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# HYPERBOLIC STEFAN PROBLEM WITH NONLOCAL BOUNDARY CONDITIONS

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In this paper, we consider problems with unknown boundaries for hyperbolic equations and systems with free boundaries with two independent variables. The boundary conditions for such equations in the linear or quasilinear cases are given in nonlocal (non-separable and integral) form. The hyperbolic Stefan and Darboux-Stefan problems (the line of initial conditions degenerates to a point) are considered. There are proved the existence and uniqueness theorems of generalized solution, which are continuous solutions of equivalent systems of the second kind Volterra integral equations. The method of characteristics based on a combination of the Banach fixed point theorem allows us to obtain global generalized solutions in terms of the time variable in the case of linear hyperbolic equations with free boundaries and local solutions for quasilinear equations. Nonlocal (non-separable and integral) conditions require additional solvability conditions that are not present in the case of generally accepted boundary conditions for hyperbolic equations and systems. The paper provides examples indicating the significance of the conditions for the solvability of the corresponding problems. The corresponding solutions may have discontinuities along the characteristics of the hyperbolic equations. This additionally requires setting the conditions for matching the initial data of the problems at the corner points of the considered domains. This paper extends the results on the problems with nonlocal conditions for hyperbolic equations and systems to the case of hyperbolic equations with free boundaries.

**1. Introduction.** With this article, we complete a series of papers [21–23] devoted to problems with nonlocal (non-separable and integral) boundary conditions for linear and semilinear hyperbolic systems of equations of arbitrary order with two independent variables. In these papers, nonlocal mixed problems, Darboux and Darboux-Stefan problems for the corresponding hyperbolic equations and systems are considered. Now we will consider some problems with unknown boundaries (hyperbolic Stefan problems) for hyperbolic equations and systems. Problems with such conditions (nonlocal problems) for hyperbolic equations and systems are found in biology ([31]), ecology ([25]), mechanics ([34]), demography ([36]), etc. Applied problems which mathematical formulations require nonlocal boundary conditions for hyperbolic equations and systems are also given in [21].

In many applied problems, a situation arises when the equation of boundary of a domain or some of its parts are included among the functions to be searched for. Such problems are called problems with “free” (unknown) boundaries or Stefan problems. According to modern

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estimates, about 25% of all mathematical models of practical problems in industry and other fields are problems with free boundaries ([10]). A detailed review of the literature on this subject is given in the paper [6].

The first paper on this topic was [15]. There was considered a problem for the first order linear system of hyperbolic equations with free boundaries arising in gas dynamics (the piston problem). The mathematical model of the problem is reduced to finding the functions $s(t)$, $\rho_i(t)$, $u_i(x,t)$ $(i=1,2)$ under the following conditions:

$$\frac{\partial \rho_1}{\partial t}+u_1\frac{\partial \rho_2}{\partial t}+\rho_1\frac{\partial u_1}{\partial x}=0,\quad \frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}+\frac{1}{\rho_1}\frac{\partial u_2}{\partial x}=0,\quad -\infty<x<s(t),$$

$$\frac{\partial u_2}{\partial t}+u_2\frac{\partial u_2}{\partial x}+\frac{1}{\rho_2}\frac{\partial A\rho_2}{\partial x}=0,\quad \frac{\partial \rho_2}{\partial t}+u_2\frac{\partial \rho_2}{\partial x}+\rho_2\frac{\partial u_2}{\partial x}=0,\quad s(t)<x<\infty,$$

$$\rho_1(x,0)=f_1(x),\quad u_1(x,0)=g_1(x),\quad x\in(-\infty,0],$$

$$\rho_2(x,0)=f_2(x),\quad u_2(x,0)=g_2(x),\quad x\in[0,\infty),$$

$$u_1(s(t),t)=u_2(s(t),t)=s'(t),\quad t\geqslant 0,\ s(0)=0, s'(0)=b,$$

$$s''(t)=A\rho_1(s(t),t)-A\rho_2(s(t),t),\quad t>0, \tag{1}$$

where $s(t)$ is the position of the piston at time $t$; conditions (1) are Newton law for the piston motion. For the linearized problem, the existence of a global solution is established by the method of successive approximations and its regularity is investigated.

The problems with unknown boundaries for hyperbolic systems of first-order equations, as well as some problems for hyperbolic equations of the second order were considered in [26–28]. The papers [17, 18] generalize results of [26–28] for the cases of single-phase, two-phase and multiphase problems with unknown boundaries. Some modification of two-phase problems with unknown boundaries in the case of degeneracy of the initial condition line to the point were studied in [22]. More general results were obtained in [1, 20] for the first order quasilinear hyperbolic system of equations on a line.

A one-sided Goursat-type problem for a wave equation in a flat domain $\{(x,t)\colon t>0,\ 0<x<s(t),\ s(0)=0\}$ with an unknown boundary $s(t)$ and local integral conditions at $x=0$ is studied in [32]. The study of the piston problem is devoted to the papers by [4, 13]. Thus, the piston problem [4] is studied when the self-consistent motion of a gas and a piston in a one-dimensional channel under the action of external forces is described. In [13], the authors consider a single-phase free boundary value problem for a hyperbolic system of the first-order equations that arises when describing the motion of a piston under the action of a compressible fluid in a tube. In [24], there was considered a problem of the motion of a string oscillating when it hits a wall of arbitrary shape and an arbitrary external force acts on the string.

Some theoretical issues of hyperbolic Stefan problems have been studied in the papers of [14, 16, 41]. It is known that in many real-world environments, heat spreading is more accurately described by a hyperbolic equation than by the classical heat conduction equation. Thus, it is natural to formulate the Stefan problem for the hyperbolic equation ([3, 7, 8, 11, 29, 33, 35, 37, 38]). The hyperbolic model of heat transfer ([7]) arises from the mathematical description of heat spreading in a medium that has the relaxation property $\tau q_t+q=-kT_x$ (where $q$ is the heat flux, $T$ is the temperature) instead of the usual Fourier law $q=-kT_x$. In [8], there was studied a similar one-phase hyperbolic Stefan problem in multidimensional space for the telegraph equation $\tau T_{tt}+cT_t-k\Delta T=0$. In [37, 38], there was consider a nonlinear telegraph equation in a domain with a free unknown boundary. In [37], instead of the classical Fourier's law, an alternative model is proposed in which the flow reacts to the temperature gradient not instantaneously, but after a period of delay.

Recently, the interest in studying the properties of the system

$$\frac{\partial E}{\partial t} + \mathrm{div} q = 0, \quad \tau q_t(t) + q(t) = -\nabla\Theta(t) \tag{2}$$

has increased significantly in problems describing a phase transition, where $\Theta$ is the temperature, $E$ is the internal energy. In [5,30,35], formulations of problems with a phase transition for the system (2) were proposed. In [12], it was proposed to consider the system (2) in a generalized sense and thus only the strong discontinuity conditions (Rankine-Hugoniot conditions [14]) are fulfilled at the phase transition boundary. In the article [11], the authors considered the single-phase one-dimensional Stefan problem for the hyperbolic heat equation $\tau Q_{tt} + Q_t - Q_{xx} = 0$. Other interesting formulations of hyperbolic Stefan problems are proposed in [33,40]. Problems with unknown boundaries for quasilinear hyperbolic equations and systems are discussed in [1,2,39].

In [39], the existence of a global classical solution for one class of problems with a free boundary is investigated. Similar problems of the existence of a global solution to hyperbolic Stefan problems under conditions of monotonicity and familiarity with the initial data are considered in [1].

In the curved quadrilateral of the plane $(x, y)$ for the system of quasilinear hyperbolic equations of the form $\partial_x z + \lambda(x, y, z, Vz)\partial_y z = f(x, y, z, Vz)$, where $z = (z_1, \ldots, z_n)$, $f = (f_1, \ldots, f_n)$ and $Vz = (Vz)(x, y)$ is the operator function ([2]), the problem with an unknown boundary dividing the domain into two parts is considered.

## 2. The hyperbolic Stefan problem with nonlocal boundary conditions for a system of equations of arbitrary order.

**2.1. Formulation of the problem.** Let $\Omega_T = \{(x,t) \in \mathbb{R}^2 \colon 0 < t \leqslant T,\ a_1(t) < x < a_2(t),\ a_i(0) = a_i^0$ known constants, $i = 1, 2,\ a_1^0 < a_2^0\}$, and the functions $a_i(t)$ are unknown in advance. In $\Omega_T$, consider a matrix differential equation of order $n \geqslant 1$

$$Au \equiv \sum_{i=0}^{n} A_i\Big(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)u = f(x, t), \tag{3}$$

where $A_i$ is a linear homogeneous differential operator

$$A_i\Big(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)u \equiv \sum_{j=0}^{i} A_{ij}(x, t)\frac{\partial^i u}{\partial x^j \partial t^{i-j}},$$

which coefficients $A_{ij}$ are square matrices of order $m\times m$, and $A_{n0}(x, t) \equiv E_m$, $u = \mathrm{col}(u^1, \ldots, u^m)$, $f = \mathrm{col}(f^1, \ldots, f^m)$.

The equation (3) is strictly hyperbolic in $\Omega_T$, that is, the roots $\lambda$ of the equation $\det A_n(x, t, 1, \lambda) = 0$ are real and different for all $(x, t) \in \overline{\Omega}_T$. These roots $-\lambda_1(x, t), \ldots, -\lambda_{mn}(x, t)$ will be called the characteristic roots of the equation (3). It is known [9] that if $\lambda_i \neq \lambda_j$ at $i \neq j$, then the set of eigenvectors corresponding to the eigenvalues can be divided into $n$ groups of $m$ each so that each group forms a basis in $\mathbb{R}^m$

$$(h_1^1, \ldots, h_1^m), (h_2^1, \ldots, h_2^m), \ldots, (h_n^1, \ldots, h_n^m).$$

In accordance with the numbering of the eigenvectors, we renumber the characteristic roots (the vector $h_i^s$ corresponds to the eigenvalue $-\lambda_i^s$).

