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M. M. Sheremeta

PSEUDOSTARLIKE AND PSEUDOCONVEX DIRICHLET SERIES OF ORDER α AND TYPE β

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The concepts of the pseudostarlikeness of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$ and the pseudoconvexity of order α and type β are introduced for Dirichlet series with null abscissa of absolute convergence. In terms of coefficients, the pseudostarlikeness and the pseudoconvexity criteria of order α and type β are proved. Let $h \geq 1$, $\Lambda = (\lambda_k)$ be an increasing to $+\infty$ sequence of positive numbers $(\lambda_1 > h$. We call a conformal function of the form $F(s) = e^{sh} + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}$, $s = \sigma + it$, in $\Pi_0 = \{s\colon\ \mathrm{Re}\, s < 0\}$ pseudostarlike of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$ if

$$\left|\frac{F'(s)}{F(s)} - h\right| < \beta \left|\frac{F'(s)}{F(s)} - (2\alpha - h)\right|, \quad s \in \Pi_0.$$

The main results of the article are contained in Theorems 1 and 2. Theorem 1 states: *If $\alpha \in [0, 1)$ and $\beta \in (0, 1]$ such that*

$$\sum_{k=1}^{\infty} \{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\}|f_k| \leq 2\beta(h-\alpha)$$

then the function F is pseudostarlike of order α and type β. The corresponding results for Hadamard compositions of such series are also established.

1. Introduction. Let S be the class of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} f_n z^n \tag{1}$$

univalent in $\mathbb{D} = \{z\colon\ |z| < 1\}$. Function $f \in S$ is said to be starlike if $f(\mathbb{D})$ is starlike domain concerning of the origin. It is well known [1, p. 202] that the condition $\mathrm{Re}\,\{zf'(z)/f(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the starlikeness of f. A. W. Goodman ([2], see also [3, p. 9]) proved that if $\sum_{n=2}^{\infty} n|f_n| \leq 1$ then function (1) is starlike. The concept of the starlikeness of function (1) got the series of generalizations. I.S. Jack ([4]) studied starlike functions of order $\alpha \in [0, 1)$, i. e. such functions (1), for which $\mathrm{Re}\,\{zf'(z)/f(z)\} > \alpha$ $(z \in \mathbb{D})$. It is proved [4], [3, p. 13] that if $\sum_{n=2}^{\infty}(n-\alpha)|f_n| \leq 1-\alpha$ then function (1) is starlike function

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of order α. A function f of the form (1) is called *starlike function of order* $\alpha \in [0,\, 1)$ *and type* $\beta \in (0,\, 1]$ when

$$\Big|\frac{zf'(z)}{f(z)} - 1\Big| < \beta \cdot \Big|\frac{zf'(z)}{f(z)} + 1 - 2\alpha\Big|.$$

This concept was introduced by V. P. Gupta ([5]).

We point out that the concept of p-valent starlike function $f(z) = z^p + \sum_{n=p+1}^{\infty} f_n z^n$ has appeared comparatively recently (see, for example, [6], [7] and [3, p. 14]).

Let Σ be the class of functions

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n \tag{2}$$

analytic in $\mathbb{D}_0 = \{z\colon\ 0 < |z| < 1\}$. The function $f \in \Sigma$ is said to be *meromorphic starlike of order* $\alpha \in [0,\, 1)$ if $\mathrm{Re}\,\{-zf'(z)/f(z)\} > \alpha$ $(z \in \mathbb{D}_0)$. O.P. Juneja and T.R. Reddy [8] proved (see also [3, p. 14]) that if $\sum_{n=2}^{\infty}(n+\alpha)|f_n| \le 1-\alpha$ then function (2) is *meromorphic starlike function of order* α. According to B. A. Uralegaddi ([9]) the function (2) is said meromorphic starlike function of order $\beta \in (0,\, 1]$ if $|zf'(z) + f(z)| < \beta |zf'(z) - f(z)|$ for all $z \in \mathbb{D}_0$. Finally, combining these definitions, M. L. Mogra, T. R. Reddy and O. P. Juneja ([10]) call a function $f \in \Sigma$ to be meromorphic starlike of order $\alpha \in [0,\, 1)$ and type $\beta \in (0,\, 1]$ if

