УДК 517.537.72

M. M. Sheremeta

ON THE TRANSFINITE DENSITY OF SEQUENCES AND ITS APPLICATIONS TO DIRICHLET SERIES

M. M. Sheremeta. *On the transfinite density of sequences and its applications to Dirichlet series*, Mat. Stud. **63** (2025), 115–128.

For an increasing to ∞ sequence (λ_n) of positive numbers let
$$n(t)=\sum_{\lambda_n\le t}1,\ N(x)=\int_0^x\frac{n(t)}{t}dt,\ L_k(t)=\sum_{\lambda_n\le t}\prod_{j=0}^{k-1}\frac{1}{\ln_j\lambda_n}$$
for $k\ge 1$ and $t\ge t_k=\exp_k(0)$, where $\ln_j x$ is the j-th iteration of the logarithm and $\exp_k(x)$ is the k-th iteration of the exponent. The quantities $D(0)=\varlimsup\limits_{t\to+\infty}\frac{n(t)}{t}$ and $\overline{D}^*=\varlimsup\limits_{t\to+\infty}\frac{1}{t}\int_0^t\frac{n(x)}{x}dx$ are called the upper density and upper average density of (λ_n) respectively. Moreover, let $D_k(0)=\varlimsup\limits_{t\to+\infty}\frac{L_k(t)}{\ln_k t}$ be the upper k-logarithmic density and $D=\lim\limits_{k\to\infty}D_k(0)$ be the maximal transfinite density of (λ_n). In the works of many authors devoted to lacunary power series and Dirichlet series, estimates of the canonical product $\Lambda(z)=\prod\limits_{n=0}^{\infty}(1+z^2/\lambda_n^2)$ are used, which is an entire function if $D(0)<+\infty$.

Here various properties of k-logarithmic densities are studied and the estimate
$$\varlimsup_{r\to+\infty}\frac{\ln\Lambda(r)}{r}\le\pi D$$
is proved. This allows us to replace $\overline{D}^*$ with D in many results of G. Polya, S. Mandelbrojt and other authors.

1. Introduction. Let (λ_n) be a sequence of positive numbers such that
$$\lambda_{n+1}-\lambda_n\ge p>0,\quad n\ge 1. \tag{1}$$
and $n(t)=\sum_{\lambda_n\le t}1$. For $\xi\in[0,1)$ G. Polya ([1]) introduced the quantities
$$d(\xi)=\varliminf_{t\to+\infty}\frac{n(t)-n(t\xi)}{(1-\xi)t},\quad D(\xi)=\varlimsup_{t\to+\infty}\frac{n(t)-n(t\xi)}{(1-\xi)t},$$
which are called the lower density and the upper densities of the sequence (λ_n) on the basis $\xi\in[0,1)$ respectively. In turn, the quantities $d(0),D(0),d(1)=\lim\limits_{\xi\to1}d(\xi),D(1)=\lim\limits_{\xi\to1}D(\xi)$ are called lower, upper, minimal and maximal densities respectively. If $d(0)=D(0)=\Delta$ then the sequence (λ_n) is called measurable and Δ is called its density. G. Polya ([1]) applied this quantities to various questions in the theory of analytic functions. For example, he showed

2020 *Mathematics Subject Classification:* 11B50, 30B50.
Keywords: positive sequence; k-logarithmic density; Dirichlet series.
doi:10.30970/ms.63.2.115-128

that if a unique singular point of a power series lies on the circle of convergence of its sum then the maximal density of non-zero coefficients is equal to 1. It has also been proven that if the maximal density of non-zero coefficients of an entire function f is equal to δ then in each angle with a deviation greater than $2\pi\delta$ the function f has the same order and type as in the full plane.

Let $\ln_j x$ denote the j-th iteration of the logarithm: $\ln_0 x = x$ and $\ln_j x = \ln\ln_{j-1} x$ for $j \ge 1$. Similarly: $\exp_0(x) = x$ and $\exp_j(x) = \exp(\exp_{j-1}(x))$ for $j \ge 1$. Suppose that $\xi \in [0,1)$, $t \ge t_k = \exp_k(0)$ and for $k \ge 1$ define

$$L_k(t) = \sum_{\lambda_n \le t} \prod_{j=0}^{k-1} \frac{1}{\ln_j \lambda_n},\ \ l_k(t,\xi) = \exp_k(\xi \ln_k t),\ \ L_k^*(t,\xi) = L_k(t) - L_k(l_k(t),\xi)).$$

As in [2], the quantities

$$d_k(\xi) = \varliminf_{t\to+\infty} \frac{L_k^*(t,\xi)}{(1-\xi)\ln_k t}, \quad D_k(\xi) = \varlimsup_{t\to+\infty} \frac{L_k^*(t,\xi)}{(1-\xi)\ln_k t}$$

we will call, respectively, the lower and upper k-logarithmic densities of the sequence (λ_n) on the basis $\xi \in [0,1)$ and the quantities $d_k(0)$ and $D_k(0)$ we will call, respectively, the lower and upper k-logarithmic densities of the sequence (λ_n). If $d_k(0) = D_k(0) = \Delta_k$ then the sequence (λ_n) is called k-logarithmic measurable and Δ_k is called its k-logarithmic density. In [2] it is proven that if a sequence is measurable then it is k-logarithmic measurable for each $k \ge 1$, and the converse is not true.

Finally ([2]), the functions $d_k(\xi)$ and $D_k(\xi)$ are continuous on $[0,1)$ and there are limits

$$d_k(1) = \lim_{\xi\to 1} d_k(\xi), \quad D_k(1) = \lim_{\xi\to 1} D_k(\xi),$$

which are called minimal and maximal k-logarithmic densities, respectively. The sequences $(d_k(\xi))$ and $(d_k(1))$ are non-decreasing and there exists a limit

$$d = \lim_{k\to\infty} d_k(\xi) = \lim_{k\to\infty} d_k(1).$$

The sequences $(D_k(\xi))$ and $(D_k(1))$ are non-increasing and there exists a limit

$$D = \lim_{k\to\infty} D_k(\xi) = \lim_{k\to\infty} D_k(1).$$

The numbers d and D are called ([2]) minimal and maximal transfinite densities of (λ_n), respectively. If $d = D$ then the sequence (λ_n) is called transfinite measurable.

Suppose that an entire transcendental function

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$$

with $a_n \ne 0$ of finite order has a Borel exceptional value A, i.e. $f(z) = A + F(z)\exp\{bz^p\}$, where p is a natural number equal to the order of f and F is an entire function whose growth does not exceed the minimal type of order p. A. Pfluger and G. Polya ([4]) showed that then the coefficient density (i.e. the density of the sequence of the exponents λ_n) is equal to one of the fractions of the form s/p, $s \in \{1,2,\dots,p\}$. This statement is correct ([3])

if replace coefficient density with coefficient k-logarithmic and transfinite densities. Hence, for example, it follows that in this case

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} = \frac{s}{p} \ln \lambda_n + o(\ln \lambda_n)$$

as $n \to \infty$. The previous equality defines more precisely Fejér's result ([5]) that if

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} < +\infty$$

then f does not have Borel exceptional values. Moreover, T. Murai in [6] proved that an entire function with Fejér gaps does not have a finite Nevanlinna deficient value.

Density concepts of sequences were used by S. Mandelbrojt, whose monograph [7] significantly influenced the results obtained in the proposed article.

