a^b (\Psi(x)S^{-1}(x)h_1 \mid \Psi(x)S^{-1}(x)h_2)dx = \int_a^b (\Psi(x)S^{-1}(x)Ph_1 \mid \Psi(x)S^{-1}(x)Ph_2)dx =$$

$$= \int_a^b (S^{-1}(x)S'(x)S^{-1}(x)Ph_1 \mid Ph_2)dx = -\int_a^b ([S^{-1}(x)]'(x)Ph_1 \mid Ph_2)dx =$$

$$= (S^{-1}(a)Ph_1 \mid Ph_2) - (S^{-1}(b)Ph_1 \mid Ph_2) \quad (a < b).$$

Thus, by Lemma 4, we get that for all $t \in \mathbb{R}$

$$\int_t^{+\infty} (\Psi(x)S^{-1}(x)h_1 \mid \Psi(x)S^{-1}(x)h_2)dx = (S^{-1}(t)Ph_1 \mid Ph_2) - (Ph_1 \mid Ph_2). \tag{54}$$

Let $f \in C_0(\mathbb{R}, H)$ and set $\widetilde{f}(t) := \Psi^*(t)f(t)$. From the definition of $\mathfrak{N}$, we have

$$(\mathfrak{N}^*f)(x) = \int_{-\infty}^x \Psi(x)S^{-1}(x)\Psi^*(t)f(t)dt, \quad x \in \mathbb{R}.$$

Hence,

$$\|\mathfrak{N}^* f\|^2_{\mathcal{H}} = \iiint_{t\le\tau\le x} (\Psi(x)S^{-1}(x)\widetilde{f}(t) \mid \Psi(x)S^{-1}(x)\widetilde{f}(\tau))dxdtd\tau +$$
$$+ \iiint_{\tau\le t\le x} (\Psi(x)S^{-1}(x)\widetilde{f}(t) \mid \Psi(x)S^{-1}(x)\widetilde{f}(\tau))dxdtd\tau.$$

Using (54), we find

$$\|\mathfrak{N}^* f\|^2_{\mathcal{H}} = \iint_{t\le\tau} (S^{-1}(t)\widetilde{f}(t) \mid \widetilde{f}(\tau))dtd\tau + \iint_{\tau\le t} (S^{-1}(t)\widetilde{f}(t) \mid \widetilde{f}(\tau))dtd\tau -$$
$$- \iint_{t\le\tau} (\widetilde{f}(t) \mid \widetilde{f}(\tau))dtd\tau - \iint_{\tau\le t} (\widetilde{f}(t) \mid \widetilde{f}(\tau))dtd\tau =$$
$$= \iint_{t\le\tau} (S^{-1}(t)\Psi^*(t)f(t) \mid \Psi^*(\tau)f(\tau))dtd\tau + \iint_{\tau\le t} (S^{-1}(t)\Psi^*(t)f(t) \mid \Psi^*(\tau)f(\tau))dtd\tau -$$
$$- \iint_{\mathbb{R}^2} (\Psi^*(t)f(t) \mid \Psi^*(\tau)f(\tau))dtd\tau = (f \mid \mathfrak{N}f)_{\mathcal{H}} + (\mathfrak{N}f \mid f)_{\mathcal{H}} - \|V^*f\|^2_{\mathcal{H}}.$$

□

Corollary 5. *The equality $UU^* + VV^* = \mathcal{I}$ holds.*

Proof. The equality (53) implies that for all $f \in C_0(\mathbb{R}, H)$

$$\|U^*f\|^2_{\mathcal{H}} = \|(\mathcal{I} - \mathfrak{N}^*)f\|^2_{\mathcal{H}} = \|f\|^2_{\mathcal{H}} + \|\mathfrak{N}^*f\|^2_{\mathcal{H}} - (f \mid \mathfrak{N}f)_{\mathcal{H}} - (\mathfrak{N}f \mid f)_{\mathcal{H}} = \|f\|^2_{\mathcal{H}} - \|V^*f\|^2_{\mathcal{H}}.$$

Hence,

$$((UU^* + VV^*)f \mid f)_{\mathcal{H}} = (f \mid f)_{\mathcal{H}}, \ f \in C_0(\mathbb{R}, H).$$

Since $C_0(\mathbb{R}, H)$ is everywhere dense in $\mathcal{H}$, we get $UU^* + VV^* = \mathcal{I}$. □

Proof of Theorem 7. Assume that the conditions of the theorem are satisfied. In view of Corollaries 4 and 5, it remains to show that the operator U maps the Sobolev space $W_2^2(\mathbb{R}, H)$ into itself and the equality (47) holds. Put

$$\Phi(x) := Re^{-xK}, \ x \in \mathbb{R}.$$

Then (see (13))

$$\Psi(x) = \Phi(x)S(x), \qquad x \in \mathbb{R}. \tag{55}$$

Let $f \in W_2^2(\mathbb{R}, H)$ and set

$$g(x) := f(x) - (Uf)(x) = \int_x^\infty \Psi(x)\Phi^*(t)f(t)dt, \qquad x \in \mathbb{R}.$$

The functions Ψ and Φ are analytic in the strip $|\operatorname{Im} z| < \delta$ for some $\delta > 0$, and hence $g \in W^2_{2,\mathrm{loc}}(\mathbb{R}, H)$. Note that by Corollary 4, we also have $g \in \mathcal{H}$. Direct calculations yield

$$g''(x) = \Psi''(x) \int_x^\infty \Phi^*(t)f(t)dt - 2(\Psi'\Phi^*f)(x) - [\Psi(\Phi^*f)'](x).$$