$$|zf'(z) + f(z)| < \beta |zf'(z) + (2\alpha - 1) f(z)|, \quad z \in \mathbb{D}_0,$$

and prove that if

$$\sum_{n=1}^{\infty}((1+\beta)n + \beta(2\alpha - 1) + 1)|f_n| \le 2\beta(1-\alpha),$$

then the function (2) is meromorphic starlike of order α and type β. Using this statement, O. M. Mulyava and Yu. S. Trukhan ([11]) indicated conditions on the parameters $a_1^{(1)}, a_2^{(1)}, a^{(0)}, a_2^{(0)}, a_3^{(0)}$ of the differential equation of S. Shah $z^2 w'' + (a_1^{(1)} z^2 + a_2^{(1)} z) w' + {}+(a_1^{(0)} z^2 + a_2^{(0)} z + a_3^{(0)}) w = 0$, under which this equation has a meromorphic starlike solution of order α and type β.

For power series $f_j(z) = \sum_{k=0}^{\infty} f_{k,j} z^k$ $(j = 1, 2)$ the series $(f_1 * f_2)(z) = \sum_{k=0}^{\infty} f_{k,1} f_{k,2} z^k$ is called the Hadamard composition (product) [12, 13]. Properties of this composition obtained by J. Hadamard found applications ([13, 14]) in the theory of the analytic continuation of the functions represented by power series. We remark also that singular points of the Hadamard composition are investigated in the article [15].

L. Zalzman [16] studied Hadamard compositions of univalent functions (1). For the functions $f_j(z) = 1/z + \sum_{k=1}^{\infty} f_{k,j} z^k \in \Sigma$ $(j = 1, 2)$ M. L. Mogra ([17]) defined Hadamard composition as $(f_1 * f_2)(z) = 1/z + \sum_{k=1}^{\infty} f_{k,1} f_{k,2} z^k$ and proved, for example, that if the functions f_j are meromorphically starlike of order $\alpha_j \in [0,\, 1)$ and $f_{k,j} \ge 0$ for all $k \ge 1$ then $f_1 * f_2$ is meromorphically starlike of order $\alpha = \max\{\alpha_1,\, \alpha_2\}$. Hadamard compositions of functions from the classes S and Σ were studied also by J. H. Choi, Y. C. Kim and S. Owa ([18]), M. K. Aouf and H. Silverman ([19]), J. Liu and R. Srivastava ([20]) and many other mathematicians.

Since Dirichlet series with positive increasing to $+\infty$ exponents are direct generalizations of power series, here was a necessity of a construction of the geometrical theory for the class of Dirichlet series, absolutely convergent in the half-plane $\Pi_0 = \{s\colon\ \mathrm{Re}\, s < 0\}$. The paper [21] is devoted to solve this problem (see also [3, p. 135–154]).

So, let $h \geq 1$, $\Lambda = (\lambda_k)$ be an increasing to $+\infty$ sequence of positive numbers ($\lambda_1 > h$) and $SD(\Lambda, 0)$ be the class of Dirichlet series

$$F(s) = e^{sh} + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}, \quad s = \sigma + it, \tag{3}$$

with the exponents Λ and the abscissa of absolute convergence $\sigma_a[F] = 0$. It is known [21] that each function $F \in SD(\Lambda, 0)$ is non-univalent in Π_0, but there exist conformal in Π_0 functions (3), and if $\sum\limits_{k=2}^{\infty} \lambda_k |f_k| \leq \lambda_1$ then function (3) is conformal in Π_0. A conformal function (3) in Π_0 is said to be *pseudostarlike* if $\mathrm{Re}\{F'(s)/F(s)\} > 0$ for $s \in \Pi_0$. In [21] (see also [3, p. 139]) it is proved that if $\sum\limits_{k=2}^{\infty} \lambda_k |f_k| \leq \lambda_1$ then function (3) is pseudostarlike.