2. K-logarithmic functions of excess and deficiency. Let (λ_n) be an increasing sequence of positive numbers and

$$\mathfrak{N}_k(t) = \int_{t_k}^{t} \frac{n(x)}{x} d \ln_k x,$$

where as above $n(t) = \sum_{\lambda_n \le t} 1$ and $t_k = \exp_k(0)$. We will call

$$\Delta_k(t) = \frac{\mathfrak{N}_k(t)}{\ln_k t}, \quad \Delta_k^*(x) = \sup_{t \ge x} \Delta_k(t), \quad \Delta_{*k}(x) = \inf_{t \ge x} \Delta_k(t)$$

the function, the upper and lower functions of the k-logarithmic density of the sequence (λ_n) respectively.

In what follows we will assume that the upper density of the sequence (λ_n) is bounded, i.e. $D(0) < +\infty$. Then $D_k(0) < +\infty$ for all $k \ge 1$, $n(t) = O(t)$ as $t \to +\infty$, and it is easy to show that

$$D_k(0) = \varlimsup_{t \to +\infty} \Delta_k(t) = \lim_{t \to +\infty} \Delta_k^*(t), \quad d_k(0) = \varliminf_{t \to +\infty} \Delta_k(t) = \lim_{t \to +\infty} \Delta_{*k}(t).$$

The function $\Delta_k^*(t)$ is the smallest of the continuous non-increasing functions $\varphi(t)$ such that $\varphi(t) \ge \Delta_k(t)$, and the function $\Delta_{*k}(t)$ is the greatest of the continuous non-decreasing functions, which do not exceed $\Delta_k(t)$.

We will call

$$\nu_k(\Delta) = \sup_{t \ge t_k} \{(\Delta_k(t) - \Delta) \ln_k t\} = \sup_{t \ge t_k} \int_{t_k}^{t} \left(\frac{n(x)}{x} - \Delta \right) d \ln_k x$$

the k-logarithmic function of the excess of the sequence (λ_n). It's easy to see that $\nu_k(\Delta)$ is a non-negative and non-increasing function. For every $\Delta < D_k(0)$ we have $\nu_k(\Delta) = +\infty$. If $\Delta > D_k(0)$ then $\nu_k(\Delta) < +\infty$. Put $M_k = \sup_{t \ge t_k} \Delta_k(t)$. If $\Delta \ge M_k$ then $\nu_k(\Delta) = 0$. If $D_k(0) < M_k$ then the function $\nu_k(\Delta)$ is convex on the interval $(D_k(0), M_k]$, because the functions $f_t(\Delta) = \int_{t_k}^{t} \frac{n(x)}{x} d \ln_k x - \Delta \ln_k t$, for which $\nu_k(\Delta)$ is the upper envelope, are linear and collectively bounded for $D_k(0) < \Delta' \le \Delta \le M_k$ for any $\Delta' \in (D_k(0), M_k]$. If $\Delta \to D_k(0)$ then $\nu_k(\Delta) \to \nu_k(D_k(0) + 0) = \nu_k(D_k(0))$. If $D_k(0) < M_k$ then there exits a function $\nu_k^{-1}(x)$

inverse to $\nu_k(\Delta)$, which is defined on the $[0, \nu_k(D_k(0))]$, continuous, convex and decreasing from M_k to $D_k(0)$. If $\nu_k(D_k(0)) < +\infty$ then we put $\nu_k^{-1}(x) = D_k(0)$ for $x \geq \nu_k(D_k(0))$. If $D_k(0) = M_k$ then we put $\nu_k^{-1}(x) = D_k(0)$ for $x > 0$.

We will call

$$\mu_k(\Delta) = \sup_{t \geq t_k}\{(\Delta - \Delta_k(t)) \ln_k t\} = \sup_{t \geq t_k} \int_{t_k}^{t} \left(\Delta - \frac{n(x)}{x}\right) d\ln_k x$$

the k-logarithmic function of the deficiency of the sequence (λ_n). The function $\mu_k(\Delta)$ is a non-negative and non-decreasing. If $\Delta > d_k(0)$ then $\mu_k(\Delta) = +\infty$ and if $\Delta < d_k(0)$ then $\mu_k(\Delta) < +\infty$. For $\Delta \leq m_k := \inf_{t \geq t_k} \Delta_k(t)$ we have $\mu_k(\Delta) = 0$. If $d_k(0) > m_k$ then $\mu_k(\Delta)$ is increasing and convex on the interval $[m_k, d_k(0))$.

Let $(\lambda_n^{(1)})$ and $(\lambda_n^{(2)})$ be increasing sequences of positive numbers that have no common members, and let (λ_n) be an increasing sequence such that $\{\lambda_n\} = \{\lambda_n^{(1)}\} \bigcup \{\lambda_n^{(2)}\}$. If there exist constants $l > 0$ and $p \geq 0$ such that $|\lambda_n - ln| \leq p$ for all $n \geq 1$ then the sequences $(\lambda_n^{(1)})$ and $(\lambda_n^{(2)})$ are called ([7, p. 61]) *complementary by index* (l, p). If $l = 1$ and $p = 0$ then $\{\lambda_n\} = \{\lambda_n^{(1)}\} \bigcup \{\lambda_n^{(2)}\}$ is the sequence of positive integers. From the definition we get $nl - p \leq \lambda_n \leq nl + p$, i.e.

$$(x - p)/l \leq n(x) < (x + p)/l. \tag{2}$$

From hence in view of Theorem 4.2 from [3] it follows that

$$D_k^{(1)}(0) + d_k^{(2)}(0) = 1/l \tag{3}$$

for all $k \geq 1$ and, therefore, $D^{(1)} + d^{(2)} = 1/l$, where $D^{(1)}$ and $d^{(2)}$ are maximal and minimal transfinite densities of the sequences $(\lambda_n^{(1)})$ and $(\lambda_n^{(2)})$, respectively. Equalities (3) complement the equalities $D^{(1)}(0) + d^{(2)}(0) = 1/l$ and $\overline{D}^{*(1)} + \overline{D}_*^{(2)} = 1/l$ proved in [6, p. 61–62], where $\overline{D}^*$ and $\overline{D}_*$ are the upper and lower averaged densities defined as

$$\overline{D}^* = \varlimsup_{t \to +\infty} \frac{1}{t} \int_0^t \frac{n(x)}{x} dx, \quad \overline{D}_* = \varliminf_{t \to +\infty} \frac{1}{t} \int_0^t \frac{n(x)}{x} dx.$$

From (2) for the sequences $(\lambda_n^{(1)})$ and $(\lambda_n^{(2)})$ we get

$$\frac{1}{l} - \left(1 + \frac{p}{l}\right) \frac{1}{t} \leq \frac{n^{(1)}(t)}{t} + \frac{n^{(2)}(t)}{t} \leq \frac{1}{l} + \frac{p}{lt}. \tag{4}$$

Let (r) denote the smallest integer greater than or equal to r. Then the following assertion follows from (4).

Proposition 1. *If the sequences* $(\lambda_n^{(1)})$ *and* $(\lambda_n^{(2)})$ *are complementary by index* (l, p) *and* $\lambda_n^{(3)} = \lambda_{n+(p/l)}^{(1)}$ *then* $D_k^{(3)}(0) + d_k^{(2)}(0) = 1/l$, $\Delta^{*(3)}(t) + \Delta_*^{(2)}(t) \leq 1/l$ *for* $t > \max\{\lambda_{(p/l)}^{(1)}, t_k\}$ *and* $\nu_k^{(3)}(1/l - \Delta) \leq \mu^{(2)}(\Delta) + A$, *where* $D_k^{(3)}(0)$, $\Delta^{*(3)}(t)$, $\nu_k^{(3)}(\Delta)$ *are the corresponding characteristics of the sequence* $(\lambda_n^{(3)})$, $d_k^{(2)}(0)$, $\Delta_*^{(2)}(t)$, $\mu^{(2)}(\Delta)$ *are the corresponding characteristics of the sequence* $(\lambda_n^{(3)})$ *and* $A \equiv \mathrm{const} \geq 0$.