Since (see (11)) $\Psi''(x) = q(x)\Psi(x) + \Psi(x)K^2$, $(\Phi^*)''(x) = K^2\Phi^*(x)$, we get

$$g''(x) = q(x)g(x) + \Psi(x) \int_x^\infty (\Phi^*)''(t)f(t)dt - 2(\Psi'\Phi^*f)(x) - [\Psi(\Phi^*f)'](x).$$

Integrating by parts, we get

$$\int_x^\infty (\Phi^*)''(t)f(t)dt = [-(\Phi^*)'f + \Phi^* f'](x) + \int_x^\infty \Phi^*(t)f''(t)dt.$$

Thus,

$$g''(x) = q(x)g(x) + \Psi(x)\int_x^\infty \Phi^*(t)f''(t)dt - 2(\Psi\Phi^*)'(x)f(x).$$

Using (5), we obtain

$$2(\Psi\Phi^*)' = 2(\Phi S\Phi^*)' = 2\Phi(-KS - SK + S')\Phi^* = -4\Phi SKS\Phi^* = -4\Psi K\Psi^* = q,$$

and hence,

$$g''(x) = q(x)g(x) + \Psi(x)\int_x^\infty \Phi^*(t)f''(t)dt - q(x)f(x) = (qg + f'' - Uf'' - qf)(x). \quad (56)$$

It follows that $g'' \in \mathcal{H}$, which implies that $g \in W_2^2(\mathbb{R}, H)$, and therefore $Uf \in W_2^2(\mathbb{R}, H)$. From (56), it also follows that

$$(T_qUf)(x) = (T_qf - T_qg)(x) = (-f'' + qf + g'' - qg)(x) = -(Uf'')(x) = (UT_0f)(x).$$

□

Corollary 6. *Let $q \in \mathcal{Q}_{\text{reg}}(K)$. Then*

(I) $\mathcal{H}_+ = \operatorname{ran} U$;

(II) $UU^* = \mathcal{P}_+, \quad VV^* = \mathcal{P}_-$;

(III) the operator T_q^+ is unitarily equivalent to the operator T_0.

Proof. Let $\mathfrak{H}_+ := \operatorname{ran} U$. Consider the operator $U_1 : \mathcal{H} \to \mathfrak{H}_+$ defined by the formula

$$U_1 g := Ug, \quad g \in \mathcal{H}.$$

Since the operator U is isometric, the operator U_1 maps $\mathcal{H}$ unitarily onto $\mathfrak{H}_+$.

From Theorem 7, it follows that the linear space $W_2^2(\mathbb{R}, H) \cap \mathfrak{H}_+$ is ewerywhere dense in $\mathfrak{H}_+$ and $T_q\mathfrak{H}_+ \subset \mathfrak{H}_+$, $U_1^* T_q U_1 = T_0$. Threfore, the operator $T_q|_{\mathfrak{H}_+}$ is unitarily equivalent to the operator T_0. In particular, this implies that $T_q|_{\mathfrak{H}_+} > 0$. Hence, $\mathfrak{H}_+ \subset \mathcal{H}_+$. Moreover, from (48), it follows that $\mathcal{H} = \mathfrak{H}_+ + \mathfrak{H}_-$, where $\mathfrak{H}_- = \operatorname{ran} V$. Since (see Corollaries 2 and 3) $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, $\mathfrak{H}_- \subset \mathcal{H}_-$, we conclude that $\mathcal{H}_+ = \mathfrak{H}_+$ and $\mathcal{H}_- = \mathfrak{H}_-$. From this, the corollary follows. □

Let us show that the operator U is an analog of the classical transformation operator. Denote by $L_{2,\text{loc}}(\mathbb{R}, H)$ the set of all functions $f\colon \mathbb{R} \to H$ that belong locally to $L_2(\mathbb{R}, H)$, and equip the linear space

$$\mathcal{L}_+ := \left\{ f \in L_{2,\text{loc}}(\mathbb{R}, H) \mid \forall n \in \mathbb{N} \quad \int_{-n}^\infty \|f(x)\|^2 dx < \infty \right\}$$

with the locally convex topology generated by the seminorms

$$\rho_n(f) := \left(\int_{-n}^\infty \|f(x)\|^2 dx \right)^{1/2}, \quad f \in \mathcal{L}_+, \quad n \in \mathbb{N}.$$

Denote by e_λ the operator-valued function acting from $\mathbb{R}$ to $\mathcal{B}(H)$ defined by
$$e_\lambda(x) := e^{i\lambda x} I,\ x \in \mathbb{R}.$$

Let $S \in \mathscr{S}_{\mathrm{reg}}(K)$ and set $q := q_S$, $\Psi := \Psi_S$. Note that, according to Theorem 7, for an arbitrary $f \in \mathcal{L}_+$ the formula

$$(Uf)(x) = f(x) - \operatorname*{s\text{-}lim}_{a\to+\infty} \int_x^a \Psi(x) S^{-1}(t)\Psi^*(t) f(t)dt, \quad x \in \mathbb{R}, \tag{57}$$

defines a function $Uf \in \mathcal{L}_+$. Moreover, the extended operator $U\colon \mathcal{L}_+ \to \mathcal{L}_+$ is continuous. In view of (55), the formula (57) can be rewritten as

$$(Uf)(x) = f(x) - \operatorname*{s\text{-}lim}_{a\to+\infty} \int_x^a \Psi(x)\Phi^*(t) f(t)dt, \quad x \in \mathbb{R}. \tag{58}$$