The proposed paper continues the study of the geometric properties of Dirichlet series that are absolutely convergent in the half-plane.

2. Pseudostarlikeness of functions from $SD(\Lambda, 0)$. A conformal function (3) in Π_0 is said to be *pseudostarlike* of order α if

$$\mathrm{Re}\{F'(s)/F(s)\} > \alpha \in [0,\, 1), \quad s \in \Pi_0. \tag{4}$$

Since the inequality $|w - h| < |w - (2\alpha - h)|$ holds if and only if $\mathrm{Re}\, w > \alpha$, function (3) is pseudostarlike of order α if and only if

$$\left| \frac{F'(s)}{F(s)} - h \right| < \left| \frac{F'(s)}{F(s)} - (2\alpha - h) \right|, \quad s \in \Pi_0. \tag{5}$$

In view of (5) we call conformal function (3) in Π_0 *pseudostarlike* of order $\alpha \in [0,\, 1)$ and type $\beta \in (0,\, 1]$ if

$$\left| \frac{F'(s)}{F(s)} - h \right| < \beta \left| \frac{F'(s)}{F(s)} - (2\alpha - h) \right|, \quad s \in \Pi_0. \tag{6}$$

Theorem 1. *Let $F \in SD(\Lambda, 0)$ be a function of the form (3). If $\alpha \in [0,\, 1)$ and $\beta \in (0,\, 1]$ such that*

$$\sum_{k=1}^{\infty} \{(1 + \beta)\lambda_k - 2\beta\alpha - h(1 - \beta)\} |f_k| \leq 2\beta(h - \alpha) \tag{7}$$

then the function F is pseudostarlike of order α and type β.

Proof. Clearly, (6) holds if and only if

$$|F'(s) - hF(s)| - \beta |F'(s) - (2\alpha - h)F(s)| < 0, \quad s \in \Pi_0. \tag{8}$$

On the other hand,

$$|F'(s) - hF(s)| - \beta |F'(s) - (2\alpha - h)F(s)| =$$

$$= \left| he^{sh} + \sum_{k=1}^{\infty} \lambda_k f_k \exp\{s\lambda_k\} - he^{sh} - h \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\} \right| -$$

$$-\beta \left| he^{sh} + \sum_{k=1}^{\infty} \lambda_k f_k \exp\{s\lambda_k\} - (2\alpha - h)e^{sh} - (2\alpha - h) \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\} \right| =$$

$$= \left|\sum_{k=1}^{\infty}(\lambda_k - h)f_k \exp\{s\lambda_k\}\right| - \beta\left|2(h-\alpha)e^{sh} + \sum_{k=1}^{\infty}(\lambda_k - 2\alpha + h)f_k \exp\{s\lambda_k\}\right|.$$

Since $-|a+b| \le -|a| + |b|$ and $\sigma < 0$, in view of (7) we get

$$|F'(s) - hF(s)| - \beta|F'(s) - (2\alpha - h)F(s)| \le$$
$$\le \left|\sum_{k=1}^{\infty}(\lambda_k - h)f_k \exp\{s\lambda_k\}\right| - \left|2\beta(h-\alpha)e^{sh}\right| + \left|\beta\sum_{k=1}^{\infty}(\lambda_k - 2\alpha + h)f_k \exp\{s\lambda_k\}\right| \le$$
$$\le \sum_{k=1}^{\infty}(\lambda_k - h)|f_k| \exp\{\sigma\lambda_k\} - 2\beta(h-\alpha)e^{\sigma h} + \beta\sum_{k=1}^{\infty}(\lambda_k - 2\alpha + h)|f_k| \exp\{\sigma\lambda_k\} =$$
$$= e^{\sigma h}\left(\sum_{k=1}^{\infty}\{(1+\beta)\lambda_k - 2\beta\alpha + h(1-\beta)\}|f_k| \exp\{\sigma(\lambda_k - h)\} - 2\beta(h-\alpha)\right) <$$
$$< \sum_{k=1}^{\infty}\{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\}|f_k| - 2\beta(h-\alpha) \le 0,$$