Proof. Let us prove, for example, the third statement. Indeed, (4) implies

$$\nu_k^{(3)}(1/l - \Delta) = \sup_{t \geq t_k} \int_{t_k}^{t} \left(\frac{n^{(3)}(x)}{x} - \frac{1}{l} + \Delta\right) d\ln_k x =$$

$$= \sup_{t\ge t_k}\int_{t_k}^{t}\left(\frac{n^{(1)}(x)+p/l}{x}-\frac{1}{l}+\Delta\right)d\ln_k x \le \sup_{t\ge t_k}\int_{t_k}^{t}\left(\Delta-\frac{n^{(2)}(x)}{x}+\frac{(p/l)-p/l}{x}\right)d\ln_k x \le$$
$$\le \sup_{t\ge t_k}\int_{t_k}^{t}\left(\Delta-\frac{n^{(2)}(x)}{x}\right)d\ln_k x + A = \mu^{(2)}(\Delta)+A. \qquad \square$$

Note that $\overline{D}_* \le d_k(0) \le D_k(0) \le \overline{D}^*$ for each $k \ge 1$. Indeed, it is enough to prove that $\overline{D}_* \le d_1(0) \le D_1(0) \le \overline{D}^*$, and the previous inequalities follow from the equality

$$\mathfrak{N}_1(t) = \int_1^t \frac{n(x)}{x}d\ln x = \int_1^t \frac{dN(x)}{x} = \frac{N(t)}{t} - N(1) + \int_1^t \frac{N(x)}{x}d\ln x,$$

where $N(x) = \int_0^x \frac{n(t)}{t}dt$.

Let us show that the inequalities $\overline{D}_* < d_1(0)$ and $D_1(0) < \overline{D}^*$ are possible. To do this, let's put $\omega(x) = (3+\sin(\ln x))/4$. Then ([2]) there exists a sequence (λ_n), $\lambda_n > 1$, such that $n(x) = x\omega(x) - \alpha(x)$, where $0 \le \alpha(x) < 1$. For this sequence

$$N(t) = \frac{3}{4}t + \frac{\sin(\ln t)-\cos(\ln t)}{8}t + O(\ln t),\ \ \mathfrak{N}_1(t) = \frac{3}{4}\ln t + \frac{-\cos(\ln t)}{4}t + O(1)$$

as $t \to +\infty$ and, thus, $\overline{D}_* = 3/4 - \sqrt{2}/8$ and $\overline{D}^* = 3/4 + \sqrt{2}/8$, but $d_1(0) = D_1(0) = 3/4$.

3. Estimates of a certain canonical product. As above, let (λ_n) be an increasing sequence of positive numbers with $D(0) < +\infty$ and $t_k = \exp_k\{0\}$. Then the canonical product

$$\Lambda(z) = \prod_{n=0}^{\infty}(1+z^2/\lambda_n^2)$$

is an entire function and for all $r > 0$

$$\ln\Lambda(r) = \int_0^\infty \ln\left(1+\frac{r^2}{t^2}\right)dn(t) = 2r^2\int_0^\infty \frac{n(t)}{t}\frac{dt}{t^2+r^2}. \tag{5}$$

Function Λ plays an important role in the study of the lacunarity of power series and representation of analytic functions by Dirichlet series (see, for example, [1] and [8]).

Since $\Delta_k(t) < D_k(0)+\varepsilon$ for every $\varepsilon > 0$ and all $t > t_* = t_*(\varepsilon) \ge t_k$, i.e. $\mathfrak{N}_k(t) < (D_k(0)+\varepsilon)\ln_k t$, from (5) we obtain

$$\ln\Lambda(r) = 2r^2\left(\int_0^{t_*}+\int_{t_*}^{\infty}\right)\frac{n(t)}{t}\frac{dt}{t^2+r^2} \le 2\int_0^{t_*}\frac{n(t)}{t}dt + 2r^2\int_{t_*}^{\infty}\frac{n(t)}{t}\prod_{j=0}^{k-1}\ln_j t\frac{d\ln_k t}{t^2+r^2} =$$
$$= K_1 + 2r^2\int_{t_*}^{\infty}\prod_{j=0}^{k-1}\ln_j t\frac{d\mathfrak{N}_k(t)}{t^2+r^2} \le K_1 - 2r^2\int_{t_*}^{\infty}\mathfrak{N}_k(t)d\left(\frac{1}{t^2+r^2}\prod_{j=0}^{k-1}\ln_j t\right) \le$$
$$\le K_1 - 2(D_k(0)+\varepsilon)r^2\int_{t_*}^{\infty}\ln_k t d\left(\frac{1}{t^2+r^2}\prod_{j=0}^{k-1}\ln_j t\right) =$$
$$= K_1 + \frac{2(D_k(0)+\varepsilon)r^2}{t_*^2+r^2}\prod_{j=0}^{k}\ln_j t_0 + 2(D_k(0)+\varepsilon)r^2\int_{t_*}^{\infty}\frac{dt}{t^2+r^2} \le$$
$$\le K_1 + K_2 + 2(D_k(0)+\varepsilon)r^2\left(\frac{\pi}{2r}-\frac{1}{r}\operatorname{arctg}\frac{t_*}{r}\right) \le K_1 + K_2 + (D_k(0)+\varepsilon)\pi r,$$

where K_1 and K_2 are positive constants depending on ε, whence in view of the arbitrariness of ε we get

$$\varlimsup_{r\to+\infty}\frac{\ln\Lambda(r)}{r}\le\pi D_k(0). \tag{6}$$

Since the maximal transfinite densities $D=\lim\limits_{k\to\infty}D_k(0)$, from (6) in view of the arbitrariness of k it follows that

$$\varlimsup_{r\to+\infty}\frac{\ln\Lambda(r)}{r}\le\pi D, \tag{7}$$

i.e. the following statement is true.

Proposition 2. *The function Λ has growth no higher than the first order and type πD.*

Let

$$\Lambda_j(z)=(1+z^2/\lambda_j^2)^{-1}\Lambda(z)=\prod_{n\ne j}(1+z^2/\lambda_n^2)=\sum_{n=0}^{\infty}c_n^{(j)}z^{2n}. \tag{8}$$

Clearly, Λ_j is an entire function, $c_0^{(j)}=1$ and $c_n^{(j)}>0$ for all $n\ge1$. As in [7, p. 72] we put

$$L_j(u)=\int_0^{\infty}e^{-ur}\Lambda_j(r)dr. \tag{9}$$

Then

$$L_j(\pi\Delta)=\int_0^{\infty}e^{-u\Delta r}\Lambda(r)\frac{\lambda_j^2dr}{\lambda_j^2+r^2}=\lambda_j\int_0^{\infty}e^{-u\Delta\lambda_jt}\Lambda(\lambda_jt)\frac{dt}{1+t^2}\le$$
$$\le\frac{\pi\lambda_j}{2}\sup_{x\ge0}\exp\{-\pi\Delta x+\ln\Lambda(x)\},$$

whence in view of (5) we get

$$\ln L_j(\pi\Delta)\le\ln\frac{\pi\lambda_j}{2}+\sup_{x\ge0}\left\{2x^2\left(\int_0^{\infty}\frac{n(t)}{t}\frac{dt}{x^2+t^2}-\int_0^{\infty}\Delta\frac{dt}{x^2+t^2}\right)\right\}=$$
$$=\ln\frac{\pi\lambda_j}{2}+\sup_{x\ge0}\left\{2x^2\left(\int_0^{t_k}\left(\frac{n(t)}{t}-\Delta\right)\frac{dt}{x^2+t^2}+\int_{t_k}^{\infty}\frac{n(t)}{t}\frac{dt}{x^2+t^2}-\int_{t_k}^{\infty}\frac{\Delta dt}{x^2+t^2}\right)\right\}.$$