Let the expressions $\omega_i^{s1}(t) \equiv \lambda_i^s(a_1(t), t) - a_1'(t)$ and $\omega_i^{s2}(t) \equiv \lambda_i^s(a_2(t), t) - a_2'(t)$ have no zeros ($i = \overline{1, n},\ s = \overline{1, m}$) at $t = 0$. By $I_l^{s+}$ ($I_l^{s-}$) we denote the set of indices $i$ for which $\omega_i^{sl}(0) > 0$ ($\omega_i^{sl}(0) < 0$), $l = 1, 2$. Moreover, $I_l^{\pm} = \{(i, s) \colon i \in I_l^{s\pm}, s = \overline{1, m}\}$ ($l = 1, 2$).

Consider the following problem: for some $T_1 > 0$ find the vector function $a(t) = (a_1(t), a_2(t))$ and in the corresponding domain $\Omega_{T_1}$ a solution $u(x,t)$ of the system (3) such that the following conditions are satisfied

$$\frac{\partial^i u^s}{\partial t^i}(x,0) = g_i^s(x), \quad x \in [a_1^0, a_2^0], \quad i = \overline{0, n-1}, \quad s = \overline{1, m}, \tag{4}$$

$$\sum_{s=1}^{m}\sum_{i=0}^{n-1}\sum_{j=0}^{i}\Big\{\sum_{q=1}^{2} B_{qik}^{sj}(t)\frac{\partial^i u^s}{\partial x^j \partial t^{i-j}}\Big|_{x=a_q(t)} + \int_{a_1(t)}^{a_2(t)} C_{ik}^{sj}(\xi,t)\frac{\partial^i u^s(\xi,t)}{\partial \xi^j \partial t^{i-j}}d\xi\Big\} = h_k(t),$$
$$k = \overline{1, N_0}, \quad t \in [0, T_1], \tag{5}$$

$$\sum_{s=1}^{m}\sum_{i=0}^{n-1}\sum_{j=0}^{i}\int_{a_1(t)}^{a_2(t)} C_{ik}^{sj}(\xi,t)\frac{\partial^i u^s(\xi,t)}{\partial \xi^j \partial t^{i-j}}d\xi = h_k(t), \quad k = \overline{N_0+1, N}, \quad t \in [0, T_1], \tag{6}$$

$$a_l''(t) = F_l(t, a_1(t), a_2(t), a_1'(t), a_2'(t), u(a_1(t),t), u(a_2(t),t)),$$
$$a_l(0) = a_l^0, \quad a_l'(0) = \alpha_l^0, \quad l = 1, 2, \quad \forall t \in [0, T_1], \tag{7}$$

$$\max_{(i,s)\in I_l^-} \lambda_i^s(a_l(t),t) < a_l'(t) < \min_{(i,s)\in I_l^+} \lambda_i^s(a_l(t),t), \quad l = 1, 2, \quad t \in [0, T_1], \tag{8}$$

where $g_i^s(x)$, $B_{qik}^{sj}(t)$, $C_{ik}^{sj}(x,t)$, $h_k(t)$ and $F_i$ are known functions, $0 \leqslant N_0 \leqslant N$, $N$ is the number of elements of $I_1^+ \cup I_2^-$.

**2.2. Auxiliary transformations.** Before defining the generalized solution of the problem (3)–(8), we will perform its preliminary transformation under the assumption that $u \in [C^n(\overline{\Omega}_T)]^m$ and all equalities (3)–(6) are satisfied. Let us introduce the matrices $\Lambda_i = -\operatorname{diag}(\lambda_i^1, \dots, \lambda_i^m)$ $(i = \overline{1, n})$ and consider the block matrix

$$A(x,t) = \begin{Vmatrix} A_{n1}(x,t) & A_{n2}(x,t) & \dots & A_{nn-1}(x,t) & A_{nn}(x,t) \\ -E_m & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & -E_m & 0 \end{Vmatrix}$$

Since $\det(A + \lambda E_{mn}) = \det A_n(x,t,1,\lambda)$, $-\lambda_i^s(x,t)$ are the eigenvalues of the matrix $A$. Thus, this matrix is similar to a diagonal matrix, and therefore there exists a matrix $P(x,t)$ such that $P^{-1}AP = \operatorname{diag}(\Lambda_1, \dots, \Lambda_n)$. Let us multiply the left side of the obtained ratio and write the result element by element. Then for each $i = \overline{1, n}$ we get

$$\sum_{k=1}^{n}\sum_{l=1}^{n} P^{ik}\tilde{A}_{kl}P_{lr} = \Lambda_i \delta_{ir}, \quad r = \overline{1, n},$$

where $\|\tilde{A}_{kl}\|_{i,k=1}^n = A$, $\|P_{lr}\|_{l,r=1}^n = P$, $\|P^{ik}\|_{i,k=1}^n = P^{-1}$, where $P^{ik}$, $P_{lr}$ are square matrices of order $m$, $\delta_{ir}$ is the Kronecker symbol. Multiply this expression on the right by $P^{rj}$ and summarize in $r$:

$$\sum_{k=1}^{n}\sum_{l=1}^{n} P^{ik}\tilde{A}_{kl}\sum_{r=1}^{n} P_{lr}P^{rj} = \sum_{r=1}^{n}\Lambda_i\delta_{ir}P^{rj}, \quad \sum_{k=1}^{n}\sum_{l=1}^{n} P^{ik}\tilde{A}_{kl}\delta_{lj} = \Lambda_i P^{ij}, \text{ or } \sum_{k=1}^{n} P^{ik}\tilde{A}_{kj} = \Lambda_i P^{ij}$$

$j = \overline{1, n}$, then $P^{i1}A_{nj} - P^{i,j+1} = \Lambda_i P^{ij}$, $j = \overline{1, n-1}$, $P^{i1}A_{nn} = \Lambda_i P^{in}$. Hence we get

$$A_{nj} = (P^{i1})^{-1}P^{i,j+1} + (P^{i1})^{-1}\Lambda_i P^{ij}, \quad j = \overline{1, n-1}, \quad A_{nn} = (P^{i1})^{-1}\Lambda_i P^{in}. \tag{9}$$

Let us introduce operators

$$M_i\Big(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)u \equiv \sum_{j=1}^{n} P^{ij}(x,t)\frac{\partial^{n-1}u}{\partial t^{n-j}\partial x^{j-1}}, \quad i = \overline{1, n}, \tag{10}$$

that form a basis in the space of linear homogeneous matrix differential operators of order $n-1$ and

$$\frac{\partial^{n-1}u}{\partial t^{n-i}\partial x^{i-1}} = \sum_{j=1}^{n} P_{ij}M_j\Big(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)u, \quad i = \overline{1, n}. \tag{11}$$

Using the relations (9), we obtain a decomposition that is valid for each $i=\overline{1,n}$:

$$A\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u=\sum_{j=0}^{n}A_{nj}(x,t)\frac{\partial^n u}{\partial t^{n-j}\partial x^j}=E\frac{\partial^n u}{\partial t^n}+\sum_{j=1}^{n-1}\Big[(P^{i1})^{-1}P^{i,j+1}+(P^{i1})^{-1}\Lambda_i P^{ij}\Big]\times$$

$$\times\frac{\partial^n u}{\partial t^{n-j}\partial x^j}+(p^{i1})^{-1}\Lambda_i P^{in}\frac{\partial^n u}{\partial x^n}=E\frac{\partial^n u}{\partial t^n}+\sum_{j=1}^{n-1}(P^{i1})^{-1}P^{i,j+1}\frac{\partial^n u}{\partial t^{n-j}\partial x^j}+$$

$$+\sum_{j=1}^{n}(P^{i1})^{-1}\Lambda_i P^{ij}\frac{\partial^n u}{\partial t^{n-j}\partial x^j}=\sum_{j=0}^{n-1}(P^{i1})^{-1}P^{i,j+1}\frac{\partial^n u}{\partial t^{n-j}\partial x^j}+\sum_{j=1}^{n}(P^{i1})^{-1}\Lambda_i P^{ij}\frac{\partial^n u}{\partial t^{n-j}\partial x^j}=$$

$$=(P^{i1})^{-1}\Big[E_m\frac{\partial}{\partial t}\Big(\sum_{j=1}^{n}P^{ij}\frac{\partial^{n-1}u}{\partial t^{n-j}\partial x^{j-1}}\Big)+\Lambda_i\Big(\sum_{j=1}^{n}P^{ij}\frac{\partial^{n-1}u}{\partial t^{n-j}\partial x^{j-1}}\Big)-$$

$$-E_m\sum_{j=1}^{n}\frac{\partial P^{ij}}{\partial t}\frac{\partial^{n-1}u}{\partial t^{n-j}\partial x^{j-1}}-\Lambda_i\sum_{j=1}^{n}\frac{\partial P^{ij}}{\partial t}\frac{\partial^{n-1}u}{\partial t^{n-j}\partial x^{j-1}}\Big],$$

and, given (11), we write the result

$$A_n\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u=\left(P^{i1}\right)^{-1}\Big(E_m\frac{\partial}{\partial t}+\Lambda_i\frac{\partial}{\partial x}\Big)M_i\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u-$$

$$-(P^{i1})^{-1}\Big[E_m\sum_{j=1}^{n}\frac{\partial P^{ij}}{\partial t}\frac{\partial^{n-1}u}{\partial t^{n-j}\partial x^{j-1}}+\Lambda_i\sum_{j=1}^{n}\frac{\partial P^{ij}}{\partial t}\frac{\partial^{n-1}u}{\partial t^{n-j}\partial x^{j-1}}\Big],\quad i=\overline{1,n}. \tag{12}$$

Let us put

$$V_i(x,t)\equiv M_i\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u(x,t),\quad i=\overline{1,n} \tag{13}$$

and denote by $v_i^1,\dots,v_i^m$, the components of the vector $V_i$. Then, by virtue of (11)–(13), the system (3) can be rewritten in one of $n$ equivalent forms

$$\frac{\partial V_i}{\partial t}+\Lambda_i\frac{\partial V_i}{\partial x}=\sum_{l=1}^{n}b_{il}(x,t)V_l(x,t)+P^{i1}\left(f(x,t)-\sum_{r=0}^{n-2}\sum_{j=0}^{r}A_{rj}(x,t)\frac{\partial^r u}{\partial t^{r-j}\partial x^j}\right), \tag{14}$$

where $i=\overline{1,n}$, and $b_{il}(x,t)=\sum_{j=1}^{n}[\frac{\partial P^{ij}}{\partial t}+\Lambda_i\frac{\partial P^{ij}}{\partial x}-P^{i1}A_{n-1,j-1}]P_{jl}(x,t)$.