i. e. (8) holds. □

Theorem 2. *If a function $F \in SD(\Lambda, 0)$ of form* (3) *is pseudostarlike of order $\alpha \in [0,1)$ and type $\beta \in (0,1]$ and $f_k \le 0$ for all $k \ge 1$ then inequality* (7) *holds.*

Proof. Since function (3) is pseudostarlike of order α and type β and $f_k = -|f_k|$ for all $k \ge 1$ in view of (8) as above we have for all $s \in \Pi_0$

$$\left|\frac{-\sum_{k=1}^{\infty}(\lambda_k - h)|f_k| \exp\{s\lambda_k\}}{2(h-\alpha)e^{sh} - \sum_{k=1}^{\infty}(\lambda_k - 2\alpha + h)|f_k| \exp\{s\lambda_k\}}\right| = \left|\frac{F'(s) - hF(s)}{F'(s) - (2\alpha - h)F(s)}\right| < \beta.$$

Therefore,

$$\mathrm{Re}\frac{\sum_{k=1}^{\infty}(\lambda_k - h)|f_k| \exp\{s\lambda_k\}}{2(h-\alpha)e^{sh} - \sum_{k=1}^{\infty}(\lambda_k - 2\alpha + h)|f_k| \exp\{s\lambda_k\}} < \beta,$$

whence for all $\sigma < 0$ we obtain

$$\frac{\sum_{k=1}^{\infty}(\lambda_k - h)|f_k| \exp\{\sigma\lambda_k\}}{2(h-\alpha)e^{\sigma h} - \sum_{k=1}^{\infty}(\lambda_k - 2\alpha + h)|f_k| \exp\{\sigma\lambda_k\}} < \beta.$$

Letting $\sigma \to 0$ from here we get

$$\sum_{k=1}^{\infty}(\lambda_k - h)|f_k| \le \beta\left(2(h-\alpha) - \sum_{k=1}^{\infty}(\lambda_k - 2\alpha + h)|f_k|\right),$$

whence (7) follows. □

Choosing $\beta = 1$ from Theorems 1 and 2 we obtain the following statement.

Corollary 1. *In order for function* (3) *to be pseudostarlike of order* $\alpha \in [0,\ 1)$ *it is sufficient and in the case when* $f_k \le 0$ *for all* $k \ge 1$ *it is necessary that*

$$\sum_{k=1}^{\infty} (\lambda_k - \alpha)|f_k| \le h - \alpha. \tag{9}$$

3. Pseudostarlikeness of functions from $\Sigma D(\Lambda, 0)$**.** Let $h \ge 1$, $\Lambda = (\lambda_k)$ be an increasing to $+\infty$ sequence of positive numbers and $\Sigma D(\Lambda, 0)$ be the class of Dirichlet series

$$F(s) = e^{-sh} + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}, \quad s = \sigma + it, \tag{10}$$

absolutely convergent in Π_0. Dirichlet series (9) is called Σ-*pseudostarlike of order* $\alpha \in [0,\ 1)$ if

$$\mathrm{Re}\{F'(s)/F(s)\} < -\alpha \in [0,\ 1), \quad s \in \Pi_0. \tag{11}$$

Since the inequality $|w + h| < |w + (2\alpha - h)|$ holds if and only if $\mathrm{Re}\, w < -\alpha$, function (10) is Σ-pseudostarlike of order α if and only if

$$\left|\frac{F'(s)}{F(s)} + h\right| < \left|\frac{F'(s)}{F(s)} + (2\alpha - h)\right|, \quad s \in \Pi_0. \tag{12}$$