Since

$$\sup_{x\ge0}\left\{2x^2\int_0^{t_k}\left(\frac{n(t)}{t}-\Delta\right)\frac{dt}{x^2+t^2}\right\}\le2\sup_{0\le t\le t_k}\left\{\frac{n(t)}{t}\right\}t_k,$$
$$\sup_{x\ge0}\left\{2x^2\left(\int_{t_k}^{\infty}\frac{n(t)}{t}\frac{dt}{x^2+t^2}-\int_{t_k}^{\infty}\frac{\Delta dt}{x^2+t^2}\right)\right\}=$$
$$=\sup_{x\ge0}\left\{2x^2\left(\int_{t_k}^{\infty}\frac{n(t)}{t}\prod_{j=0}^{k-1}\ln_j t\frac{d\ln_k t}{x^2+t^2}-\int_{t_k}^{\infty}\Delta\prod_{j=0}^{k-1}\ln_j t\frac{d\ln_k t}{x^2+t^2}\right)\right\}=$$
$$=\sup_{x\ge0}\left\{2x^2\left(\int_{t_k}^{\infty}\prod_{j=0}^{k-1}\ln_j t\frac{d\mathfrak{N}_k(t)}{x^2+t^2}-\int_{t_k}^{\infty}\prod_{j=0}^{k-1}\ln_j t\frac{d(\Delta\ln_k t)}{x^2+t^2}\right)\right\}\le$$
$$\le\sup_{x\ge0}\left\{2x^2\left(-\int_{t_k}^{\infty}\mathfrak{N}_k(t)d\left(\frac{1}{x^2+t^2}\prod_{j=0}^{k-1}\ln_j t\right)+\int_{t_k}^{\infty}\Delta\ln_k td\left(\frac{1}{x^2+t^2}\prod_{j=0}^{k-1}\ln_j t\right)\right)\right\}=$$

$$= \sup_{x\ge 0}\left\{-2x^2\int_{t_k}^{\infty}(\Delta_k(t)-\Delta)\ln_k t d\left(\frac{1}{x^2+t^2}\prod_{j=0}^{k-1}\ln_j t\right)\right\}\le$$

$$\le \sup_{x\ge 0}\{\Delta_k(t)-\Delta)\ln_k t\}\sup_{x\ge 0}\left\{-2x^2\int_{t_k}^{\infty} d\left(\frac{1}{x^2+t^2}\prod_{j=0}^{k-1}\ln_j t\right)\right\} = \nu_k(\Delta)2\prod_{j=0}^{k-1}\ln_j t_k,$$

we get

$$\ln L_j(\pi\Delta)\le \ln\frac{\pi\lambda_j}{2}+2t_k\sup_{0\le t\le t_k}\left\{\frac{n(t)}{t}\right\}+2\nu_k(\Delta)\prod_{j=0}^{k-1}\ln_j t_k. \tag{10}$$

Since $\nu_k(\Delta)<+\infty$ for $\Delta>D_k(0)$ and $\nu_k(\Delta)=+\infty$ for $\Delta<D_k(0)$ and $k\ge 1$, from (10) it follows that integral (9) converges for $u>\pi D_k(0)$. In view of the arbitrariness of k integral (9) converges for $u>\pi D$, where D is the maximal transfinite density of the sequence (λ_n). Substituting (8) into (9) we get

$$L_j(u)=\sum_{n=0}^{\infty}\frac{(2n)!c_n^{(j)}}{u^{2n+1}}. \tag{11}$$

The radius πP of convergence of this series is independent of the j and

$$\pi P=\varlimsup_{n\to+\infty}\left(\Lambda_j^{(2n)}(0)\right)^{1/2n}=\varlimsup_{n\to+\infty}\left(\Lambda^{(2n)}(0)\right)^{1/2n}\le\pi D\le\pi D_k(0).$$

4. Asymptotic Dirichlet series. Let $s=\sigma+it$ and a domain G such that the intersection of G with any half-plane $\{s\colon \sigma>\sigma_0\}$ is non-empty. Let F be a single-valued holomorphic function in G, (d_n) be a sequence of complex numbers and (λ_n) be a sequence of positive numbers. Let us assume that for sufficiently large x in the domain G

$$\inf_{m\ge j}\sup_{\sigma\ge x}\left|F(s)-\sum_{n=1}^{m}d_ne^{-s\lambda_n}\right|\le e^{-p_j(x)},\quad j\in\mathbb{N}, \tag{12}$$

where p_j is a non-decreasing to $+\infty$ function (it can be equal to $+\infty$ for sufficiently large x). They say ([7, p. 73]) that amounts $\sum_{n=1}^{m}d_ne^{-s\lambda_n}$ for $m\ge j$ represent a function F in the domain G with logarithmic precision $p_j(\sigma)$.

Let $C(a,R)=\{s\colon |s-a|<R\}$. For a Jordan curve J, the set $C=\bigcup_{s\in J}C(s,R)$ is called a channel of width $2R$ with a central line J. Let's assume that a_1 and a_2 are the ends of the central line of the channel C of width $2R$ and the domains G_1 and G_2 such that $C(a_j,R)\subset G_j$ $(j\in\{1,2\})$. Assume also that the function F_j is holomorphic function in G_j $(j\in\{1,2\})$. If in the domain $G_1\bigcup C\bigcup G_2$ there exists a holomorphic function F such that $F(s)\equiv F_j(s)$ for $s\in G_j$ then they say ([7, p. 83]) that the function F_1 continues analytically from G_1 to G_2 along the channel of width $2R$.

In what follows we will assume that a positive for sufficiently large σ the function $g(\sigma)$ has the bounded variation for $\sigma\ge\sigma_0$ and, therefore, $\lim\limits_{\sigma\to+\infty}g(\sigma)=g<+\infty$. Let $p(\sigma)$ be a positive non-decreasing to $+\infty$ function (it can be equal to $+\infty$ for sufficiently large x). If there exists a positive continuous non-increasing function $h(\sigma)\to h$ as $\sigma\to+\infty$ such that

$$g>h,\quad \ln L_j(\pi h(\sigma))<p(\sigma)+M,\quad (M=\text{const}>0,\ \sigma\ge\sigma_0), \tag{13}$$

and
$$\int^{\infty}(p(\sigma)-\ln L_j(\pi h(\sigma)))\exp\left\{-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)-h(u)}\right\}d\sigma=+\infty \tag{14}$$
then they say ([7, p. 76]) that there is an assumption $A^{(j)}[g(\sigma),p(\sigma),(\lambda_n)]$. The following theorem refines Theorem 7.I from [7].

Theorem 1. *Let us assume that the following three conditions are met.*

I. An increasing sequence (λ_n) of positive numbers has a bounded upper density, D is the maximal transfinite density (λ_n) and $G=\{s=\sigma+it\colon \sigma>a, |t|<\pi g(\sigma)\}$, where g is continuous function of bounded variation such that $g(\sigma)>D$.