We denote by $\check{P}_{ij}^{sr}(x,t)$ (respectively $\check{P}_{sr}^{ij}(x,t)$, $s,r=\overline{1,m}$) the elements of the matrix $P_{ij}(x,t)$ (respectively $P^{ij}(x,t)$) and for each $r=\overline{1,m}$ we introduce the matrices

$$\alpha^{r1}(t)=\|\alpha_{kl}^{r1}\|=\left\|\sum_{j=0}^{n-1}\sum_{s=1}^{m}B_{1,n-1,k}^{sj}(t)\check{P}_{j+1,l}^{sr}(a_1(t),t)\right\|,\quad k=\overline{1,N_0},\quad l\in I_1^{r+},$$

$$\alpha^{r2}(t)=\|\alpha_{kl}^{r2}\|=\left\|\sum_{j=0}^{n-1}\sum_{s=1}^{m}B_{2,n-1,k}^{sj}(t)\check{P}_{j+1,l}^{sr}(a_2(t),t)\right\|,\quad k=\overline{1,N_0},\quad l\in I_2^{r-},$$

$$\alpha^{r3}(t)=\|\alpha_{kl}^{r3}\|=\left\|\sum_{j=0}^{n-1}\sum_{s=1}^{m}C_{n-1,k}^{sj}(a_1(t),t)\check{P}_{j+1,l}^{sr}(a_1(t),t)\omega_l^{r1}(t)\right\|,\quad k=\overline{N_0+1,N},\quad l\in I_1^{r+},$$

$$\alpha^{r4}(t)=\|\alpha_{kl}^{r4}\|=\left\|-\sum_{j=0}^{n-1}\sum_{s=1}^{m}C_{n-1,k}^{sj}(a_2(t),t)\check{P}_{j+1,l}^{sr}(a_2(t),t)\omega_l^{r2}(t)\right\|,\quad k=\overline{N_0+1,N},\quad l\in I_2^{r-}.$$

Let us construct a square matrix $\beta(t)$ of order $N$

$$\beta(t)=\begin{Vmatrix}\alpha^{11}(t) & \dots & \alpha^{m1}(t) & \alpha^{12}(t) & \dots & \alpha^{m2}(t)\\ \alpha^{13}(t) & \dots & \alpha^{m3}(t) & \alpha^{14}(t) & \dots & \alpha^{m4}(t)\end{Vmatrix}.$$

Let us assume that the condition

$$\det \beta(t) \neq 0, \quad \forall t \in [0, T_1] \tag{15}$$

is fulfilled.

**2.3. Existence and uniqueness theorem for the solution of the problem in the case $n > 2$.**

**Theorem 1.** *Let $n > 2$ and*

1) *the equation (3) is strictly hyperbolic, the coefficients of the operator $A_n$ in (3) are continuously differentiable and the coefficients of the operators $A_i$ ($i < n$), and the free member $f$ are continuous in $\overline{U}_{T_1} = \{(x,t)\colon x \in \mathbb{R},\ 0 \leqslant t \leqslant T_1\}$;*
2) *$g_i^s \in \mathcal{C}^{n-i-1}([a_1^0, a_2^0])$, $i = \overline{0, n-1}$, $s = \overline{1, m}$;*
3) *$B_{qik}^{sj}$, $h_k \in \mathcal{C}([0, T_1])$, $\mathcal{C}_{ik}^{sj} \in \mathcal{C}(\overline{U}_{T_1})$, $i = \overline{0, n-1}$, $j = \overline{0, i}$, $s = \overline{1, m}$, $q = 1, 2$, $k = \overline{1, N_0}$;*
4) *$C_{n-1,k}^{sj} \in C^1(\overline{U}_{T_1})$, $C_{ik}^{sj} \in C^{0,1}(\overline{U}_{T_1})$, $i = \overline{0, n-2}$, $h_k \in C^1([0, T_1])$, $k = \overline{N_0 + 1, N}$;*
5) *the functions $F_i(t, x_1, x_2, y_1, y_2, z_1, z_2)$ are defined to be continuous in all arguments in $\Pi = [0, T_1] \times \mathbb{R}^{2m+4}$ and satisfy the Lipschitz condition for all variables except $t$ with constant $M$;*
6) *the conditions of agreement at the points $(a_1^0, 0)$ and $(a_2^0, 0)$:*
$$\sum_{s=1}^{m} \sum_{i=0}^{n-1} \sum_{j=0}^{i} \left\{ \sum_{q=1}^{2} B_{qik}^{sj}(0) \frac{d^j g_{i-j}^s(a_q^0)}{dx^j} + \int_{a_1^0}^{a_2^0} \mathcal{C}_{ik}^{sj}(\xi, 0) \frac{d^j g_{i-j}^s(\xi)}{d\xi^j} d\xi \right\} = h_k(0), \quad k = \overline{1, N_0};$$
$$\sum_{s=1}^{m} \sum_{i=0}^{n-1} \sum_{j=0}^{i} \int_{a_1^0}^{a_2^0} \mathcal{C}_{ik}^{sj}(\xi, 0) \frac{d^j g_{i-j}^s(\xi)}{d\xi^j} d\xi = h_k(0), \quad k = \overline{N_0 + 1, N}$$
*are fulfilled, that ensure the fulfillment of the ratio $a_i''(0) = F_i(0, a^0, \alpha^0, g_0(a^0))$, $i = 1, 2$, where $a^0 = (a_1^0, a_2^0)$, $\alpha^0 = (\alpha_1^0, \alpha_2^0)$;*
7) *(15) are fulfilled.*

*Then there exists a $\varepsilon \in (0, T_1]$ such that the problem (3)–(8) has a unique generalized solution in $\overline{\Omega}_\varepsilon$ defined for all $t \in [0, \varepsilon]$ (the detailed definition of the solution see below on p.154).*

*Proof.* Let $[\mathcal{C}^1[0, T_1]]^2$ be the space of continuously differentiable vector functions $a(t)$ ($0 \leq t \leq T_1$). In $[\mathcal{C}^1[0, T_1]]^2$ we enter the set

$$Q_{T_1}^h = \{a(t) = (a_1(t), a_2(t))\colon a \in [\mathcal{C}^2[0, T_1]]^2,\ |a_i(t) - a_i^0| \leqslant T_1(\alpha_i^0 + 1),\ |a_i'(t) - \alpha_i^0| \leqslant h,$$
$$0 \leqslant t \leqslant T_1,\ i = 1, 2\},$$

with metrics

$$\rho(a^1(t), a^2(t)) = \sum_{i=1}^{2} \left( \max_t |a_i^1(t) - a_i^2(t)| + \max_t |a_i^{1'}(t) - a_i^{2'}(t)| \right).$$

Let us assume that $T_1$ and $h$ are so small that for all $a \in Q_{T_1}^h$ conditions (8), (15) are satisfied.

For each fixed vector function $a \in Q_{T_1}^h$ we have the problem (3)–(6), which solution $u(x,t) = U(x,t;a)$ is the value of some operator on $a(t)$.

For convenience, we write the equalities (14) in the following form

$$\frac{\partial v_i^s}{\partial t} + \lambda_i^s(x,t) \frac{\partial v_i^s}{\partial x} = \sum_{l=1}^{n} \sum_{r=1}^{m} b_{il}^{sr}(x,t) v_l^r(x,t) -$$

$$-\sum_{l=0}^{n-2}\sum_{j=0}^{l}\sum_{r=1}^{m}\gamma_{ilj}^{sr}(x,t)\frac{\partial^l u^r}{\partial t^{l-j}\partial x^j}+\sum_{r=1}^{m}\hat{P}_{sr}^{i1}(x,t)f^r(x,t),\ i=\overline{1,n},\ s=\overline{1,m}, \tag{16}$$

where $\gamma_{ilj}^{sr}$ are the elements of the matrix $(P^{i1}\cdot A_{lj})$.

Let $l\colon\ y=\psi(\tau;x,t)$ be a smooth curve connecting the point $(x,t)\in\overline{\Omega}_{T_1}$ with the interval $[a_1^0,a_2^0]$, completely lies in $\overline{\Omega}_{T_1}$, and $\psi(t;x,t)=x$, for example [23]

$$\psi(\tau;x,t)=a_1(\tau)+\frac{a_2(\tau)-a_1(\tau)}{a_2(t)-a_1(t)}(x-a_1(t)),\quad 0\leqslant\tau\leqslant t. \tag{17}$$

Let us express all derivatives of the functions $u^s(s=\overline{1,m})$ up to and including the $n-2$-th order in terms of $v_j^s$. To do this, in the obvious equality

$$\frac{\partial^i u^s}{\partial t^{i-j}\partial x^j}=\frac{\partial^i u^s(\psi(0;x,t),0)}{\partial t^{i-j}\partial x^j}+\int_0^t\frac{d}{d\tau}\left(\frac{\partial^i u^s(\psi(\tau;x,t),\tau)}{\partial t^{i-j}\partial x^j}\right)d\tau,\quad i=\overline{0,n-2},\ j=\overline{0,i},$$

write the integral term using the formula for the derivative of a complex function. Then, for each of the obtained $(i+1)$-order derivatives, we apply a similar transformation, and so on to the $(n-1)$-order derivatives, which are expressed in terms of $v_i^s$ using the formulas (11). After that, using the standard permutation of the integration order, we will transform multiple integrals into single integrals. As a result, we get the following image

$$\frac{\partial^i u^s}{\partial t^{i-j}\partial x^j}=\sum_{r=1}^{n-2}\sum_{l=j}^{r-i+j}\frac{d^l g_{r-l}^s(\psi(0;x,t))}{\partial x^l}\delta_{ij}^{rls}(x,t)+\int_0^t\sum_{l=1}^{n}\sum_{r=1}^{m}Q_{ij}^{rls}(\tau,x,t)v_l^r(\psi(\tau;x,t),\tau)d\tau,$$
$$i=\overline{0,n-2},\ j=\overline{0,i},\ s=\overline{1,m}, \tag{18}$$

where $\delta_{ij}^{rls}$, $Q_{ij}^{rls}$ are expressions composed of known smooth functions.