In view of (11) we call function (10) Σ-*pseudostarlike of order* $\alpha \in [0,\ 1)$ *and type* $\beta \in (0,\ 1]$ if

$$\left|\frac{F'(s)}{F(s)} + h\right| < \beta\left|\frac{F'(s)}{F(s)} + (2\alpha - h)\right|, \quad s \in \Pi_0. \tag{13}$$

Theorem 3. *Let* $F \in \Sigma D(\Lambda, 0)$ *be a function of form* (10)*. If* $\alpha \in [0,\ 1)$ *and* $\beta \in (0,\ 1]$ *such that*

$$\sum_{k=1}^{\infty} \{(1+\beta)\lambda_k + 2\beta\alpha + h(1-\beta)\}|f_k| \le 2\beta(h - \alpha) \tag{14}$$

then F *is* Σ*-pseudostarlike of order* α *and type* β*.*

Proof. Clearly, (13) holds if and only if

$$|F'(s) + hF(s)| - \beta|F'(s) + (2\alpha - h)F(s)| < 0, \quad s \in \Pi_0. \tag{15}$$

On the other hand, as in the proof of Theorem 1

$$|F'(s) + hF(s)| - \beta|F'(s) + (2\alpha - h)F(s)| = e^{-\sigma h}\left(\left|\sum_{k=1}^{\infty}(\lambda_k + h)f_k \exp\{s(\lambda_k + h)\}\right| -\right.$$

$$\left. -\beta\left|2(\alpha - h) + \sum_{k=1}^{\infty}(\lambda_k + 2\alpha - h)f_k \exp\{s(\lambda_k + h)\}\right|\right) \le$$

$$\le e^{-\sigma h}\left(\sum_{k=1}^{\infty}(\lambda_k + h)|f_k| \exp\{\sigma(\lambda_k + h)\} - \right.$$

$$-\beta\left(2(h-\alpha)-\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)|f_k|\exp\{\sigma(\lambda_k+h)\}\right)\Bigg) =$$

$$= e^{-\sigma h}\left(\sum_{k=1}^{\infty}\{(1+\beta)\lambda_k+2\beta\alpha+h(1-\beta)\}|f_k|\exp\{\sigma(\lambda_k+h)\}-2\beta(h-\alpha)\right) <$$

$$< e^{-\sigma h}\left(\sum_{k=1}^{\infty}\{(1+\beta)\lambda_k+2\beta\alpha+h(1-\beta)\}|f_k|\}-2\beta(h-\alpha)\right)\le 0$$

i.e. (15) holds. □

Theorem 4. *If the function $F\in\Sigma D(\Lambda,0)$ of form* (10) *is Σ-pseudostarlike of order α and type β and $f_k\ge 0$ for all $k\ge 1$ then inequality* (14) *holds.*

Proof. Since function (10) is Σ-pseudostarlike of order α and type β and $f_k=|f_k|$ for all $k\ge 1$ in view of (15) as above we have for all $s\in\Pi_0$

$$\mathrm{Re}\,\frac{\sum_{k=1}^{\infty}(\lambda_k+h)f_k\exp\{s\lambda_k\}}{2(h-\alpha)e^{-sh}-\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_k\exp\{s\lambda_k\}}\le$$

$$\le\frac{\left|\sum_{k=1}^{\infty}(\lambda_k+h)f_k\exp\{s\lambda_k\}\right|}{\left|2(h-\alpha)e^{-sh}-\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_k\exp\{s\lambda_k\}\right|}=$$

$$=\frac{\left|\sum_{k=1}^{\infty}(\lambda_k+h)f_k\exp\{s\lambda_k\}\right|}{\left|2(\alpha-h)e^{-sh}+\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_k\exp\{s\lambda_k\}\right|}=\left|\frac{F'(s)+hF(s)}{F'(s)+(2\alpha-h)F(s)}\right|<\beta,$$

and, therefore, for all $\sigma<0$ we obtain

$$\frac{\sum_{k=1}^{\infty}(\lambda_k+h)f_k\exp\{\sigma\lambda_k\}}{2(h-\alpha)e^{-\sigma h}-\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_k\exp\{\sigma\lambda_k\}}<\beta.$$