II. A holomorphic in G function F continues analytically from G to $C(s_0,\pi R)$ along the channel of width $2\pi D$.

III. A sequence (d_n) and $j\in\mathbb{N}$ are such that the amounts $\sum_{n=1}^{m}d_ne^{-s\lambda_n}$ for $m\geq j$ represent F in G with logarithmic precision $p_j(\sigma)$.

Then the assumption $A^{(j)}[g(\sigma),p(\sigma),(\lambda_n)]$ implies
$$|d_j|\leq\frac{1}{2}\pi^2\lambda_jR\Lambda_j^*M(s_0,\pi R)\exp\left\{2t_k\sup_{0\leq t\leq t_k}\left\{\frac{n(t)}{t}\right\}+2\nu_k(R)\prod_{j=0}^{k-1}\ln_j t_k+\lambda_jRes_0\right\}, \tag{15}$$
where $\nu_k(\Delta)$ is the k-logarithmic function of the excess of the sequence (λ_n),
$$\Lambda_j^*=\prod_{n\neq j}\frac{\lambda_n^2}{|\lambda_j^2-\lambda_n^2|}\quad\text{and}\quad M(s_0,\pi R)=\sup_{s\in C(s_0,\pi R)}|F(s)|.$$

Proof. Let $h(\sigma)$, $\sigma\geq a_0>a$, be the function from (13) and (14). We put $\alpha=\pi(h(a_0)+e^{-a_0})$, $\gamma(\sigma)=\min\limits_{|x-\sigma|\leq\alpha}g(x)$, $h_1(\sigma)=h(\sigma)+e^{-\sigma}$ and $G(\sigma)=\pi(\gamma(\sigma)-h(\sigma-\varrho-\alpha))$, where $\varrho>16(g-h)$. In [7, p. 85] it is proved that there exists $\sigma^*>\varrho+\alpha+a_0$ such that in $G^*=\{s=\sigma+it\colon \sigma>\sigma^*, |t|<G(\sigma)\}$ we have
$$F_j(s)=d_j\Lambda_j(i\lambda_j)e^{-s\lambda_j}, \tag{16}$$
where
$$F_j(s)=\sum_{n=0}^{\infty}(-1)^nc_n^{(j)}F^{(2n)}(s) \tag{17}$$
and $c_n^{(j)}$ are the coefficients of the power development (8). In addition to (16), we need the following lemma ([7, p. 73]).

Lemma 1. *If a function $\Phi(z)$ is holomorphic and bounded in the disk $\{z\colon |z-z^0|<\pi R\}$ for $R>P$, where πP is radius of convergence of series (11), then the series $\sum\limits_{n=0}^{\infty}(-1)^nc_n^{(j)}\Phi^{(2n)}(s)$ converges uniformly in each disk $\{z\colon |z-z_0|<\pi\varrho\}$, $0<\varrho<R-P$, and represents in this disk a function Φ_k satisfying the inequality $|\Phi_k(z_0)|\leq\pi RL_j(\pi R)M$, where $M=\sup\limits_{|z-z^0|<\pi R}|\Phi(z)|$.*

Let us continue the proof of the theorem. According to the condition II, function F is holomorphic in the domain $G\bigcup C_1\bigcup C(s_0,\pi R)$, where C_1 is a channel of width $2\pi R_1>2\pi D$, the center line of which has ends s_1 and s_2, and $C(s_1,\pi R_1)\subset C(s_0,\pi R)$, $C(s_2,\pi R_1)\subset G$. It is obvious that $R_1\leq R$ and for any sufficiently large σ_2' there is a curve $J'\subset J$ with ends s_0

and σ_2' such that the function F is holomorphic in the domain $G\bigcup C_2\bigcup C(s_0,\pi R)$, where C_2 is a channel of width $2\pi R_2 > 2\pi D$ with the central line J'. Let $0<\varrho<R_2-D$. Then, in view of the inequalities $D\geq P$ and $R_2-P\geq R_2-D$, by Lemma 1, series (17) converges uniformly in each disk $C(s',\pi R)(s'\in J')$ and, thus, represents a holomorphic function in the channel C' of width $2\pi\varrho$ with the central line J'. If σ_2' is large enough then part of the channel lies in the domain G^*, where this series according to (16) and (17) represents the function $F_j(s)=d_j\Lambda_j(i\lambda_j)e^{-s\lambda_j}$. By the uniqueness theorem, this equality holds in the entire channel C' and in the disk with center s_0 belonging to this channel. Then by Lemma 1 we have $|F_k(s_0)|\leq \pi RL_j(\pi R)M(s_0,\pi R)$. Since $|\Lambda_j(i\lambda_j)|=1/\Lambda_j^*$, from the previous inequality and (16) we obtain

$$\frac{|d_j|}{\Lambda_j^*}e^{-\lambda_j\mathrm{Re}s_0}\leq \pi RL_j(\pi R)M(s_0,\pi R), \tag{18}$$

whence in view of (10) we get (15). □

Also the following theorem is true.

Theorem 2. *Let us assume that the conditions I, II and III of Theorem 1 are met. If the functions $g(\sigma)$, $p_j(\sigma)$ and the sequence (λ_n) satisfy the assumption $A^{(j)}[g(\sigma),p_j(\sigma),(\lambda_n)]$ then*

$$|d_j|\leq \frac{1}{2}\pi^2\lambda_jR\Lambda_j^*M(s_0,\pi R)\exp\{2\nu(R)+\lambda_jRes_0\} \tag{19}$$

where $\nu(\Delta)=\sup\limits_{t\geq 0}\int_0^t(\frac{n(x)}{x}-\Delta)dx$ is the function of the excess of the sequence (λ_n).

The validity of this theorem follows from (16) and the inequality $\ln L_j(\pi\Delta)\leq \ln\frac{\pi\lambda_j}{2}+2\nu(\Delta)$ proven in [6, p. 72].

Theorem 2 generalizes the well-known theorem of S. Mandelbrojt ([6, p. 84]), in which instead of D there is the upper averaged density $\overline{D}^*$.

5. Other assumptions and corollaries. Now we put $\nu_0(\Delta)=\nu(\Delta)$,

$$T_k=\prod_{j=0}^{k-1}\ln_j t_k$$

for $k\geq 1$ and $T_0=1$. The assumption $A[g(\sigma),p(\sigma),(\lambda_n),k]$, $k\geq 0$, means that $\lim\limits_{\sigma\to+\infty}g(\sigma)=g$ and there is a positive continuous non-increasing fumction $h(\sigma)\to h$ as $\sigma\to+\infty$ such that

$$g>h,\quad 2T_k\nu_k(h(\sigma))<p(\sigma)+M,\quad (M\equiv\text{const}>0,\sigma\geq\sigma_0), \tag{20}$$

and

$$\int^\infty(p(\sigma)-2T_k\nu_k(h(\sigma)))\exp\left\{-\frac{1}{2}\int^\sigma\frac{du}{g(u)-h(u)}\right\}d\sigma=+\infty. \tag{21}$$

Lemma 2. *For all $k\geq 0$ and $j\geq 1$ the assumption $A[g(\sigma),p(\sigma),(\lambda_n),k]$ contains the assumption $A^{(j)}[g(\sigma),p(\sigma),(\lambda_n)]$.*