Let us fix $\lambda \in \mathbb{C}_+$ and $h \in H$. Then

$$\int_x^a e^{i\lambda t}\Phi^*(t)hdt = \int_x^a e^{-t(K-i\lambda I)}R^*hdt = e^{-x(K-i\lambda I)}K_\lambda R^*h - e^{-a(K-i\lambda I)}K_\lambda R^*h =$$
$$= e^{-x(K-i\lambda I)}PK_\lambda R^*h - e^{-a(K-i\lambda I)}PK_\lambda R^*h.$$

Since $\operatorname*{s\text{-}lim}_{a\to+\infty} e^{-aK}Ph = 0$, the integral $\int_x^\infty e^{-t(K-i\lambda I)}R^*dt$ converges in the strong operator topology and (see (9))

$$\int_x^\infty e^{i\lambda t}\Phi^*(t)dx = e^{-x(K-i\lambda I)}PK_\lambda R^* = e^{i\lambda x}e^{-xK}K_\lambda R^*.$$

Therefore,

$$U(e_\lambda(\cdot)h)(x) = e^{i\lambda x}\left(I - \Psi(x)e^{-xK}K_\lambda R^*\right)h, \quad x \in \mathbb{R}, \quad \lambda \in \mathbb{C}_+, \quad h \in H. \tag{59}$$

In [2], it was proven that the formula

$$e(\lambda, x) = e^{i\lambda x}\left[I - \Psi(x)e^{-xK}K_\lambda R^*\right], \quad x \in \mathbb{R}, \quad \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\},$$

defines the right Jost solution of the equation $-y'' + qy = \lambda^2 y$. Thus, the equality (59) means that the operator U maps the right $\mathcal{B}(H)$-valued Jost solution of the equation $-y'' = \lambda^2 y$ with $\lambda \in \mathbb{C}_+$ into the right $\mathcal{B}(H)$-valued Jost solution of the equation $-y'' + qy = \lambda^2 y$. It is well-known [6] that this is a characteristic property of the classical transformation operator for the Schrödinger operator.

Another property of the classical transformation operator is its triangularity (see [8]) with respect to the chain $\mathfrak{E} := \{E_\xi \mid \xi \in \mathbb{R}\}$ of orthoprojectors $E_\xi \in \mathcal{B}(\mathcal{H})$ defined by

$$(E_\xi f)(x) := \chi_\xi(x) f(x), \quad x \in \mathbb{R}, \quad f \in \mathcal{L}_+,$$

where χ_ξ is the characteristic function of the half-line $(-\infty, \xi)$. The transformation operator U is lower-triangular with respect to the chain $\mathfrak{E}$, i.e., $(\mathcal{I} - E_\xi)UE_\xi = 0,\ \xi \in \mathbb{R}$.

In this respect, the following question arises.

Question 2. *For which potentials $q \in \mathcal{Q}_\pi$ do there exist lower-triangular isometric operators $U\colon \mathcal{H} \to \mathcal{H}$ that for all $\lambda \in \mathbb{C}_+$ map the functions e_λ to the Jost solution of the equation $-y'' + qy = \lambda^2 y$?*

6. The operator $\mathfrak{A}$. Let $q \in \mathcal{Q}(K)$. Consider the function

$$f(\lambda, x) := e^{i\lambda x}[I - \Psi(x)D(\lambda, x)\Psi^*(0)]M^{-1}(\lambda), \qquad \lambda \in \mathcal{O}_M(K),$$

where

$$D(\lambda, x) := K_\lambda e^{-xK} + K_{-\lambda} e^{xK}, \qquad K_\lambda := (K - i\lambda I)^{-1},$$
$$M(\lambda) := I - 2\Psi(0)K(K^2 + \lambda^2 I)^{-1}\Psi^*(0), \qquad \lambda \in \mathcal{O}(K),$$

and $\mathcal{O}(K) := \{\lambda \in \mathbb{C} \mid \pm i\lambda \notin \sigma(K)\}$, $\mathcal{Q}_M(K) := \{\lambda \in \mathcal{O}(K) \mid M(\lambda) \in \mathcal{B}_{\text{inv}}(H)\}$.

As shown in [1], the set $\mathbb{C} \setminus \mathcal{O}_M(K)$ is a compact subset of the imaginary axis and

$$f(\lambda, \cdot) = \begin{cases} f_+(\lambda^2, \cdot), & \lambda \in \mathbb{C}_+ \setminus i\mathbb{R}; \\ f_-(\lambda^2, \cdot), & \lambda \in \mathbb{C}_- \setminus i\mathbb{R}. \end{cases} \tag{60}$$

Denote by $\mathscr{T}$ the self-adjoint operator in $\mathcal{H}$ defined by the formula

$$(\mathscr{T} g)(x) := x^2 g(x), \quad x \in \mathbb{R}, \quad g \in \operatorname{dom} \mathscr{T} := \Big\{\varphi \in \mathcal{H} \mid \int_{\mathbb{R}} x^2 \|\varphi(x)\|^2 dx < \infty\Big\}. \tag{61}$$

The main result of this section is

Theorem 8. *Let $q \in \mathcal{Q}(K)$ and $T = T_q$. Then the formula*

$$(\mathfrak{A}\varphi)(x) := \lim_{a \to +\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(\xi, x) M^{1/2}(\xi)\varphi(\xi) d\xi, \quad x \in \mathbb{R}, \tag{62}$$

defines an isometric operator $\mathfrak{A}\colon \mathcal{H} \to \mathcal{H}$ for which the following equalities hold

$$T\mathfrak{A} = \mathfrak{A}\mathscr{T}, \qquad \mathfrak{A}\mathfrak{A}^* = \mathcal{P}_+.$$

First, we prove the following lemma.