Letting $\sigma\to 0$ from here we get

$$\sum_{k=1}^{\infty}(\lambda_k+h)f_k\le\beta\left(2(h-\alpha)-\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)|f_k|\right),$$

whence (14) follows. Theorem 4 is proved. □

Choosing $\beta=1$ from Theorems 3 and 4 we obtain the following statement.

Corollary 2. *In order for function* (10) *to be Σ-pseudostarlike of order $\alpha\in[0,1)$ it is sufficient and in the case when $f_k\ge 0$ for all $k\ge 1$ it is necessary that*

$$\sum_{k=1}^{\infty}(\lambda_k+\alpha)|f_k|\le h-\alpha. \qquad (16)$$

4. Hadamard compositions. For Dirichlet series $F_j(s) = \sum_{k=0}^{\infty} f_{k,j}\exp\{s\lambda_k\}$ $(j = 1, 2)$ in [22] a composition of Hadamard $(F_1 * F_2)(s) = \sum_{k=0}^{\infty} f_{k,1}f_{k,2}\exp\{s\lambda_k\}$ is defined and some its properties are studied. Suppose that $F_j \in SD(\Lambda, 0)$. Then the Hadamard composition has the form

$$(F_1 * F_2)(s) = e^{sh} + \sum_{k=1}^{\infty} f_{k,1}f_{k,2}\exp\{s\lambda_k\}. \tag{17}$$

Corollary 1 implies the following statement.

Corollary 3. *If the functions $F_j \in SD(\Lambda, 0)$ are pseudostarlike of orders $\alpha_j \in [0,\ 1)$ and $f_{k,j} \le 0$ for all $k \ge 1$ and $j = 1,\ 2$ then Hadamard composition $F_1 * F_2$ is pseudostarlike of order $\alpha = \max\{\alpha_j\colon\ j = 1,\ 2\}$.*

Indeed, from (9) it follows that $|f_{k,j}| < (h-\alpha_j)/(\lambda_k-\alpha_j) < 1$ for all $k \ge 1$ and, therefore,

$$\sum_{k=1}^{\infty} \frac{\lambda_k - \alpha_1}{h - \alpha_1}|f_{k,1}f_{k,2}| \le \sum_{k=1}^{\infty} \frac{\lambda_k - \alpha_1}{h - \alpha_1}|f_{k,1}| < 1$$

and similarly $\sum\limits_{k=1}^{\infty} \frac{\lambda_k-\alpha_2}{h-\alpha_2}|f_{k,1}f_{k,2}| < 1$, i. e.

$$\sum_{k=1}^{\infty} \frac{\lambda_k - \max\{\alpha_1, \alpha_2\}}{h - \max\{\alpha_1, \alpha_2\}}|f_{k,1}f_{k,2}| = \sum_{k=1}^{\infty} \max\left\{\frac{\lambda_k - \alpha_1}{h - \alpha_1},\ \frac{\lambda_k - \alpha_2}{h - \alpha_2}\right\}|f_{k,1}f_{k,2}| < 1,$$

and by Corollary 1 the function $F_1 * F_2$ is pseudostarlike of order $\alpha = \max\{\alpha_j\colon\ j = 1,\ 2\}$.

Using Theorem 1 and 2 we get the following statement.