Proof. For $k=0$ Lemma 2 is proved in [7, p. 80]. If $k\geq 1$ then in view of (10) from (20) and (21) we get respectively

$$\ln L_j(\pi h(\sigma))\leq \ln\frac{\pi\lambda_j}{2}+2t_k\sup_{0\leq t\leq t_k}\left\{\frac{n(t)}{t}\right\}+2T_k\nu_k(h(\sigma))\leq p(\sigma)+M_1$$

for $\sigma \geq \sigma_0$ and $M_1 \equiv \text{const} > 0$, and

$$\int^{\infty} (p(\sigma) - \ln L_j(\pi h(\sigma))) \exp\left\{-\frac{1}{2}\int^{\sigma} \frac{du}{g(u)-h(u)}\right\} d\sigma \geq$$
$$\geq \int^{\infty} (p(\sigma) - 2T_k\nu_k(h(\sigma))) \exp\left\{-\frac{1}{2}\int^{\sigma} \frac{du}{g(u)-h(u)}\right\} d\sigma -$$
$$-\int^{\infty} \left(\ln\frac{\pi\lambda_j}{2} + 2t_k \sup_{0\leq t\leq t_k}\left\{\frac{n(t)}{t}\right\}\right) \exp\left\{-\frac{1}{2}\int^{\sigma} \frac{du}{g(u)-h(u)}\right\} d\sigma = +\infty,$$

because in view of the inequality $h < g$ we have $\int^{\infty} \exp\{-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)-h(u)}\}d\sigma < +\infty$. □

The assumption $A_1[g(\sigma), p(\sigma), (\lambda_n), k]$, $k \geq 0$, means that $\lim\limits_{\sigma\to+\infty} g(\sigma) = g > D_k(0)$ and there is a positive number h such that $D_k(0) < h < g$ and

$$\int^{\infty} p(\sigma) \exp\left\{-\frac{1}{2}\int^{\sigma} \frac{du}{g(u)-h}\right\} d\sigma = +\infty. \tag{22}$$

Lemma 3. *The assumption $A_1[g(\sigma), p(\sigma), (\lambda_n), k]$ for all $k \geq 0$ contains the assumption $A[g(\sigma), p(\sigma), (\lambda_n), k]$.*

Proof. If just put the $h(\sigma) = h$ and notice that $\nu_k(h) < +\infty$ then (22) implies (21). □

Replacing $D_k(0)$ with D under the assumption $A_1[g(\sigma), p(\sigma), (\lambda_n), k]$, we obtain an assumption that we denote by $A_1^*[g(\sigma), p(\sigma), (\lambda_n), k]$.

Lemma 4. *The assumption $A_1^*[g(\sigma), p(\sigma), (\lambda_n), k]$ contains for all $j \geq 1$ the assumption $A^{(j)}[g(\sigma), p(\sigma), (\lambda_n)]$.*

Proof. If $g > D$ and $D < h < g$ then in view of the equality $\lim\limits_{k\to\infty} D_k(0) = D$ there exists $k = k(h)$ such that $D_k(0) < h < g$, i.e. $A_1[g(\sigma), p(\sigma), (\lambda_n), k]$ holds. Therefore, Lemmas 2 and 3 imply Lemma 4. □

The assumption $A_2[g(\sigma), p(\sigma), (\lambda_n), k]$, $k \geq 0$, means that $\lim\limits_{\sigma\to+\infty} g(\sigma) = g > D_k(0)$ and there is $\alpha \in (0, 1/(2T_k))$ such that

$$\int^{\infty} p(\sigma) \exp\left\{-\frac{1}{2}\int^{\sigma} \frac{du}{g(u)-\nu_k^{-1}(\alpha p(u))}\right\} d\sigma = +\infty. \tag{23}$$

Lemma 5. *The assumption $A_2[g(\sigma), p(\sigma), (\lambda_n), k]$ for all $k \geq 0$ contains the assumption $A[g(\sigma), p(\sigma), (\lambda_n), k]$.*

Proof. If $p(\sigma) = +\infty$ for σ enough large then taking $\nu_k^{-1}(\sigma) = h$, where $D_k(0) < h < g$, we get that (23) implies (22), that is $A_2[g(\sigma), p(\sigma), (\lambda_n), k]$ contains $A_1[g(\sigma), p(\sigma), (\lambda_n), k]$ and, thus, $A[g(\sigma), p(\sigma), (\lambda_n), k]$.

If $p(\sigma) < +\infty$ for all σ and $\alpha \in (0, 1/(2T_k))$ we put $h(\sigma) = \nu_k^{-1}(\alpha p(\sigma))$. Then

$$2T_k\nu_k(h(\sigma)) - p(\sigma) \leq 2T_k\alpha p(\sigma)) - p(\sigma) < -\gamma p(\sigma)$$

for some $\gamma > 0$. Therefore, if $A_2[g(\sigma), p(\sigma), (\lambda_n), k]$ holds then $A[g(\sigma), p(\sigma), (\lambda_n), k]$ holds. □

The assumption $A_3[g(\sigma),p(\sigma),(\lambda_n),k]$, $k\ge 0$, means that $\lim\limits_{\sigma\to+\infty} g(\sigma)=g>D_k(0)$ and there is $\gamma\in(0,+\infty)$ such that

$$\int^{\infty} p(\sigma)\exp\left\{-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)-\Delta_k^*(\exp_k\{\gamma p(u)\})}\right\}d\sigma=+\infty, \tag{24}$$

where $\Delta_0^*(x)=\sup\limits_{t\ge x}\frac{n(x)}{x}$ and, as above, $\Delta_k^*(x)=\sup\limits_{t\ge x}\frac{\mathfrak{N}_k(t)}{\ln_k t}$ for $k\ge 1$.

Lemma 6. *The assumption $A_3[g(\sigma),p(\sigma),(\lambda_n),k]$ for all $k\ge 0$ contains the assumption $A_2[g(\sigma),p(\sigma),(\lambda_n),k]$.*

Proof. Since $\Delta_k^*(x)\searrow D_k(0)$ as $x\to+\infty$, for $x>t_k$ we have

$$\nu_k(\Delta_k^*(x))=\sup_{t\ge t_k}\{(\Delta_k(t)-\Delta_k^*(x))\ln_k t\}\le$$
$$\le\sup_{t\ge t_k}\{(\Delta_k^*(t)-\Delta_k^*(x))\ln_k t\}=\sup_{t_k\le t\le x}\{(\Delta_k^*(t)-\Delta_k^*(x))\ln_k t\}=$$
$$=\max\left\{\sup_{t_k\le t\le\exp_k\sqrt{\ln_k x}}\{(\Delta_k^*(t)-\Delta_k^*(x))\ln_k t\},\sup_{\exp_k\sqrt{\ln_k x}\le t\le x}\{(\Delta_k^*(t)-\Delta_k^*(x))\ln_k t\}\right\}=$$
$$=\max\left\{\sqrt{\ln_k x}(\Delta_k^*(t_k)-\Delta_k^*(x)),(\Delta_k^*(\exp_k\sqrt{\ln_k x})-\Delta_k^*(x))\ln_k x\right\}=o(\ln_k x)$$

as $x\to+\infty$. Thus, $\nu_k(\Delta_k^*(\exp_k\{\gamma p(\sigma)\}))=o(p(\sigma))$ as $\sigma\to+\infty$ and, therefore, for σ enough large and $\alpha\in(0,1/2)$ we get $\nu_k(\Delta_k^*(\exp_k\{\gamma p(\sigma)\}))\le\alpha p(\sigma)$, whence $\nu_k^{-1}(\alpha p(\sigma))\le\Delta_k^*(\exp_k\{\gamma p(\sigma)\})$. Thus, (24) implies (23) and, since $\lim\limits_{\sigma\to+\infty}\Delta_k^*(\exp_k\{\gamma p(\sigma)\})=D_k(0)$. Lemma 6 is proved. □

Using Lemmas 2–6, from Theorems 1 and 2 we obtain the following statement.