Lemma 7. *Let $q \in \mathcal{Q}(K)$. Then*

$$m_-(\lambda^2) - m_+(\lambda^2) = -2i\lambda M^{-1}(\lambda), \quad \lambda \in \mathbb{C}_+ \setminus i\mathbb{R}. \tag{63}$$

Proof. By definition, $m_\pm(\lambda^2) = f'_\pm(\lambda^2, 0)$, $\lambda \in \mathbb{C}_+ \setminus i\mathbb{R}$. Therefore, in view of (60), we have

$$m_-(\lambda^2) - m_+(\lambda^2) = f'(-\lambda, 0) - f'(\lambda, 0), \quad \lambda \in \mathbb{C}_+ \setminus i\mathbb{R}.$$

Since $f(\lambda, 0) = I$, $D'(\lambda, 0) = -i\lambda D(\lambda, 0)$, we obtain

$$f'(\lambda, 0) = i\lambda I - [\Psi'(0)D(\lambda, 0) + \Psi(0)D'(\lambda, 0)]\Psi^*(0)M^{-1}(\lambda) =$$
$$= i\lambda I - [\Psi'(0)D(\lambda, 0) - i\lambda\Psi(0)D(\lambda, 0)]\Psi^*(0)M^{-1}(\lambda).$$

Taking into account that $D(\lambda, 0) = D(-\lambda, 0)$, we get

$$f'(-\lambda, 0) - f'(\lambda, 0) = -2i\lambda I - 2i\lambda\Psi(0)D(\lambda, 0)\Psi^*(0)M^{-1}(\lambda) =$$
$$= -2i\lambda I + 2i\lambda[M(\lambda) - I]M^{-1}(\lambda) = -2i\lambda M^{-1}(\lambda), \tag{64}$$

and hence, $m_-(\lambda^2) - m_+(\lambda^2) = -2i\lambda M^{-1}(\lambda)$. □

Remark 2. It follows from (63) and (34) that

$$-2i\lambda W^{-1}(\lambda^2) = M(\lambda) = \left(I + \int_{\mathbb{R}} \frac{d\nu_q(t)}{t^2+\lambda^2}\right)^{-1}, \qquad \lambda \in \mathcal{Q}_M(K). \tag{65}$$

Proof of Theorem 8. In view of the equalities (60) and (65), the formula (23) for the resolvent of the operator T can be rewritten as follows

$$[(T-\lambda^2\mathcal{I})^{-1}g](x) =$$
$$= -\frac{1}{2i\lambda}\left(\int_{-\infty}^{x} f(\lambda,x)M(\lambda)[f(\bar\lambda,t)]^* g(t)dt + \int_{x}^{\infty} f(-\lambda,x)M(\lambda)[f(-\bar\lambda,t)]^* g(t)dt\right). \tag{66}$$

Let us consider the $\mathcal{B}(H)$-valued function

$$\gamma(\lambda,x,t) := f(\lambda,x)M(\lambda)[f(\bar\lambda,t)]^*, \qquad x,t\in\mathbb{R}, \quad \lambda\in\mathcal{O}_M(K).$$

It is clear that this function depends continuously on the variables λ, x, t. Note also that $\gamma(\lambda,x,t) = \gamma^*(\bar\lambda,t,x)$, $M(\lambda) = M(-\lambda)$. Let $\varphi \in C_0(\mathbb{R}\setminus\{0\},H)$. Then (66) implies

$$((T-\lambda^2\mathcal{I})^{-1}\varphi \mid \varphi)_{\mathcal{H}} =$$
$$= -\frac{1}{2i\lambda}\left(\iint_{t\le x}(\gamma(\lambda,x,t)\varphi(t)\mid\varphi(x))_{\mathcal{H}}dtdx + \iint_{t\ge x}(\gamma(-\lambda,x,t)\varphi(t)\mid\varphi(x))_{\mathcal{H}}dtdx\right). \tag{67}$$

Since the right-hand side of (67) is continuous in the domain $\mathcal{O}_M(K)$, for an arbitrary $\xi\in\mathbb{R}_+$ the following limits exist

$$((T-(\xi^2\pm i0)\mathcal{I})^{-1}\varphi\mid\varphi)_{\mathcal{H}} = \lim_{\varepsilon\to+0}((T-(\pm\xi+i\varepsilon)^2\mathcal{I})^{-1}\varphi\mid\varphi)_{\mathcal{H}} =$$
$$= \mp\frac{1}{2i\xi}\left(\iint_{t\le x}(\gamma(\pm\xi,x,t)\varphi(t)\mid\varphi(x))_{\mathcal{H}}dtdx + \iint_{t\ge x}(\gamma(\mp\xi,x,t)\varphi(t)\mid\varphi(x))_{\mathcal{H}}dtdx\right).$$