Substituting (18) into equation (16), we come to a system of integro-differential equations of the Volterra type

$$\frac{\partial v_i^s}{\partial t}+\lambda_i^s(x,t)\frac{\partial v_i^s}{\partial x}=\sum_{l=1}^{n}\sum_{r=1}^{m}b_{il}^{sr}(x,t)v_l^r(x,t)+\sum_{l=1}^{n}\sum_{r=1}^{m}\int_0^t D_{il}^{sr}(\tau,x,t)v_l^r(\psi(\tau;x,t),\tau)d\tau+$$
$$+E_i^s(x,t),\quad i=\overline{1,n},\ s=\overline{1,m}, \tag{19}$$

where

$$D_{il}^{sr}(\tau,x,t)=-\sum_{p=0}^{n-2}\sum_{j=0}^{p}\sum_{g=1}^{m}\gamma_{ipj}^{sg}(x,t)Q_{pj}^{lgr}(\tau,x,t),$$

$$E_i^s(x,t)=\sum_{r=1}^{m}\hat{P}_{sr}^{il}(x,t)f^r(x,t)-\sum_{p=0}^{n-2}\sum_{j=0}^{p}\sum_{l=1}^{m}\gamma_{ipj}^{sl}(x,t)\sum_{g=p}^{n-2}\sum_{r=j}^{g-p+j}\frac{d^r g_{p-r}^l(\psi(0;x,t))}{dx^r}\delta_{pj}^{grl}(x,t).$$

Given (10), (13) and (18), we rewrite the initial conditions (4) and the boundary conditions (5), (6) as follows

$$v_i^s(x,0)=\sum_{j=1}^{n}\sum_{r=1}^{m}\hat{P}_{sr}^{ij}(x,0)\frac{d^{j-1}g_{n-j}^r(x)}{dx^{j-1}}=\psi_i^s(x),\quad i=\overline{1,n},\ s=\overline{1,m}, \tag{20}$$

$$\sum_{l=1}^{n}\sum_{r,s=1}^{m}\Bigg\{\sum_{q=1}^{2}\Bigg(\sum_{j=0}^{n-1}B_{q,n-1,k}^{sj}(t)\check{P}_{j+1,l}^{sr}(x,t)v_l^r(x,t)+$$
$$+\sum_{i=0}^{n-2}\sum_{j=0}^{i}B_{qik}^{sj}(t)\int_0^t Q_{ij}^{rls}(\tau,x,t)v_l^r(\psi(\tau;x,t),\tau)d\tau\Bigg)\Bigg|_{x=a_q(t)}+$$

$$+\sum_{j=0}^{n-1}\int_{a_1(t)}^{a_2(t)} C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)v_l^r(\xi,t)d\xi+$$

$$\left.+\sum_{i=0}^{n-2}\sum_{j=0}^{i}\int_{a_1(t)}^{a_2(t)} C_{ik}^{sj}(\xi,t)\int_0^t Q_{ij}^{rls}(\tau,\xi,t)v_l^r(\psi(\tau;\xi,t),\tau)d\tau d\xi\right\} = H_k^1(t), \quad k=\overline{1,N_0}, \tag{21}$$

$$\sum_{l=1}^{n}\sum_{r,s=1}^{m}\left\{\sum_{j=0}^{n-1}\int_{a_1(t)}^{a_2(t)} C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)v_l^r(\xi,t)d\xi+\right.$$

$$\left.+\sum_{i=0}^{n-2}\sum_{j=0}^{i}\int_{a_1(t)}^{a_2(t)} C_{ik}^{sj}(\xi,t)\int_0^t Q_{ij}^{rls}(\tau,\xi,t)v_l^r(\psi(\tau;\xi,t),\tau)d\tau d\xi\right\} = H_k^2(t), \; k=\overline{N_0+1,N}, \tag{22}$$

where

$$H_k^1(t) = h_k(t) - \sum_{i=0}^{n-2}\sum_{j=0}^{i}\sum_{s=1}^{m}\sum_{r=i}^{n-2}\sum_{l=j}^{r-i+j}\left(\sum_{q=1}^{2} B_{qik}^{sj}(t)\times\right.$$

$$\left.\times\frac{d^l g_{r-l}^s(\psi(0;x,t))}{dx^l}\delta_{ij}^{rls}(x,t)\right|_{x=a_q(t)} + \left.\int_{a_1(t)}^{a_2(t)} C_{ik}^{sj}(\xi,t)\frac{d^l g_{r-l}^s(\psi(0;\xi,t))}{d\xi^l}\delta_{ij}^{rls}(\xi,t)d\xi\right),$$

$$H_k^2(t) = h_k(t) - \sum_{i=0}^{n-2}\sum_{j=0}^{i}\sum_{s=1}^{m}\sum_{r=i}^{n-2}\sum_{l=j}^{r-i+j}\int_{a_1(t)}^{a_2(t)} C_{ik}^{sj}(\xi,t)\frac{d^l g_{r-l}^s(\psi(0;\xi,t))}{d\xi^l}\delta_{ij}^{rls}(\xi,t)d\xi.$$

For each $s=\overline{1,m}$ we introduce the notation

$$\mu_i^s(t) = v_i^s(a_1(t),t), \quad i\in I_1^{s+}, \qquad \kappa_i^s(t) = v_i^s(a_2(t),t), \quad i\in I_2^{s-}. \tag{23}$$

Let $x=\varphi_i^s(t;\xi,\tau)$ be the solution of the characteristic equation $dx/dt = -\lambda_i^s(x,t)$, with initial conditions $x(\tau)=\xi$, where $(\xi,\tau)\in\overline{\Omega}_{T_1}$. By $L_i^s(\xi,\tau)$ we denote the corresponding characteristic passing through the point $(\xi,\tau)$ and extended in the direction of decreasing $t$ until it intersects the boundary $\Omega_{T_1}$. Let $t_i^s(\xi,\tau)$ be the least-squares ordinate for points of this characteristic. It is obvious that $0\leqslant t_i^s(\xi,\tau)\leqslant\tau$. Let's take $\varepsilon\in(0,T_1]$ such that the characteristics emitted from the corner points $(a_i^0,0)$ in the domain of $\Omega_\varepsilon$ do not intersect. Then these characteristics divide $\Omega_\varepsilon$ into three subdomains:

$$\Omega_i^{s0} = \{(\xi,\tau)\colon t_i^s(\xi,\tau)=0\}, \quad \Omega_i^{s1} = \{(\xi,\tau)\colon t_i^s(\xi,\tau)>0, \;\; \varphi_i^s(t_i^s(\xi,\tau);\xi,\tau) = a_1(t_i^s(\xi,\tau))\},$$
$$\Omega_i^{s2} = \{(\xi,\tau)\colon t_i^s(\xi,\tau)>0, \;\; \varphi_i^s(t_i^s(\xi,\tau);\xi,\tau) = a_2(t_i^s(\xi,\tau))\}.$$

Any of the sets $\Omega_j^{s1}$ or $\Omega_j^{s2}$ can be empty.

Further, we will need the following derivatives ([19, 23]):

$$\frac{\partial\varphi_i^s(\tau;x,t)}{\partial x} = \exp\left(-\int_\tau^t \lambda_{ix}^{s'}(\varphi_i^s(\sigma;x,t),\sigma)d\sigma\right),$$

$$\frac{\partial\varphi_i^s(\tau;x,t)}{\partial t} = \lambda_i^s(x,t)\exp\left(-\int_\tau^t \lambda_{ix}^{s'}(\varphi_i^s(\sigma;x,t),\sigma)d\sigma\right),$$

$$\frac{\partial t_i^s(x,t)}{\partial x} = \frac{\exp\left(-\int_{t_i^s(x,t)}^t \lambda_{ix}^{s'}(\varphi_i^s(\sigma;x,t),\sigma)d\sigma\right)}{\lambda_i^s(a_l(t_i^s(x,t)),t_i^s(x,t)) - a_l'(t_i^s(x,t))},$$

$$\frac{\partial t_i^s(x,t)}{\partial t} = \frac{\lambda_i^s(x,t)\exp\left(-\int_{t_i^s(x,t)}^t \lambda_{ix}^{s'}(\varphi_i^s(\sigma;x,t),\sigma)d\sigma\right)}{\lambda_i^s(a_l(t_i^s(x,t)),t_i^s(x,t)) - a_l'(t_i^s(x,t))}, \tag{24}$$

where $l=1$ at $(x,t)\in\Omega_i^{s1}$, $l=2$ at $(x,t)\in\Omega_i^{s2}$.

Assuming that $v_i^s \in C^1(\overline{\Omega}_{T_1})$, and integrating (19) along the characteristics ([21]), we obtain the system of integro-functional equations

$$v_i^s(x,t) = W_i^s(x,t) + \int_{t_i^s(x,t)}^{t} \Bigg( \sum_{l=1}^{n}\sum_{r=1}^{m} \Big[ b_{il}^{sr}(\varphi_i^s(\tau;x,t),\tau) v_l^r(\varphi(\tau;x,t),\tau) +$$

$$+ \int_0^\tau D_{il}^{sr}(\eta,\varphi_i^s(\tau;x,t),\tau) v_l^r(\psi(\eta;\varphi_i^s(\tau;x,t),\tau),\eta)d\eta \Big] + E_i^s(\varphi_i^s(\tau;x,t),\tau)\Bigg)d\tau,$$

$$i=\overline{1,n},\ s=\overline{1,m}, \tag{25}$$

$$W_i^s(x,t) = \begin{cases} \psi_i^s(\varphi_i^s(0;x,t)), & \text{at } (x,t)\in\Omega_i^{s0};\\ \mu_i^s(t_i^s(x,t)), & \text{at } (x,t)\in\Omega_i^{s1};\\ \kappa_i^s(t_i^s(x,t)), & \text{at } (x,t)\in\Omega_i^{s2}.\end{cases}$$

**Definition.** *A solution of the problem* (3)–(8) we call a set of functions $a_i \in C^2([0,\varepsilon])$ $(i=1,2)$ and the generalized solution $u$ in the domain $\Omega_\varepsilon$ of the problem (3)–(6), which satisfy the conditions (7), (8). In turn, the *generalized solution of the problem* (3)–(6) is the vector function $u \in [C^{n-2}(\Omega_\varepsilon)]^m$, which, together with its derivatives up to $n-2$ order inclusive, is given as (18), where $v$ is a continuous vector function satisfying for all $(x,t)$ the integro-functional equation (25) and the conditions (20)–(22).