Corollary 4. *Let the functions $F_j \in SD(\Lambda, 0)$ are pseudostarlike of orders $\alpha \in [0,\ 1)$ and type $\beta_j \in (0,\ 1]$ for $j = 1,\ 2$. If $f_{k,j} \le 0$ for all $k \ge 1$ and $j = 1,\ 2$ then Hadamard composition $F_1 * F_2$ is pseudostarlike of order α and type $\beta = \min\{\beta_1,\ \beta_2\}$.*

Indeed, from (7) it follows that $|f_{k,j}| < \frac{2\beta_j h - 2\beta_j\alpha}{(1+\beta_j)\lambda_k - 2\beta_j\alpha + h(1-\beta_j)} < 1$ for all $k \ge 1$ and, therefore, as above we have

$$\sum_{k=1}^{\infty} \frac{(1+\beta_j)\lambda_k - 2\beta_j\alpha - h(1-\beta_j)}{2\beta_j(h-\alpha)}|f_{k,1}f_{k,2}| \le 1, \quad j = 1,\ 2.$$

Hence

$$\sum_{k=1}^{\infty} \max\left\{\frac{(1+\beta_j)\lambda_k - 2\beta_j\alpha - h(1-\beta_j)}{2\beta_j(h-\alpha)}\colon\ j = 1,\ 2\right\}|f_{k,1}f_{k,2}| \le 1.$$

Since

$$\max\left\{\frac{(1+\beta_j)\lambda_k - 2\beta_j\alpha - h(1-\beta_j)}{2\beta_j(h-\alpha)}\colon\ j = 1,\ 2\right\} =$$

$$= \max\left\{\frac{\lambda_k - 2\alpha - h + (\lambda_k + h)/\beta_j}{2(h-\alpha)}\colon\ j = 1,\ 2\right\} = \frac{\lambda_k - 2\alpha - h + (\lambda_k + h)/\min\{\beta_1,\ \beta_2\}}{2(h-\alpha)},$$

by Theorem 1 the function $F_1 * F_2$ is pseudostarlike of order α and type $\beta = \min\{\beta_1,\ \beta_2\}$.

Let us move on to the functions from the class $\Sigma D(\Lambda, 0)$. If $F_j(s) = e^{-hs}+\sum_{k=1}^{\infty} f_{k,j}\exp\{s\lambda_k\}$ $(j = 1, 2)$ then the Hadamard composition is defined by the equality

$$(F_1 * F_2)(s) = e^{-sh} + \sum_{k=1}^{\infty} f_{k,1}f_{k,2}\exp\{s\lambda_k\}. \tag{18}$$

Corollary 2 implies the following statement.

Corollary 5. *If the functions $F_j \in \Sigma D(\Lambda, 0)$ are Σ-pseudostarlike of orders $\alpha_j \in [0,\ 1)$ and $f_{k,j} \ge 0$ for all $k \ge 1$ and $j = 1,\ 2$ then Hadamard composition $F_1 * F_2$ is Σ-pseudostarlike of order $\alpha = \max\{\alpha_j\colon\ j = 1,\ 2\}$.*

Indeed, from (16) it follows that $|f_{k,j}| < 1$ for all $k \ge 1$ and, therefore, as above we have $\sum\limits_{k=1}^{\infty} \frac{\lambda_k+\alpha_1}{h-\alpha_1}|f_{k,1}f_{k,2}| < 1$ and $\sum\limits_{k=1}^{\infty} \frac{\lambda_k+\alpha_2}{h-\alpha_2}|f_{k,1}f_{k,2}| < 1$, i. e.

$$\sum_{k=1}^{\infty} \frac{\lambda_k + \max\{\alpha_1, \alpha_2\}}{h - \max\{\alpha_1, \alpha_2\}}|f_{k,1}f_{k,2}| = \sum_{k=1}^{\infty} \max\left\{\frac{\lambda_k + \alpha_1}{h-\alpha_1},\ \frac{\lambda_k+\alpha_2}{h-\alpha_2}\right\}|f_{k,1}f_{k,2}| < 1,$$

and by Corollary 2 the function $F_1 * F_2$ is Σ-pseudostarlike of order $\alpha = \max\{\alpha_j\colon\ j = 1,\ 2\}$.