Hence,

$$((T-(\xi^2+i0)\mathcal{I})^{-1}\varphi\mid\varphi)_{\mathcal{H}} - ((T-(\xi^2-i0)\mathcal{I})^{-1}\varphi\mid\varphi)_{\mathcal{H}} =$$
$$= -\frac{1}{2i\xi}\left(\iint_{\mathbb{R}^2}(\gamma(\xi,x,t)\varphi(t)\mid\varphi(x))_{\mathcal{H}}dtdx + \iint_{\mathbb{R}^2}(\gamma(-\xi,x,t)\varphi(t)\mid\varphi(x))_{\mathcal{H}}dtdx\right). \tag{68}$$

Put

$$\delta(\xi,\varphi) := \frac{1}{2\pi i}\left[((T-(\xi+i0)\mathcal{I})^{-1}\varphi\mid\varphi)_{\mathcal{H}} - ((T-(\xi-i0)\mathcal{I})^{-1}\varphi\mid\varphi)_{\mathcal{H}}\right],\ \xi>0.$$

According to Stone's formula (see [9]),

$$(\mathcal{P}_+\varphi\mid\varphi)_{\mathcal{H}} = \int_{\mathbb{R}_+}\delta(\xi,\varphi)d\xi = \int_{\mathbb{R}_+}2\xi\delta(\xi^2,\varphi)d\xi.$$

It follows from (68) that

$$2\xi\delta(\xi^2,\varphi) = \frac{1}{2\pi}\left(\iint_{\mathbb{R}^2}(\gamma(\xi,x,t)\varphi(t)\mid\varphi(x))_{\mathcal{H}}dtdx + \iint_{\mathbb{R}^2}(\gamma(-\xi,x,t)\varphi(t)\mid\varphi(x))_{\mathcal{H}}dtdx\right).$$

Thus,

$$(\mathcal{P}_+\varphi\mid\varphi)_{\mathcal{H}} = \frac{1}{2\pi}\iiint_{\mathbb{R}^3}(\gamma(\xi,x,t)\varphi(t)\mid\varphi(x))_{\mathcal{H}}dtdxd\xi. \tag{69}$$

Consider the auxiliary operator

$$(\mathfrak{D}\varphi)(\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} M^{1/2}(\xi)f^*(\xi,x)\varphi(x)dx, \qquad \xi\in\mathbb{R}\setminus\{0\}, \quad \varphi\in C_0(\mathbb{R}\setminus\{0\},H),$$

on the domain $\operatorname{dom}\mathfrak{D} := C_0(\mathbb{R}\setminus\{0\}, H)$. By (69), we obtain

$$\|\mathfrak{D}\varphi\|^2 = \frac{1}{2\pi}\iiint_{\mathbb{R}^3}(\gamma(\xi,x,t)\varphi(t) \mid \varphi(x))_{\mathcal{H}}dtdxd\xi = (\mathcal{P}_+\varphi \mid \varphi)_{\mathcal{H}} \leq \|\varphi\|^2,$$

which means that the operator $\mathfrak{D}$ is densely defined and bounded, moreover $\|\mathfrak{D}\| \leq 1$. It is easy to see that

$$(\mathfrak{A}\varphi \mid \psi)_{\mathcal{H}} = (\varphi \mid \mathfrak{D}\psi)_{\mathcal{H}} = (\mathfrak{D}^*\varphi \mid \psi)_{\mathcal{H}},\ \varphi,\psi \in C_0(\mathbb{R}\setminus\{0\}, H).$$

Hence, the formula (62) defines a bounded operator $\mathfrak{A} = \mathfrak{D}^*$, and

$$(\mathfrak{A}\mathfrak{A}^*\varphi \mid \varphi) = \|\mathfrak{D}\varphi\|^2 = (\mathcal{P}_+\varphi \mid \varphi)_{\mathcal{H}},\ \varphi \in C_0(\mathbb{R}\setminus\{0\}, H).$$

Therefore,

$$\mathfrak{A}\mathfrak{A}^* = \mathcal{P}_+, \qquad \mathfrak{A}^* = \mathfrak{A}^*\mathcal{P}_+, \qquad \mathfrak{A} = \mathcal{P}_+\mathfrak{A}. \tag{70}$$

As shown in [1], $-f''(\lambda,x) + q(x)f(\lambda,x) = \lambda^2 f(\lambda,x)$, $x \in \mathbb{R}$, for all $\lambda \in \mathcal{O}_M(K)$, hence $\mathfrak{A}\varphi \in \operatorname{dom} T$ for all $\varphi \in C_0(\mathbb{R}\setminus\{0\}, H)$ and $T\mathfrak{A}\varphi = \mathfrak{A}\mathscr{T}\varphi$. From this and the closedness of the operators T and $\mathscr{T}$, we obtain

$$T\mathfrak{A}\varphi = \mathfrak{A}\mathscr{T}\varphi, \qquad \varphi \in \operatorname{dom}\mathscr{T}. \tag{71}$$

Thus, it remains to prove the isometricity of the operator $\mathfrak{A}$.