In order for the functions $v_j^s(x,t)$ to be continuous when moving from $\Omega_j^{s0}$ to $\Omega_j^{s1}$ and $\Omega_j^{s2}$, it is necessary that for each $s=\overline{1,m}$ the following conditions

$$\mu_j^s(0)=\psi_j^s(a_1^0),\quad j\in I_1^{s+},\quad \kappa_j^s(0)=\psi_j^s(a_2^0),\quad j\in I_2^{s-} \tag{26}$$

are satisfied. In the first and third members of the condition (21), we substitute the image $v_l^r(x,t)$ from (25):

$$H_k^1(t) = \sum_{l=1}^{n}\sum_{r,s=1}^{m}\Bigg\{ \sum_{q=1}^{2}\Bigg[ \sum_{j=0}^{n-1} B_{q,n-1,k}^{sj}(t)\check{P}_{j+1,l}^{sr}(x,t)W_l^r(x,t) +$$

$$+\sum_{j=0}^{n-1} B_{q,n-1,k}^{sj}(t)\check{P}_{j+1,l}^{sr}(x,t)\int_{t_l^r(x,t)}^{t}\Bigg(\sum_{i=1}^{n}\sum_{p=1}^{m}\Big[b_{li}^{rp}(\varphi_l^r(\tau;x,t),\tau)v_i^p(\varphi_l^r(\tau;x,t),\tau)+$$

$$+\int_0^\tau D_{li}^{rp}(\eta,\varphi_l^r(\tau;x,t),\tau)v_i^p(\psi(\eta;\varphi_l^r(\tau;x,t),\tau),\eta)d\eta\Big] + E_l^r(\varphi_l^r(\tau;x,t),\tau)\Bigg)d\tau +$$

$$+\sum_{i=0}^{n-2}\sum_{j=0}^{i} B_{qik}^{sj}(t)\int_0^t Q_{ij}^{rls}(\tau,x,t)v_l^r(a(\tau),\tau)d\tau\Bigg]\Bigg|_{x=a_q(t)} +$$

$$+\sum_{j=0}^{n-1}\int_{a_1(t)}^{a_2(t)} C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)W_l^r(\xi,t)d\xi +$$

$$+\sum_{j=0}^{n-1}\int_{a_1(t)}^{a_2(t)} C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)\int_{t_l^r(\xi,t)}^{t}\Bigg(\sum_{i=1}^{n}\sum_{p=1}^{m}\Big[b_{li}^{rp}(\varphi_l^r(\tau;\xi,t),\tau)v_i^p(\varphi_l^r(\tau;\xi,t),\tau)+$$

$$+\int_0^\tau D_{li}^{rp}(\eta,\varphi_l^r(\eta;\xi,t),\tau)v_i^p(\psi(\eta;\varphi_l^r(\tau;\xi,t),\tau)\eta)d\eta\Big] + E_l^r(\varphi_l^r(\tau;\xi,t),\tau)d\tau d\xi +$$

$$+\sum_{i=0}^{n-2}\sum_{j=0}^{i}\int_{a_1(t)}^{a_2(t)} C_{ik}^{sj}(\xi,t)\int_0^t Q_{ij}^{rls}(\tau,\xi,t)v_l^r(\psi(\tau;\xi,t),\tau)d\tau d\xi\Bigg)\Bigg\} \equiv$$

$$\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \quad k = \overline{1, N_0}. \tag{27}$$

Given the matrices introduced in 2.2 and the fact that for each $s = \overline{1, m}$, $t_l^s(a_1(t), t) \equiv t$ at $l \in I_1^{s+}$, $t_l^s(a_2(t), t) \equiv t$ at $l \in I_2^{s-}$, we obtain

$$I_1 = \sum_{s,r=1}^{m} \sum_{j=0}^{n-1} \Bigg\{ \sum_{l \in I_1^{r+}} B_{1,n-1,k}^{sj}(t) \check{P}_{j+1,l}^{sr}(a_1(t), t) \mu_l^r(t_l^r(a_1(t), t)) +$$

$$+ \sum_{l \in I_2^{r-}} B_{2,n-1,k}^{sj}(t) \check{P}_{j+1,l}^{sr}(a_2(t), t) \kappa_l^r(t_l^r(a_2(t), t)) +$$

$$+ \sum_{l \in I_1^{r-}} B_{1,n-1,k}^{sj}(t) \check{P}_{j+1,l}^{sr}(a_1(t), t) \psi_l^r(\varphi_l^r(0; a_1(t), t)) +$$

$$+ \sum_{l \in I_2^{r+}} B_{2,n-1,k}^{sj}(t) \check{P}_{j+1,l}^{sr}(a_2(t), t) \psi_l^r(\varphi_l^r(0; a_2(t), t)) \Bigg\} =$$

$$= \sum_{r=1}^{m} \Bigg\{ \sum_{l \in I_1^{r+}} \alpha_{kl}^{r1}(t) \mu_l^r(t) + \sum_{l \in I_2^{r-}} \alpha_{kl}^{r2}(t) \kappa_l^r(t) \Bigg\} +$$

$$+ \sum_{r,s=1}^{m} \sum_{j=0}^{n-1} \Bigg\{ \sum_{l \in I_1^{r-}} B_{1,n-1,k}^{sj}(t) \check{P}_{j+1,l}^{sr}(a_1(t), t) \psi_l^r(\varphi_l^r(0; a_1(t), t)) +$$

$$+ \sum_{l \in I_2^{r+}} B_{2,n-1,k}^{sj}(t) \check{P}_{j+1,l}^{sr}(a_2(t), t) \psi_l^r(\varphi_l^r(0; a_2(t), t)) \Bigg\}, \ k = \overline{1, N_0}. \tag{28}$$

Denote

$$S_l^r(\varphi_l^r(\tau; x, t), \tau; v) = \sum_{i=1}^{n} \sum_{p=1}^{m} \Big[ b_{li}^{rp}(\varphi_l^r(\tau; x, t), \tau) v_i^p(\varphi_l^r(\tau; x, t), \tau) +$$

$$+ \int_0^\tau D_{li}^{rp}(\eta, \varphi_l^r(\tau; x, t), \tau) v_i^p(\psi(\eta; \varphi_l^r(\tau; x, t), \tau), \eta) d\eta \Big] + E_l^r(\varphi_l^r(\tau; x, t), \tau).$$

Then

$$I_2 = \sum_{s,r=1}^{m} \sum_{j=0}^{n-1} \Bigg\{ \sum_{l \in I_1^{r-}} B_{1,n-1,k}^{sj}(t) \check{P}_{j+1,l}^{sr}(a_1(t), t) \int_{t_l^r(a_1(t), t)}^{t} S_l^r(\varphi_l^r(\tau; a_1(t), t), \tau; v) d\tau +$$

$$+ \sum_{l \in I_2^{r+}} B_{2,n-1,k}^{sj}(t) \check{P}_{j+1,l}^{sr}(a_2(t), t) \int_{t_l^r(a_2(t), t)}^{t} S_l^r(\varphi_l^r(\tau; a_2(t), t), \tau; v) d\tau \Bigg\}, \ k = \overline{1, N_0}, \tag{29}$$

and $I_3$, taking into account (23), (25) and the introduced notation, is rewritten as

$$I_3 = \sum_{r,s=1}^{m} \sum_{i=0}^{n-2} \sum_{j=0}^{i} \Bigg\{ \sum_{l \in I_1^{r+}} B_{1ik}^{sj}(t) \int_0^t Q_{ij}^{rls}(\tau, a_1(t), t) \mu_l^r(\tau) d\tau +$$

$$+ \sum_{l \in I_2^{r-}} B_{2ik}^{sj}(t) \int_0^t Q_{ij}^{rls}(\tau, a_2(t), t) \kappa_l^r(\tau) d\tau + \sum_{l \in I_1^{r-}} B_{1ik}^{sj}(t) \int_0^t Q_{ij}^{rls}(\tau, a_1(t), t) \times$$

$$\times \Big( \psi_l^r(\varphi_l^r(0; a_1(t), t)) + S_l^r(\varphi_l^r(\eta; a_1(\tau), \tau), \eta; v) \Big) +$$

$$+ \sum_{l \in I_2^{r+}} B_{2ik}^{sj}(t) \int_0^t Q_{ij}^{rls}(\tau, a_2(t), t) \Big( \psi_l^r(\varphi_l^r(0; a_2(t), t)) + S_l^r(\varphi_l^r(\eta; a_2(\tau), \tau), \eta; v) \Big) d\tau \Bigg\} \tag{30}$$

for $k=\overline{1,N_0}$. Let us write the expression $I_4$ as follows

$$I_4=\sum_{s,r=1}^{m}\sum_{j=0}^{n-1}\Bigg\{\sum_{l\in I_1^{r+}\setminus I_2^{r-}}\Bigg[\int\limits_{a_1(t)}^{\varphi_l^r(t;a_1^0,0)}C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)\mu_l^r(t_l^r(\xi,t))d\xi+$$
$$+\int\limits_{\varphi_l^r(t;a_1^0,0)}^{a_2(t)}C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)\psi_l^r(\varphi_l^r(0;\xi,t))d\xi\Bigg]+$$
$$+\sum_{l\in I_1^{r+}\cap I_2^{r-}}\Bigg[\int\limits_{a_1(t)}^{\varphi_l^r(t;a_1^0,0)}C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)\mu_l^r(t_l^r(\xi,t))d\xi+$$
$$+\int\limits_{\varphi_l^r(t;a_1^0,0)}^{\varphi_l^r(t;a_2^0,0)}C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)\psi_l^r(\varphi_l^r(0;\xi,t))d\xi+\int\limits_{\varphi_l^r(t;a_2^0,0)}^{a_2(t)}C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)\kappa_l^r(t_l^r(\xi,t))d\xi\Bigg]+$$
$$+\sum_{l\in I_2^{r-}\setminus I_1^{r+}}\Bigg[\int\limits_{a_1(t)}^{\varphi_l^r(t;a_2^0,0)}C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)\psi_l^r(\varphi_l^r(0;\xi,t))d\xi+\int\limits_{\varphi_l^r(t;a_2^0,0)}^{a_2(t)}C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)\nu_l^r(t_l^r(\xi,t))d\xi\Bigg]\Bigg\}.$$

In those terms that contain the unknown functions $\mu_l^r$ and $\kappa_l^r$, we replace $\tau=t_l^r(\xi,t)$, and leave the rest of the terms unchanged.