Corollary 1. *The conclusions of Theorems 1–2 hold if the assumption $A^{(j)}[g(\sigma),p_j(\sigma),(\lambda_n)]$, $j\ge 1$, replace either with one of the assumptions*

$$A[g(\sigma),p_j(\sigma),(\lambda_n),k],\ k\ge 0,\quad\text{or}\quad A_i[g(\sigma),p(\sigma),(\lambda_n),k],\ k\ge 0,\ i\in\{1,2,3\},$$

or with the assumption $A_1^*[g(\sigma),p_j(\sigma),(\lambda_n)]$.

Let us return to the asymptotic Dirichlet series. Let $A_\xi(\sigma)$ be a family of functions that are non-increasing under $\sigma>\sigma_0$ and tend to zero under $\sigma\to+\infty$, where ξ takes values from the set I that is unbounded from above. Let's denote $A(\sigma)=\inf\limits_{\xi\in I}A_\xi(\sigma)$ $(\sigma>\sigma_0)$. Then $A(\sigma)\searrow 0$ as $\sigma\to+\infty$. If $\inf\limits_{\xi\in I,\xi\ge\xi'}A_\xi(\sigma)=O(A(\sigma))$ as $\sigma\to+\infty$ for each ξ' then $A_\xi(\sigma)$ is called an asymptotic family and the function $A(\sigma)$ is called the lower envelope of this family.

As above, let G be a domain G such that the intersection of G with any half-plane $\{s\colon\sigma>\sigma_0\}$ is non-empty and F be a single-valued holomorphic function in G. Let's say that a series $\sum_{n=1}^{\infty}d_ne^{-s\lambda_n}$ represents F in G asymptotically with respect to $A(\sigma)$, if for each $j\in\mathbb{N}$ there is ξ_j such that for $\xi_j\in I,\xi>\xi_j$ there is $m>j$, for which

$$\left|F(s)-\sum_{n=1}^{m}d_ne^{-s\lambda_n}\right|\le A(\sigma)$$

for $s\in G$ and $\sigma\ge\sigma_0$. The next statements are proven in [7, p. 75–76].

Lemma 7. *If the series $\sum_{n=1}^{\infty} d_n e^{-s\lambda_n}$ represents F in G asymptotically with respect to $A(\sigma)$ then there exists a sequence (p_j) such that for every $j \ge 1$ the amounts $\sum_{n=1}^{m} d_n e^{-s\lambda_n}$ for $m \ge j$ represent a function F in the domain G with logarithmic precision $-\ln A(\sigma) + p_j(\sigma)$.*

Lemma 8. *If the series $\sum_{n=1}^{m} d_n e^{-s\lambda_n}$ converges to $F(s)$ for $\sigma > \sigma_0$ then it represents F in every domain $G = \{s\colon \sigma > \sigma_0, |t| < a\}$ asymptotically with respect to $A(\sigma) \equiv 0$.*

Using Lemma 7 we prove the following statement.

Proposition 3. *Let the conditions I and II of Theorem 1 hold and the series $\sum_{n=1}^{\infty} d_n e^{-s\lambda_n}$ represents F in G asymptotically with respect to $A(\sigma)$. If the functions $g(\sigma)$, $p(\sigma) = -\ln A(\sigma)$ and the sequence (λ_n) satisfy one of the assumptions $A[g(\sigma), p(\sigma), (\lambda_n), k]$, $k \ge 0$, then equalities* (15) *and* (19) *hold for all $j \ge 1$.*

Proof. By Lemma 7, for each $j \ge 1$ the amounts $\sum_{n=1}^{m} d_n e^{-s\lambda_n}$ for $m \ge j$ represent a function F in the domain G with logarithmic precision $p(\sigma) + p_j(\sigma)$, where p_j is a constant value for each j. But, since the assumption $A[g(\sigma), p(\sigma), (\lambda_n), k]$ contains the assumption $A[g(\sigma), p(\sigma) + p_j, (\lambda_n), k]$, from Theorem 1 and Corollary 1 we get Proposition 3. □

Remark 1. The conclusions of Proposition 3 holds if the assumption $A[g(\sigma), p(\sigma), (\lambda_n), k]$ replace either with one of the assumptions $A_i[g(\sigma), p(\sigma), (\lambda_n), k]$, $i \in \{1, 2, 3\}$, or with the assumption $A_1^*[g(\sigma), p(\sigma), (\lambda_n)]$.

Proposition 4. *If $g = \lim\limits_{\sigma\to+\infty} g(\sigma)$ and $\int^{\infty} |g(\sigma) - g| d\sigma < +\infty$ then the conclusions of Proposition 3 holds if the assumption $A[g(\sigma), p(\sigma), (\lambda_n), k]$ replace with the assumption $A[g, p(\sigma), (\lambda_n), k]$.*

Indeed, in this case it is easy to show that for σ_0 enough large we have
$$\exp\Big\{-\frac{1}{2}\int^{\sigma} \frac{du}{g - h(u)}\Big\} \asymp \exp\Big\{-\frac{1}{2}\int^{\sigma} \frac{du}{g(u) - h(u)}\Big\}, \quad \sigma \to +\infty,$$
i.e. the assumptions $A[g(\sigma), p(\sigma), (\lambda_n), k]$ and $A[g, p(\sigma), (\lambda_n), k]$ are equivalent, Q.E.D.

Remark 2. The conclusions of Proposition 4 holds if the assumption $A[g(\sigma), p(\sigma), (\lambda_n), k]$ replace either with one of the assumptions $A_i[g, p(\sigma), (\lambda_n), k]$, $i \in \{1, 2, 3\}$, or with the assumption $A_1^*[g, p(\sigma), (\lambda_n)]$.

6. Applications to entire Dirichlet series. Before exploring the properties of entire Dirichlet series, we present one generalization of Theorem 1.II from [7, p. 247]. Let $s(u) = \sigma(u) + it(u)$ be a continuous complex-valued function, $-\infty < u < +\infty$, such that $\lim\limits_{u\to+\infty} \sigma(u) = +\infty$, $\lim\limits_{u\to-\infty} \sigma(u) = -\infty$ and $t(u) = t_0$ for $\sigma \ge \sigma_0$, where t_0 and σ_0 are constants. Uniting all disks $C(s(u), R)$ with centers at $s(u)$ and radius R let's call a curved strip of width $2R$ horizontal to the right and continuing to $-\infty$.

Theorem 3. *Let (λ_n) be an increasing to $+\infty$ sequence of positive numbers with bounded upper density and D be its maximal transfinite density. Suppose that a Dirichlet series*
$$F(s) = \sum_{n=1}^{\infty} d_n e^{-s\lambda_n}, \quad s = \sigma + it, \tag{25}$$
has the abscissa of convergence $\sigma_c < +\infty$ and function F can be analytically continued along a curvilinear strip B of width $2\pi R > 2\pi D$ extending up to $-\infty$. If F is bounded in B then $F(s) \equiv 0$ and all $d_n = 0$.

The proof of this theorem is the same as the proof of Theorem 1.II from [7, p. 247] using Proposition 3.