Let the function $F\colon [0,\infty) \to \mathbb{C}$ be continuous with compact support in $\mathbb{R}_+$. We show that the equality

$$F(T)\mathfrak{A} = \mathfrak{A}F(\mathscr{T}) \tag{72}$$

holds. Let us consider the operators

$$B_1 := (T\mathcal{P}_+ + \mathcal{I})^{-1},\ B_2 := (\mathscr{T} + \mathcal{I})^{-1}$$

and the continuous function $F_1\colon [0,1] \to \mathbb{C}$ defined by

$$F_1(x) := \begin{cases} F(x^{-1}-1), & x \in (0,1]; \\ 0, & x = 0. \end{cases}$$

From (70) and (71), it follows that $(T\mathcal{P}_+ + \mathcal{I})\mathfrak{A} = \mathfrak{A}(\mathscr{T} + \mathcal{I})$, and hence, $(T\mathcal{P}_+ + \mathcal{I})^{-1}\mathfrak{A} = \mathfrak{A}(\mathscr{T} + \mathcal{I})^{-1}$. So,

$$B_1\mathfrak{A} = \mathfrak{A}B_2. \tag{73}$$

It is clear that the operators B_j ($j \in \{1,2\}$) are self-adjoint, and their spectrum lies in the interval $[0,1]$. From the equality (73), it follows that $\mathscr{P}(B_1)\mathfrak{A} = \mathfrak{A}\mathscr{P}(B_2)$ for arbitrary polynomials $\mathscr{P}(x) = \sum_{j=0}^n a_j x^j$. Hence, $F_1(B_1)\mathfrak{A} = \mathfrak{A}F_1(B_2)$. It is easy to see that

$$F_1(B_1) = F(T),\ F_1(B_2) = F(\mathscr{T}).$$

Thus, the equality (72) is proved.

Now we prove the isometricity of the operator $\mathfrak{A}$. To do this, it suffices to show that $\ker\mathfrak{A} = \{0\}$. Assume that $g \in \ker\mathfrak{A}$, and let $F\colon [0,\infty) \to \mathbb{C}$ be an arbitrary continuous function with compact support in $\mathbb{R}_+$. In view of (72), we have

$$0 = F(T)\mathfrak{A}g = \mathfrak{A}F(\mathscr{T})g.$$

Note that

$$[F(\mathscr{T})g](\xi) = F(\xi^2)g(\xi),\ \xi \in \mathbb{R},$$

and thus

$$\int_{\mathbb{R}} F(\xi^2)f(\xi,x)M^{1/2}(\xi)g(\xi)d\xi = 0.$$

for almost every $x \in \mathbb{R}$. From the continuity of the function $\mathbb{R}_+ \times \mathbb{R} \ni (\xi,x) \mapsto f(\xi,x) \in \mathcal{B}(H)$, it follows that for all $x \in \mathbb{R}$

$$\int_{\mathbb{R}} F(\xi^2)f(\xi,x)M^{1/2}(\xi)g(\xi)d\xi = 0. \tag{74}$$

Since the function $\xi \mapsto M(\xi)$ is even and $f(\xi,0) = I$, by setting $x = 0$ in (74), we obtain

$$0 = \int_{\mathbb{R}} F(\xi^2) M^{1/2}(\xi) g(\xi) d\xi = \int_0^\infty F(\xi^2) M^{1/2}(\xi) \big(g(\xi) + g(-\xi)\big) d\xi. \tag{75}$$

The equality (75) holds for an arbitrary continuous functions $F\colon [0,\infty) \to \mathbb{C}$ with compact support in $\mathbb{R}_+$. This implies that $M^{1/2}(\xi)(g(\xi) + g(-\xi)) = 0$ for almost every $\xi \in \mathbb{R}_+$. Since $M(\xi) > 0$ for all $\xi \in \mathbb{R}_+$, it follows that g must be an odd function. In view of the oddness of g, the formula (74) can be rewritten as

$$\int_0^\infty F(\xi^2)(f(\xi,x) - f(-\xi,x)) M^{1/2}(\xi) g(\xi) d\xi = 0, \quad x \in \Pi_K.$$

It is easy to see that the left-hand side of the equality can be differentiated with respect to the variable x. Hence,

$$\int_0^\infty F(\xi^2)(f'(\xi,0) - f'(-\xi,0)) M^{1/2}(\xi) g(\xi) d\xi = 0.$$

Since (see (64)) $f'(\xi,0) - f'(-\xi,0) = 2i\xi M^{-1}(\xi)$, we obtain

$$\int_0^\infty \xi F(\xi^2) M^{-1/2}(\xi) g(\xi) d\xi = 0. \tag{76}$$

From the arbitrariness of the function F, it follows that $\xi M^{-1/2}(\xi) g(\xi) = 0$ almost everywhere on $\mathbb{R}_+$, and therefore $g = 0$, i.e., $\ker \mathfrak{A} = \{0\}$. □

7. Completion of the proof of Theorem 3. In this section, we complete the proof of Theorem 3 and prove Proposition 4.

Lemma 8. *Let $S \in \mathscr{S}(K)$, $q := q_S$, and let V be the operator defined by the formula* (40). *Then $VV^* = \mathcal{P}_-$.*

Proof. Let $S \in \mathscr{S}(K)$, and let $S_\varepsilon \in \mathscr{S}_{\mathrm{reg}}(K)$ be as in Proposition 2. Define $V_\varepsilon := V_{S_\varepsilon}$. Let us show that

$$\operatorname*{s-lim}_{\varepsilon\to+0} V_\varepsilon = V, \qquad \operatorname*{s-lim}_{\varepsilon\to+0} V_\varepsilon^* = V^*.$$