Since $\frac{\partial t_l^r}{\partial\xi}\neq 0$ is in $\overline{\Omega}_l^{r1}$, we can express $\xi$ from the relation $\tau=t_l^r(\xi,t)$ based on the implicit function theorem in $\overline{\Omega}_l^{r1}$. Let us denote the resulting function by $\xi=\rho_t^{r1}(\tau,t)$; by the same theorem, it is continuously differentiable and defined at $0\leqslant\tau<\infty$, with $(\rho_l^{r1}(\tau,t)\in\overline{\Omega}_l^{r1})$. Similarly, in $\overline{\Omega}_l^{r2}$, we can solve the equation $\tau=t_l^r(\xi,t)$ with respect to $\xi$, which will give a continuously differentiable function $\xi=\rho_l^{r2}(\tau,t)$ defined at $0\leqslant\tau<\infty$, with $(\rho_l^{r2}(\tau,t),t)\in\overline{\Omega}_l^{r2}$.

Using the introduced functions $\rho_l^{r1}$, $\rho_l^{r2}$ and considering that $t_l^r(\varphi_l^r(t;a_1^0,0),t)\equiv 0$ for $l\in I_1^{r+}$, $t_l^r(\varphi_l^r(t;a_2^0,0),t)\equiv 0$ for $l\in I_2^{r-}$, we obtain

$$I_4=\sum_{r,s=1}^{m}\sum_{j=0}^{n-1}\Bigg\{-\sum_{l\in I_1^{r+}}\int_0^t C_{n-1,k}^{sj}(\rho_l^{r1}(\tau,t),t)\check{P}_{j+1,l}^{sr}(\rho_l^{r1}(\tau,t),t)\frac{\partial\rho_l^{r1}(\tau,t)}{\partial\tau}\mu_l^r(\tau)d\tau+$$
$$+\sum_{l\in I_2^{r-}}\int_0^t C_{n-1,k}^{sj}(\rho_l^{r2}(\tau,t),t)\check{P}_{j+1,l}^{sr}(\rho_l^{r2}(\tau,t),t)\frac{\partial\rho_l^{r2}(\tau,t)}{\partial\tau}\kappa_l^r(\tau)d\tau\Bigg\}+\Psi(t),\ k=\overline{1,N_0},\quad(31)$$

where $\Psi(t)$ is an expression that contains the known smooth functions $C_{n-1,k}^{sj}$, $\check{P}_{j+1,l}^{sr}$ and $\psi_l^r$.

In $I_5$ we change the order of integration

$$I_5=\sum_{s,r=1}^{m}\sum_{j=0}^{n-1}\Bigg\{\sum_{l\in I_1^{r+}\setminus I_2^{r-}}\int_0^t d\tau\int_{\varphi_l^r(t;a_1(\tau),\tau)}^{a_2(t)}C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)S_l^r(\varphi_l^r(\tau;\xi,t),\tau;v)d\xi+$$
$$+\sum_{l\in I_1^{r+}\cap I_2^{r-}}\int_0^t d\tau\int_{\varphi_l^r(t;a_1(\tau),\tau)}^{\varphi_l^r(t;a_2(\tau),\tau)}C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)S_l^r(\varphi_l^r(\tau;\xi,t),\tau,v)d\xi+$$
$$+\sum_{l\in I_2^{r-}\setminus I_1^{r+}}\int_0^t d\tau\int_{a_1(t)}^{\varphi_l^r(t;a_2(\tau),\tau)}C_{n-1,k}^{sj}(\xi,t)\check{P}_{j+1,l}^{sr}(\xi,t)S_l^r(\varphi_l^r(\tau l\xi,t),\tau;v)d\xi\Bigg\},$$

and replace the variables $y = \varphi_l^r(\tau;\xi,t)$, given that $\xi = \varphi_l^r(t;y,\tau)$ and there is

$$\varphi_l^r(\tau;\varphi_l^r(t;a_1(\tau),\tau),t) = a_1(\tau) l \in I_1^{r+}, \quad \varphi_l^r(\tau;\varphi_l^r(t;a_2(\tau),\tau),t) = a_2(\tau) l \in I_2^{r-}, \ r = \overline{1,m}.$$

As a result, we get

$$I_5 = \sum_{r,s=1}^{m}\sum_{j=0}^{n-1}\Bigg\{ \sum_{l\in I_1^{r+}\setminus I_2^{r-}} \int_0^t d\tau \int_{a_1(\tau)}^{\varphi_l^r(\tau;a_2(t),t)} C_{n-1,k}^{sj}(\varphi_l^r(t;y,\tau),t)\times$$
$$\times \check{P}_{j+1,l}^{sr}(\varphi_l^r(t;y,\tau),t) S_l^r(y,\tau;v)\frac{\partial\varphi_l^r(t;t,\tau)}{\partial y}dy+$$
$$+\sum_{l\in I_1^{r+}\cap I_2^{r-}} \int_0^t d\tau \int_{a_1(\tau)}^{a_2(\tau)} C_{n-1,k}^{sj}(\varphi_l^r(t;y,\tau),t)\check{P}_{j+1,l}^{sr}(\varphi_l^r(t;y,\tau),t)\times$$
$$\times S_l^r(y,\tau;v)\frac{\partial\varphi_l^r(t;y,\tau)}{\partial y}dy+$$
$$+\sum_{l\in I_2^{r-}\setminus I_1^{r+}} \int_0^t d\tau \int_{\varphi_l^r(\tau;a_1(t),t)}^{a_2(\tau)} C_{n-1,k}^{sj}(\varphi_l^r(t;y,\tau),t)\check{P}_{j+1}^{sr}(\varphi_l^r(t;y,\tau),t)\times$$
$$\times S_{lr}^r(y,\tau;v)\frac{\partial\varphi_l^r(t;y,\tau)}{\partial y}dy\Bigg\}, \ k = \overline{1,N_0}. \tag{32}$$

In $I_6$, we change the order of integration and replace the variables $\psi(\tau;\xi,t) = y$, given that $\xi = \psi(t;y,\tau)$. As a result, we get for $k = \overline{1,N_0}$

$$I_6 = \sum_{r,s=1}^{m}\sum_{l=1}^{n}\sum_{i=0}^{n-2}\sum_{j=0}^{i}\int_0^t d\tau \int_{a_1(\tau)}^{a_2(\tau)} C_{ik}^{sj}(\psi(t;u,\tau),t)Q_{ij}^{rls}(\tau,\psi(t;y,\tau),t)v_l^r(e,\tau)\frac{\partial\psi(t;y,\tau)}{\partial y}dy. \tag{33}$$

To find $\frac{\partial\rho_l^{ri}(\tau,t)}{\partial\tau}$ $(l = \overline{1,n}, \ r = \overline{1,m}, \ i = 1,2)$, we use the fact that

$$\text{at } \xi = \rho_l^{r1}(\tau,t) \quad \varphi_l^r(\tau;\rho_l^{r1}(\tau,t),t) = a_1(\tau), \quad l \in I_1^{r+},$$
$$\text{at } \xi = \rho_l^{r2}(\tau,t) \quad \varphi_l^r(\tau;\rho_l^{r2}(\tau,t),t) = a_2(\tau), \quad l \in I_2^{r-}.$$

Differentiating these identities in terms of $\tau$, we obtain

$$\frac{\partial\varphi_l^r(\tau;\rho_l^{ri}(\tau,t),t)}{\partial\tau} + \frac{\partial\varphi_l^r(\tau;\rho_l^{ri}(\tau,t),t)}{\partial\xi}\frac{\partial\rho_l^{ri}(\tau,t)}{\partial\tau} = a_i'(\tau),$$

where $i = 1$, for $l \in I_1^{r+}$, $i = 2$, for $l \in I_2^{r-}$. Hence, taking into account (24), we have

$$\frac{\partial\rho_l^{ri}(\tau,t)}{\partial\tau} = -(\lambda_l^r(a_i(\tau),\tau) - a_i'(\tau))\exp\left(\int_\tau^t {\lambda_{lx}^r}'(\varphi_l^r(\sigma;\rho_l^{ri}(\tau,t),\tau),\sigma)d\sigma\right).$$

For each $r = \overline{1,m}$ we denote

$$R_{kl}^{rq}(\tau,t) = \sum_{s=1}^{m}\sum_{j=0}^{n-1} C_{n-1,k}^{sj}(\rho_l^{rq}(\tau,t),t)\check{P}_{j+1,l}^{sr}(\rho_l^{rq}(\tau;t),t)\times$$
$$\times(\lambda_l^r(a_q(\tau),\tau) - a_q'(\tau))\exp\left(\int_\tau^t {\lambda_{lx}^r}'(\varphi_l^r(\sigma;\rho_l^{rq}(\tau,t),\tau),\sigma)d\sigma\right),$$
$$\Gamma_{kl}^r(y,\tau,t) = \sum_{s=1}^{m}\sum_{j=0}^{n-1} C_{n-1,k}^{sj}(\varphi_l^r(t;y,\tau),t)\check{P}_{j+1,l}^{sr}(\varphi_l^r(t;y,\tau),t)\times$$
$$\times\exp\left(-\int_\tau^t {\lambda_{lx}^r}'(\varphi_l^r(\sigma;y,\tau),\sigma)d\sigma\right), \ l = \overline{1,n}, \ k = \overline{1,N},$$

$$L_{kl}^{rq}(\tau,t)=\sum_{s=1}^{m}\sum_{i=0}^{n-2}\sum_{j=0}^{i}B_{qik}^{sj}(t)Q_{ij}^{rls}(\tau,a_q(t),t),\ k=\overline{1,N_0},\ q=1\ \text{ at }\ l\in I_1^{r+},\ q=2\text{ at }\ l\in I_2^{r+}. \tag{34}$$