From Theorem 3 it follows that if a power series $\sum_{n=1}^{\infty} a_n z^{\lambda_n}$ in the neighbourhood of the origin converges to $f(z)$ whose the continuation is holomorphic and bounded inside an angle with a vertex at the origin and an opening greater than $2\pi D$ then $f(z) \equiv 0$.

Now for entire Dirichlet series (25) we put

$$M(\sigma) = \sup_{t\in\mathbb{R}} |F(\sigma + it)|, \quad M_a(\sigma) = \max_{|t-t_0|\le\pi a} |F(\sigma + it)|, \quad \Phi(\sigma) = \ln M(\sigma).$$

Theorem 4. *If the sequence* (λ_n) *satisfies* (1) *and*

$$\varlimsup_{\sigma\to-\infty} \frac{\ln\ln M(\sigma)}{\ln^2(-\sigma)} = +\infty \tag{26}$$

then

$$\varlimsup_{\sigma\to-\infty} \frac{\Phi^{-1}(\ln M_a(\sigma))}{-\sigma} = 1,$$

provided $a > D$, *where* D *is the maximal transfinite density of* (λ_n).

Proof. From (1) it follows that $D(0) < +\infty$. Since $M_a(\sigma) \le M(\sigma)$, for each $a > 0$ we have

$$\varlimsup_{\sigma\to-\infty} \frac{\Phi^{-1}(\ln M_a(\sigma))}{-\sigma} \le 1. \tag{27}$$

Denote $M_C(\sigma) = \max_{s\in C(s_0,\pi a)} |F(s)|$, where $s_0 = \sigma + it_0$ and $C(s_0, \pi a) = \{s\colon |s - s_0| < \pi a\}$. By Lemma 8 the conditions of Theorem 4 imply the conditions of Proposition 3 with $M(s_0, \pi R) = M_C(\sigma)$ and $R = a$. Therefore, from (19) for all $n \ge 1$ we get

$$\ln M_C(\sigma) \ge \ln d_n - \ln\lambda_n - \ln\Lambda_n^* - \lambda_n\sigma - K, \quad K = \text{const}. \tag{28}$$

In [9] it is proved that if (26) holds and

$$\ln n = o(\lambda_n\Phi^{-1}(\lambda_n)), \quad n \to \infty, \tag{29}$$

then

$$\varlimsup_{n\to\infty} \frac{\lambda_n\Phi^{-1}(\lambda_n)}{-\ln|d_n|} = 1. \tag{30}$$

Since (1) implies (29), equality (30) holds and, therefore, for every $\varepsilon > 0$ there exists an increasing sequence (n_j) such that

$$\ln|d_{n_j}| \ge -(1+\varepsilon)\lambda_{n_j}\Phi^{-1}(\lambda_{n_j}). \tag{31}$$

It is known [7, p. 67] that (1) implies $\ln\Lambda_n^* = O(\lambda_n)$ as $n \to \infty$. Therefore, from (28) and (31) we get

$$\ln M_C(\sigma) \ge -(1+\varepsilon)\lambda_{n_j}\Phi^{-1}(\lambda_{n_j}) - \lambda_n\sigma - O(\lambda_{n_j}) \ge -(1+2\varepsilon)\lambda_{n_j}\Phi^{-1}(\lambda_{n_j}) - \lambda_{n_j}\sigma$$

for all σ and $j \ge j_0(\varepsilon)$. If we choose $\sigma = \sigma_j = -(1+3\varepsilon)\Phi^{-1}(\lambda_{n_j})$ then from hence we obtain $\ln M_C(-(1+3\varepsilon)\Phi^{-1}(\lambda_{n_j})) \ge \varepsilon\lambda_{n_j}\Phi^{-1}(\lambda_{n_j})$. Since for every σ_j there exists α_j such that $|\alpha_j| \le \pi a$ and $M_a(\sigma_j + \alpha_j) \ge M_C(\sigma_j)$, from the previous inequality we get

$$\varlimsup_{\sigma\to-\infty} \frac{\Phi^{-1}(\ln M_a(\sigma))}{-\sigma} \ge \varlimsup_{j\to\infty} \frac{\Phi^{-1}(\ln M_a(-(1+3\varepsilon)\Phi^{-1}(\lambda_{n_j}) + \alpha_j))}{(1+3\varepsilon)\Phi^{-1}(\lambda_{n_j}) - \alpha_j} \ge$$

$$\geq \varlimsup_{j\to\infty} \frac{\Phi^{-1}(\varepsilon\lambda_{n_j}\Phi^{-1}(\lambda_{n_j}))}{(1+3\varepsilon)\Phi^{-1}(\lambda_{n_j})} \geq \frac{1}{1+3\varepsilon},$$

because $\varepsilon\Phi^{-1}(\lambda_{n_j}) \geq 1$ for $j \geq j_1 \geq j_0$. In view of the arbitrariness of ε and (27) Theorem 4 is proved. $\square$

Remark 3. For entire Dirichlet series of finite R-order $\varrho_R := \varlimsup_{\sigma\to-\infty} \frac{\ln\ln M(\sigma)}{-\sigma}$ the equality

$$\varlimsup_{\sigma\to-\infty} \frac{\ln\ln M_a(\sigma)}{-\sigma} = \varrho_R$$

is correct provided $a > D$.

Finally, we note that repeating the proof of Theorem 4.VI ([7]) can be obtained the following generalization of Picard's theorem for stripes.

Proposition 5. *If the sequence* (λ_n) *satisfies* (1) *and* $\varrho_R > 0$ *then in any strip* $\{s = \sigma + it\colon |t-t_0| < \pi a\}$ *with* $a > \max\{D, 1/(2\varrho_R)\}$, *entire function* (25) *takes on all values, with the possible exception of one, an infinite number of times.*

REFERENCES

1. G. Polya, *Untersuchungen über Lucken and Singularitäten von Potenzreihen*, Math. Z., **29** (1929), 549–640.
2. M.M. Sheremeta, *On k-logarithmic density of sequence and its applications to entire functions*, I. Teor. Funkts. Funkts. Anal. Prilozh., **25** (1976), 131–142. (in Russian) `https://ekhnuir.karazin.ua/handle/123456789/16410`
3. M.M. Sheremeta, *On k-logarithmic density of sequence and its applications to entire functions*, II. Teor. Funkts. Funkts. Anal. Prilozh., **25** (1976), 142–156. (in Russian) `https://ekhnuir.karazin.ua/handle/123456789/16411`
4. A. Pfluger, G. Polya, *On the power series of an integral function having an exceptional value*, Proc. Cambridge Phil. Soc., **31** (1935), 153–155.
5. R. Fejér, *Über die Wurzel von kleinsten absoluten Betrage einer algebraischen Gleichung*, Math. Ann., **65** (1908), 413–423.
6. T. Murai, *The deficiency of entire functions with Fejér gaps*, Ann. Inst. Fourier, Grenoble, **33** (1983), №3, 39–58. `http://www.numdam.org/item?id=AIF_1983__33_3_39_0`
7. S. Mandelbrojt, Séries adhérentes. Régularisation des suites. Applications, Gauthier-Villars, Paris, 1952.
8. A.F Leont'ev, Series of exponents, Moscow, Nauka, 1976.
9. B.V. Vinnitskii, M.M. Sheremeta, *Asymptotic properties of the coefficients of Dirichlet series representing entire functions*, Ukr. Math. J., **27** (1975), №2, 117–124. `https://doi.org/10.1007/BF01089992`

Ivan Franko National University of Lviv
Lviv, Ukraine
m.m.sheremeta@gmail.com

Received 15.09.2024
Revised 14.03.2025