From (40) and (10), it follows that for all $h \in H$ and $\varphi \in C_0(\mathbb{R}, H)$,

$$\lim_{\varepsilon\to+0} (V_\varepsilon h \mid \varphi)_{\mathcal{H}} = \lim_{\varepsilon\to+0} \int_{\mathbb{R}} (\Psi_\varepsilon(x) h \mid \varphi(x)) dx = \int_{\mathbb{R}} (\Psi(x) h \mid \varphi(x)) dx = (Vh \mid \varphi)_{\mathcal{H}},$$

i.e., $V_\varepsilon \to V$ in the weak operator topology as $\varepsilon \to +0$. It follows from Theorem 6 that for all $\varepsilon \in (0,1/2)$, $h \in H$ we get

$$V_\varepsilon(I-P)h = 0 = V(I-P)h, \quad \|V_\varepsilon Ph\| = \|Ph\| = \|VPh\|.$$

Hence, $\lim_{\varepsilon\to+0} V_\varepsilon h = Vh$, i.e., $\operatorname*{s-lim}_{\varepsilon\to+0} V_\varepsilon = V$. Moreover, in view of Proposition 2, we have that

$$\lim_{\varepsilon\to+0} V_\varepsilon^* \varphi = \lim_{\varepsilon\to+0} \int_{\mathbb{R}} \Psi_\varepsilon^*(x) \varphi(x) dx = \int_{\mathbb{R}} \Psi^*(x) \varphi(x) dx = V^* \varphi$$

for an arbitrary $\varphi \in C_0(\mathbb{R}, H)$. Since $C_0(\mathbb{R}, H)$ is everywhere dense in $\mathcal{H}$ and $\sup_{\varepsilon\in(0,1/2)} \|V_\varepsilon\| \le 1$, we conclude that $\operatorname*{s-lim}_{\varepsilon\to+0} V_\varepsilon^* = V^*$.

According to Corollary 6, for all $\varepsilon \in (0,1/2)$, the operator $V_\varepsilon V_\varepsilon^*$ is the spectral projection of the operator T_{q_ε} corresponding to the negative half-line $\mathbb{R}_-$. Therefore,

$$(T_{q_\varepsilon}(\mathcal{I} - V_\varepsilon V_\varepsilon^*) g \mid g)_{\mathcal{H}} \ge 0, \qquad \varepsilon \in (0,1/2), \quad g \in W_2^2(\mathbb{R}, H). \tag{77}$$

In view of Proposition 2, we can pass to the limit as $\varepsilon \to +0$ in (77). As a result, we obtain

$$(T_q(\mathcal{I} - VV^*) g \mid g)_{\mathcal{H}} \ge 0, \qquad g \in W_2^2(\mathbb{R}, H). \tag{78}$$

Since V is a partial isometry, the operator VV^* is the orthogonal projection onto the subspace $\operatorname{ran} V$. By Corollary 3, $VV^* \le \mathcal{P}_-$. Suppose that $VV^* \ne \mathcal{P}_-$. Then there exists a nonzero element $g_0 \in \mathcal{H}$ such that $\mathcal{P}_- g_0 = g_0$, $VV^* g_0 = 0$. From this, using (78) and the fact that $\mathcal{P}_-\mathcal{H} \subset W_2^2(\mathbb{R}, H)$, we get that $(T_q g_0 \mid g_0)_{\mathcal{H}} \geq 0$. This leads to a contradiction, since $T_q^- < 0$. Hence, $VV^* = \mathcal{P}_-$. □

Proof of Theorem 3. (I) In view of Corollary 2, we have $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, $\mathcal{P}_+ + \mathcal{P}_- = \mathcal{I}$.

(II) From Corollary 3 and Lemma 8, it follows that the operator T_q^- is unitarily equivalent to the operator $-K_1^2$.

(III) Let $\mathcal{F}$ denote the Fourier transform in $\mathcal{H}$ defined by

$$(\mathcal{F}f)(x) := \widehat{f}(x) = \lim_{a\to+\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ixt} f(t)dt, \quad x \in \mathbb{R}, \quad f \in \mathcal{H},$$

where the limit is understood in the topology of $\mathcal{H}$. Note that the operator $\mathcal{F}$ is unitary and (see (61)) $T_0 = \mathcal{F}^* \mathscr{T} \mathcal{F}$. According to Theorem 8, the operator $\mathfrak{B}\colon \mathcal{H} \to \mathcal{H}_+$ defined by

$$\mathfrak{B}f := \mathfrak{A}\mathcal{F}f, \ f \in \mathcal{H},$$

is also unitary and satisfies

$$T_q\mathfrak{B} = T_q\mathfrak{A}\mathcal{F} = \mathfrak{B}\mathcal{F}^*\mathscr{T}\mathcal{F} = \mathfrak{B}T_0.$$