Then the condition (27), taking into account (28)–(33) and the introduced notation (34), will be rewritten as follows

$$\sum_{r=1}^{m}\left\{\sum_{l\in I_1^{r+}}\alpha_{kl}^{r1}(t)\mu_l^r(t)+\sum_{l\in I_2^{r-}}\alpha_{kl}^{r2}(t)\kappa_l^r(t)\right\}=$$

$$=\sum_{r=1}^{m}\Bigg\{-\sum_{l\in I_1^{r+}}\int_0^t\Big[R_{kl}^{r1}(\tau,t)+L_{kl}^{r1}(\tau,t)\Big]\mu_l^r(\tau)d\tau+\sum_{l\in I_2^{r-}}\int_0^t\left[R_{kl}^{r2}(\tau,t)-L_{kl}^{r2}(\tau,t)\right]\kappa_l^r(\tau)d\tau-$$

$$-\sum_{s=1}^{m}\Bigg\{\sum_{l\in I_1^{r-}}\Bigg(\sum_{j=0}^{n-1}B_{1,n-1,k}^{sj}(t)\check{P}_{j+1,l}^{sr}(a_1(t),t)\int_{t_l^r(a_1(t),t)}^t S_l^r(\varphi_l^r(\tau;a_1(t),t),\tau;v)d\tau+$$

$$+\sum_{i=0}^{n-2}\sum_{j=0}^{i}B_{1ik}^{sj}(t)\int_0^t Q_{ij}^{rls}(\tau,a_1(t),t)v_l^r(a_1(\tau),\tau)d\tau\Bigg)+$$

$$+\sum_{l\in I_2^{r+}}\Bigg(\sum_{j=0}^{n-1}B_{2,n-1,k}^{sj}(t)\check{P}_{j+1,l}^{sr}(a_2(t),t)\int_{t_l^r(a_2(t),t)}^t S_l^r(\varphi_l^r(\tau;a_2(t),t),\tau;v)d\tau+$$

$$+\sum_{i=0}^{n-2}\sum_{j=0}^{i}B_{2ik}^{sj}(t)\int_0^t Q_{ij}^{rls}(\tau,a_2(t),t)v_l^r(a_2(\tau),\tau)d\tau\Bigg)+$$

$$+\sum_{l=1}^{n}\sum_{i=0}^{n-2}\sum_{j=0}^{i}\int\limits_0^t d\tau\int\limits_{a_1(t\tau)}^{a_2(\tau)}C_{ik}^{sj}(\psi(t;y,\tau),t)Q_{ij}^{rls}(\tau,\psi(t;y,\tau),t)\frac{a_2(t)-a_1(t)}{a_2(\tau)-a_1(\tau)}v_l^r(y,\tau)dy\Bigg\}-$$

$$-\sum_{l\in I_1^{r+}\setminus I_2^{r-}}\int_0^t d\tau\int\limits_{a_1(\tau)}^{\varphi_l^r(\tau;a_2(t),t)}\Gamma_{kl}^r(y,\tau,t)S_l^r(y,\tau;v)dy-$$

$$-\sum_{l\in I_1^{r+}\cap I_2^{r-}}\int_0^t d\tau\int\limits_{a_1(\tau)}^{a_2(\tau)}\Gamma_{kl}^r(y,\tau,t)S_l^r(y,\tau;v)dy-\sum_{l\in I_2^{r-}\setminus I_1^{r+}}\int\limits_0^t d\tau\int\limits_{\varphi_l^r(\tau;a_1(t),t)}^{a_2(\tau)}\Gamma_{kl}^r(y,\tau,t)S_l^r(y,\tau;v)dy-$$

$$-\sum_{s=1}^{m}\sum_{j=0}^{n-1}\Bigg\{\sum_{l\in I_1^{r-}}B_{1,n-1,k}^{sj}(t)\check{P}_{j+1,l}^{sr}(a_1(t),t)\psi_l^r(\varphi_l^r(0;a_1(t),t))+$$

$$+\sum_{l\in I_2^{r+}}B_{2,n-1,k}^{sj}(t)\check{P}_{j+1,l}^{sr}(a_2(t),t)\psi_l^r(\varphi_l^r(0;a_2(t),t))\Bigg\}\Bigg\}-\Psi(t)+H_k^1(t),\ k=\overline{1,N_0}. \tag{35}$$

Let us transform the condition (22) in a similar way. As a result, we get

$$\sum_{r=1}^{m}\left\{\sum_{l\in I_1^{r+}}\int\limits_0^t R_{kl}^{r1}(\tau,t)\mu_l^r(\tau)d\tau-\sum_{l\in I_2^{r-}}\int\limits_0^t R_{kl}^{r2}(\tau,t)\kappa_l^r(\tau)d\tau\right\}=$$

$$=-\sum_{r=1}^{m}\Bigg\{\sum_{s=1}^{m}\sum_{l=1}^{n}\sum_{i=0}^{n-2}\sum_{j=0}^{i}\int\limits_{0}^{t}d\tau\int\limits_{a_1(\tau)}^{a_2(\tau)}C_{ik}^{sj}(\psi(t;y,\tau),t)Q_{ij}^{rls}(\tau,\psi(t;y,\tau),t)\frac{a_2(t)-a_1(t)}{a_2(\tau)-a_1(\tau)}v_l^r(y,\tau)dy+$$

$$+\sum_{l\in I_1^{r+}\setminus I_2^{r-}}\int\limits_0^t d\tau\int\limits_{a_1(\tau)}^{\varphi_l^r(\tau;a_2(t),t)}\Gamma_{kl}^r(y,\tau,t)S_l^r(y,\tau;v)dy+\sum_{l\in I_1^{r+}\cap I_2^{r-}}\int\limits_0^t d\tau\int\limits_{a_1(\tau)}^{a_2(\tau)}\Gamma_{kl}^r(y,\tau,t)S_l^r(y,\tau;v)dy+$$

$$+\sum_{l\in I_2^{r-}\setminus I_1^{r+}}\int\limits_0^t d\tau\int\limits_{\varphi_l^r(\tau;a_1(t),t)}^{a_2(\tau)}\Gamma_{kl}^r(y,\tau,t)S_l^r(y,\tau;v)dy\Bigg\}-\Psi(t)+H_k^2(t),\ k=\overline{N_0+1,N}. \quad (36)$$

Equations (35) are the second kind Volterra equations, and equations (36) are of the first kind with respect to the functions $\mu_l^r(t)$ and $\kappa_l^r(t)$. To reduce the equations (36) to the second kind Volterra type equations, we consider that

$$R_{kl}^{r1}(t,t)=\sum_{j=0}^{n-1}C_{n-1,k}^{sj}(a_1(t),t)\check{P}_{j+1,l}^{sr}(a_1(t),t)\omega_l^{r1}(t)=\alpha_{kl}^{r3}(t)\ k=\overline{N_0+1,N},\ l\in I_1^{r+},$$

$$R_{kl}^{r2}(t,t)=\sum_{j=0}^{n-1}C_{n-1,k}^{sj}(a_2(t),t)\check{P}_{j+1,l}^{sr}(a_2(t),t)\omega_l^{r2}(t)=\alpha_{kl}^{r4}(t)\ k=\overline{N_0+1,N},\ l\in I_2^{r-},$$

and differentiate (36) in $t$. Then we get

$$\sum_{r=1}^{m}\Bigg\{\sum_{l\in I_1^{r+}}\alpha_{kl}^{r3}(t)\mu_l^r(t)+\sum_{l\in I_2^{r-}}\alpha_{kl}^{r4}(t)\kappa_l^r(t)\Bigg\}=\sum_{r=1}^{m}\Bigg\{-\sum_{l\in I_1^{r+}}\int_0^t(R_{kl}^{r1}(\tau,t))'_t\mu_l^r(\tau)d\tau+$$

$$+\sum_{l\in I_2^{r-}}\int_0^t(R_{kl}^{r2}(\tau,t))'_t\kappa_l^r(\tau)d\tau++\sum_{l=1}^{n}G_{kl}^r(S_l^r(\xi,\tau,v),\tau,t)\Bigg\},\ k=\overline{N_0+1,N}, \quad (37)$$

where

$$G_{kl}^r(S_l^r(\xi,\tau,v),\tau,t)=(H_k^2(t))'_t-\Psi'(t)-\frac{d}{dt}\Bigg\{\int_0^t\Bigg(\int_{\gamma_l^r(\tau,t)}^{\chi_l^r(\tau,t)}\Gamma_{kl}^r(y,\tau,t)S_l^r(y,\tau;v)dy+$$

$$\sum_{s=1}^{m}\sum_{i=0}^{n-2}\sum_{j=0}^{i}\int_{a_1(\tau)}^{a_2(\tau)}C_{ik}^{sj}(\psi(t;y,\tau),t)Q_{ij}^{rls}(\tau,\psi(t;y,\tau),t)\frac{a_2(t)-a_1(t)}{a_2(\tau)-a_1(\tau)}v_l^r(y,\tau)dy\Bigg)d\tau\Bigg\},$$

$$\gamma_l^r(\tau,t)=\begin{cases}a_1(\tau),\text{at } l\in I_1^{r+},\\ \varphi_l^r(\tau;a_1(t),t),\text{at } l\in I_2^{r-}\setminus I_1^{r+},\end{cases}\qquad \chi_l^r(\tau,t)=\begin{cases}\varphi_l^r(\tau;a_2(t),t),\text{at } l\in I_1^{r+}\setminus I_2^{r-},\\ a_2(\tau),\text{at } l\in I_2^{r-}.\end{cases}$$

Inequalities (35), (37) form a system of linear integro-functional equations of the Volterra type with respect to $\mu_l^r(t)$ and $\kappa_l^r(t)$, and the coefficient matrix for these functions in the left-hand sides of the equations coincides with $\beta(t)$.