Using Theorems 3 and 4 and combining the proofs of Corollaries 4 and 5, it is not difficult to prove the following statement.

Corollary 6. *Let the functions $F_j \in \Sigma D(\Lambda, 0)$ are Σ-pseudostarlike of order $\alpha \in [0,\ 1)$ and types $\beta_j \in (0,\ 1]$. If $f_{k,j} \ge 0$ for all $k \ge 1$ and $j = 1,\ 2$ then Hadamard composition $F_1 * F_2$ is Σ-pseudostarlike of order α and type $\beta = \min\{\beta_1,\ \beta_2\}$.*

5. Pseudoconvex Dirichlet series. A conformal function (3) in Π_0 is said to be *pseudoconvex* if $\mathrm{Re}\{F''(s)/F'(s)\} > 0$ for $s \in \Pi_0$. In [21] and [3, p. 139] it is proved that if $\sum_{k=2}^{\infty} \lambda_k^2|f_k| \le \lambda_1^2$ then function (3) is pseudoconvex. Here we call the function (3) *pseudoconvex of order* $\alpha \in [0,\ 1)$ if $\mathrm{Re}\{F''(s)/F'(s)\} > \alpha$, and pseudoconvex of order α and type $\beta \in (0,\ 1]$ if

$$|F''(s)/F'(s) - h| < \beta|F''(s)/F'(s) - (2\alpha - h)|$$

for all $s \in \Pi_0$.

Since $F''(s)/F'(s) = G'(s)/G(s)$, where $G(s) = e^{sh} + \sum_{k=1}^{\infty} g_k\exp\{s\lambda_k\}$ and $g_k = \lambda_k f_k/h$, the function F is pseudoconvex of order $\alpha \in [0,\ 1)$ and type $\beta \in (0,\ 1]$ if and only if the function G is pseudostarlike of order $\alpha \in [0,\ 1)$ and type $\beta \in (0,\ 1]$. Therefore, from the results proved above for pseudostarlike functions, one can easily obtain the corresponding results for pseudoconvex functions. For example, the following statement is valid.

Proposition 1. *In order for the function $F \in SD(\Lambda, 0)$ of form (3) to be pseudoconvex of order $\alpha \in [0,\ 1)$ and type $\beta \in (0,\ 1]$ it is sufficient and in the case when $f_k \le 0$ for all $k \ge 1$ it is necessary that $\sum_{k=1}^{\infty} \lambda_k\{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\}|f_k| \le 2h\beta(h - \alpha)$.*

Similarly, we call the function (10) Σ-*pseudoconvex of order* $\alpha \in [0, 1)$ if $\mathrm{Re}\{F''(s)/F'(s)\} < -\alpha$, and Σ-*pseudoconvex of order α and type* $\beta \in (0,\ 1]$ if

$$|F''(s)/F'(s) + h| < \beta|F''(s)/F'(s) + (2\alpha - h)|$$

for all $s \in \Pi_0$. Now $F''(s)/F'(s) = G'(s)/G(s)$, where $G(s) = e^{-sh} + \sum_{k=1}^{\infty} g_k\exp\{s\lambda_k\}$ and $g_k = -\lambda_k f_k/h$. Since $g_k \ge 0$ if $f_k \le 0$, from Theorem 3 and 4 we get the following statement.

Proposition 2. *In order for the function $\Sigma F \in D(\Lambda, 0)$ of form (10) to be Σ-pseudoconvex of order $\alpha \in [0,\ 1)$ and type $\beta \in (0,\ 1]$ it is sufficient and in the case when $f_k \le 0$ for all $k \ge 1$ it is necessary that $\sum_{k=1}^{\infty} \lambda_k\{(1+\beta)\lambda_k + 2\beta\alpha + h(1-\beta)\}|f_k| \le 2h\beta(h - \alpha)$.*

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Ivan Franko National University of Lviv
m.m.sheremeta@gmail.com

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