Thus, the operator T_q^+ is unitarily equivalent to the operator T_0. □

Proof of Proposition 4. Let $q \in C_{b,s}(\mathbb{R}, \mathcal{B}(H))$. Consider the operator $L := (-T_q\mathcal{P}_-)^{1/4}$. It is nonnegative and continuous ($\|L\| \leq \|q\|_\infty^{1/4}$). From the obvious equality

$$(L\varphi)'' = q(L\varphi) - T_q(L\varphi) = q(L\varphi) + L^5\varphi, \quad \varphi \in W_2^2(\mathbb{R}, H),$$

it follows that $\|(L\varphi)''\| \leq 2\|q\|_\infty^{5/4}\|\varphi\|$. Hence, the operator L acts continuously from $\mathcal{H}$ into $W_2^2(\mathbb{R}, H)$. For each $x \in \mathbb{R}$, we define the operator $\beta(x)\colon \mathcal{H} \to H$ by $\beta(x)\varphi := (L\varphi)(x)$, $\varphi \in \mathcal{H}$. From the above, the operators $\beta(x)$ are continuous and the operator-valued function $\mathbb{R} \ni x \mapsto \beta(x) \in \mathcal{B}(\mathcal{H}, H)$ is continuous and bounded. Since the operator $L\colon \mathcal{H} \to \mathcal{H}$ is self-adjoint, we can easy verify that

$$L\psi = \int_{\mathbb{R}} \beta^*(t)\psi(t)dt, \qquad \psi \in \mathcal{H},$$

where the integral converges in the strong operator topology. Combining the above, we obtain

$$\big[(-T_q^-)^{1/2}\varphi\big](x) = (L^2\varphi)(x) = \lim_{a\to+\infty} \int_{-a}^{a} \beta(x)\beta^*(t)\varphi(t)dt, \quad x \in \mathbb{R}, \quad \varphi \in \mathcal{H}_-.$$

It is easy to see that the function $\mathscr{K}_q(x,t) := \beta(x)\beta^*(t)$ is the unique function for which the equality

$$\big[(-T_q^-)^{1/2}\varphi\big](x) = \lim_{a\to+\infty} \int_{-a}^{a} \mathscr{K}_q(x,t)\varphi(t)dt, \quad x \in \mathbb{R}, \quad \varphi \in \mathcal{H}_-,$$

holds.

Next, let $q \in \mathcal{Q}_\pi$. From the equalities (41) and (8), it follows that $T_q\mathcal{P}_- = -VK^2V^*$, and hence, $(-T_q\mathcal{P}_-)^{1/2} = VKV^*$. From this, we get

$$\big[(-T_q^-)^{1/2}\varphi\big](x) = \lim_{a\to+\infty} \int_{-a}^{a} \Psi(x)K\Psi^*(t)\varphi(t)dt, \quad x \in \mathbb{R}, \quad \varphi \in \mathcal{H}_-.$$

Therefore, $\mathscr{K}_q(x,t) = \Psi(x)K\Psi^*(t)$, $x, t \in \mathbb{R}$, and (see (8)) $-4\mathscr{K}_q(x,x) = -4\Psi(x)K\Psi^*(x) = q(x)$, $x \in \mathbb{R}$. □

Appendix. Some special operator equations. From the results of [10], we have the following lemma.

Lemma 9. *Let $K, M \in \mathcal{B}(H)$, $K > 0$ and $KM + MK = 0$. Then $M = 0$.*

Using Lemma 9, we can prove the following lemma.

Lemma 10. *Let $K, B \in \mathcal{B}(H)$, $K > 0$, $B \geq 0$, and*

$$KB + BK - 2BKB = 0. \tag{79}$$

Then the operator B is an orthogonal projection that commutes with K.

Proof. Assume that $B \neq 0$. For an arbitrary $\varepsilon > 0$, denote by P_ε the spectral projection of the operator B corresponding to the half-line (ε, ∞). Suppose $P_\varepsilon \neq 0$ for some $\varepsilon > 0$. Put $H_\varepsilon := P_\varepsilon H$, and let us consider the auxiliary operators $K_\varepsilon := P_\varepsilon K P_\varepsilon|_{H_\varepsilon}$, $B_\varepsilon := B|_{H_\varepsilon}$. Clearly, $K_\varepsilon > 0$, and the operator B_ε is invertible in the algebra $\mathcal{B}(H_\varepsilon)$. Multiplying the equation (79) on both sides by P_ε, we get

$$K_\varepsilon B_\varepsilon + B_\varepsilon K_\varepsilon - 2B_\varepsilon K_\varepsilon B_\varepsilon = 0.$$

Then, multiplying this equation on both sides by B_ε^{-1}, we obtain

$$K_\varepsilon(B_\varepsilon^{-1} - I_\varepsilon) + (B_\varepsilon^{-1} - I_\varepsilon)K_\varepsilon = 0,$$

where I_ε is the identity operator in the algebra $\mathcal{B}(H_\varepsilon)$. Using this equation and Lemma 9, we conclude that $B_\varepsilon^{-1} - I_\varepsilon = 0$, i.e., $B_\varepsilon = I_\varepsilon$ for all $\varepsilon > 0$. This implies that $BP_\varepsilon = P_\varepsilon$ for all $\varepsilon > 0$. Hence, $B = P_0$, where P_0 is the spectral projection of the operator B corresponding to the set $(0, \infty)$. Consequently, applying (79), we get

$$(I - B)KB = -(I - B)(BK - 2BKB) = 0, \quad BK(I - B) = -(KB - 2BKB)(I - B) = 0,$$

and therefore, $KB = (I - B + B)KB = BKB = BK(I - B + B) = BK$. □

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