Let us write the system (35), (37) in the operator form

$$\beta\sigma(t)-(K\sigma)(t)+(Lv)(t)+\tilde{H}(t), \quad (38)$$

where $\sigma(t)$ is a column vector consisting of $\mu_l^r(t)$ at $l\in I_1^{r+}$ and $\kappa_l^r(t)$ at $l\in I_2^{r-}$, $r=\overline{1,m}$, respectively; $K$ is a matrix linear integral operator of the Volterra type, the elements of which, for each $r=\overline{1,m}$, are a linear combination of integrals of the form

$$\int_0^t\left[-R_{kl}^{r1}(\tau,t)-L_{kl}^{r1}(\tau,t)\right]\mu_l^r(\tau)d\tau,\quad l\in I_1^{r+},\quad k=\overline{1,N_0},$$

$$\int_0^t \left[-R_{kl}^{r2}(\tau,t) - L_{kl}^{r2}(\tau,t)\right]\kappa_l^r(\tau)d\tau, \quad l \in I_2^{r-}, \quad k=\overline{1,N_0},$$

$$\int_0^t \left(R_{kl}^{r1}(\tau,t)\right)_t' \mu_l^r(\tau)d\tau, \quad l \in I_1^{r+}, \quad k=\overline{N_0+1,N},$$

$$\int_0^t \left(R_{kl}^{r2}(\tau,t)\right)_t' \kappa_l^r(\tau)d\tau, \quad l \in I_2^{r-}, \quad k=\overline{N_0+1,N}$$

with the known continuous kernels $R_{kl}^{ri}$ and $(R_{kl}^{ri}(\tau,t))_t'$; $L$ is a matrix linear integral operator of Volterra type, which elements have continuous kernels acting on the vector function $v$ with components $v_j^r$; $\tilde{H}(t)$ is a column of height $N$ with elements containing the functions $h_k(t)$, $h_k'(t)$.

Based on the condition (15), we rewrite (38) in the form

$$\sigma(t) = \beta^{-1}(K\sigma + Lv + \tilde{H})(t), \quad \text{or} \quad \left(\left[I - \beta^{-1}K\right]\sigma\right)(t) = \beta^{-1}(Lv+\tilde{H})(t). \tag{39}$$

Since $K$ is an integral operator of the Volterra type, which norm is arbitrarily small for a sufficiently small $\varepsilon > 0$, we can proceed to the equation

$$\sigma(t) = [B(Lv+\tilde{H})](t), \tag{40}$$

where $B = (I-\beta^{-1}K)^{-1}\beta^{-1}$.

On the other hand, the equation (25) in the operator form will be

$$v(x,t) = (\tilde{Q}\sigma)(x,t) + (\tilde{L}v)(x,t) + \tilde{F}(x,t), \tag{41}$$

where $\tilde{Q}$ is the shift operator, which operates according to the formulas

$$(\tilde{Q}\mu_l^r)(x,t) = \mu_l^r(t_l^r(x,t)), \ \ l \in I_1^{r+}, \quad (\tilde{Q}\kappa_l^r)(x,t) = \kappa_l^r(t_l^r(x,t)), \ \ l \in I_2^{r-}, \ \ r=\overline{1,m};$$

$\tilde{L}$ is a matrix linear integral operator of Volterra type with a continuous kernel. The operator $\tilde{Q}$ is uniquely defined and for a continuous vector function $\sigma(t)$ its action $\tilde{Q}\sigma(t)$ is also a continuous vector function if and only if the consistency conditions (36) are satisfied. It follows from condition 6) that the functions $\mu_l^r(t)$ and $\kappa_l^r(t)$ given by (40) satisfy the condition (26) for any $v$.

Substituting (40) into (41), we arrive at the relation

$$([I - \tilde{Q}BL - \tilde{L}]v)(x,t) = (\tilde{Q}B\tilde{H} + \tilde{F})(x,t). \tag{42}$$

Thus, the problem (20)–(22), (25) is equivalent to the system of equations (42), in which $\sigma$ is absent.

Since $L$ and $\tilde{L}$ are integral operators of the Volterra type, (42) can be rewritten as

$$v(x,t) = \left(I - \tilde{Q}BL - \tilde{L}\right)^{-1}\left[\tilde{Q}B\tilde{H} + \tilde{F}\right](x,t). \tag{43}$$

Thus, we have found the only continuous vector function $v$ that satisfies the integro-functional equation (25) for all $(x,t)$. Therefore, taking into account (18), we obtain the image of a single generalized solution of the problem (3)–(6) for each function $a \in Q_\varepsilon^h$. It remains only to choose the one for which the conditions (7), (8) are fulfilled from the whole set of admissible vector functions $a(t)$. Each function $a \in Q_\varepsilon^h$ corresponds to a generalized solution in $\overline{\Omega}_\varepsilon = \overline{\Omega}_{\varepsilon,a}$ of the corresponding problem (3)–(6). We denote this solution by $U(x,t;a(t))$ (its value at fixed $x$, $t$ is a function of $a(t)$).

For each $r=\overline{1,m}$, the dependence $U^r(a(t),t;a)$ in the metric of uniform deviation from $a$ as an element of $\left[C^1[0,\varepsilon]\right]^2$ satisfies the Lipschitz condition: $\exists L_u \geqslant 0$, $\forall a^1, a^2 \in Q_\varepsilon^h$:

$$\max_{0\leqslant t\leqslant\varepsilon}\left|U^r(a^1(t),t;a^1) - U^r(a^2(t),t;a^2)\right| \leqslant L_u\left[\max_{0\leqslant t\leqslant}|a^1(t)-a^2(t)| + \max_{0\leqslant t\leqslant\varepsilon}|a^{1\prime}(t)-a^{2\prime}(t)|\right]. \tag{44}$$

In order to verify the relation (44), we note that from (18) and (43) we can obtain a priori estimates for the solution and its derivatives $\frac{\partial u^r}{\partial t}$, $\frac{\partial u^r}{\partial t}$ by the given functions. Hence, it follows that

$$|U^r(x,t;a)| \leqslant U_0,\ \left|U_x^{r\prime}(x,t;a)\right| \leqslant U_1,\ \left|U_t^{r\prime}(x,t;a)\right| \leqslant U_2$$
$$(U_j \equiv \text{const},\ r=\overline{1,m},\ (x,t)\in\overline{\Omega}_\varepsilon,\ a\in Q_\varepsilon^h).$$

Therefore, on the lines $x=a_i(t)$ $(i=1,2)$, the dependence of the solution of (3)–(6) on the functional parameter $a$ satisfies the Lipschitz condition, whence (44) follows.

Since $F_i$ is continuous in $\Pi$, it is bounded in any closed subdomain (e.g., for $a\in Q_\varepsilon^h$, $|U^r(x,t;a)|\leqslant U_0$), i.e., $\exists\Phi>0\colon |F_i(t,x,y,z)|\leqslant\Phi$. Let's choose $\varepsilon\in(0;T_1]$ so small that the following conditions are satisfied

1) $\varepsilon<\min\left\{\frac{h}{\Phi},\frac{2}{\Phi}\right\}$;
2) $M(\varepsilon^2/2+\varepsilon)\max\{1,L_u\}<1$.

Consider on $Q_\varepsilon^h$ the operator $\tilde{A}a=\{\tilde{A}_1a,\tilde{A}_2a\}$, which is given by the system of equations

$$(\tilde{A}_ia)(t)=a_i^0+\alpha_i^0t+\int_0^t d\tau\int_0^\tau F_i(\eta,a_i(\eta),a_i'(\eta),U(a_i(\eta),\eta;a_i(\eta)))d\eta,\ t\in[0,\varepsilon].$$

The operator $\tilde{A}$ maps $Q_\varepsilon^h$ into itself and in the metric of the space $[C^1[0,\varepsilon]]^2$ is a compression. Indeed, let $a\in Q_\varepsilon^h$, $t\in[0,\varepsilon]$. Then

$$|(\tilde{A}_ia)(t)-a_i^0|\leqslant$$
$$\leqslant\left|\alpha_i^0t+\int_0^t d\tau\int_0^\tau F_i(\eta,a_i(\eta),a_i'(\eta),U(a_i(\eta),\eta;a_i(\eta)))d\eta\right|\leqslant\alpha_i^0\varepsilon+\Phi\varepsilon^2/2\leqslant\varepsilon(\alpha_i^0+1),$$
$$|(\tilde{A}_ia)'(t)-\alpha_i^0|\leqslant\left|\int_0^t F_i(\tau,a_i(\tau),a_i'(\tau),U(a_i(\tau),\tau;a_i(\tau)))d\tau\right|\leqslant\Phi\varepsilon\leqslant h,$$

and therefore $\tilde{A}Q_\varepsilon^h\subset Q_\varepsilon^h$. In addition, $(F=(F_1,F_2))$

$$\rho(\tilde{A}a^1,\tilde{A}a^2)\leqslant\max_t\left|\int_0^t d\tau\int_0^\tau\right|F(\eta,a^1(\eta),a1'(\eta),U(a_1(\eta),\eta;a^1(\eta)))-$$
$$-F(\eta,a^2(\eta),a^{2\prime}(\eta),U(a^2(\eta),\eta;a^2(\eta)))\Big|d\eta\Big|+$$
$$+\max_t\left|\int_0^t\left|F(\tau,a^1(\tau),a^{1\prime}(\tau),U(a^1(\tau),\tau;a^1(\tau)))-F(\tau,a^2(\tau),a^{2\prime}(\tau),U(a^2(\tau),\tau;a^2(\tau)))\right|d\tau\right|\leqslant$$
$$\leqslant M(\varepsilon^2/2+\varepsilon)\max_{0\leqslant t\leqslant\varepsilon}\max\{|a^1(t)-a^2(t)|;|a^{1\prime}(t)-a^{2\prime}(t)|;|U(a^1(t),t;a^1)-U(a^2(t),t;a^2)|\}\leqslant$$
$$\leqslant M(\varepsilon^2/2+\varepsilon)\max\{1,L_u\}\rho(a^1,a^2).$$

Since $M(\varepsilon^2/2+\varepsilon)\max\{1,L_u\}<1$, $\tilde{A}$ is a compression operator. Therefore, the existence and uniqueness of the fixed point of the operator, i.e., the desired solution, which can be found by iteration, follows from the principle of compressive mappings. □

**3. Hyperbolic Stefan problems for a semilinear system of first order equations in a curved sector.** In $G_t:=\{(x,t)\colon t\in\mathbb{R}_+,a_1(t)<x<a_2(t),a_1(0)=a_2(0)=0\}$, where the functions $a_l\in C^1(\mathbb{R}_+)$, $l=1,2$ are unknown in advance, consider the system of equations

$$\frac{\partial u_i}{\partial t}+\lambda_i(x,t)\frac{\partial u_i}{\partial x}=f_i(x,t;u),\ i=\overline{1,n}, \tag{45}$$

where $u = \mathrm{col}(u_1, \dots, u_n)$. The conditions on the unknown boundaries are given in the form

$$a_l'(t) = \sum_{r=1}^{2}\sum_{i=1}^{n}\int_0^t \gamma_{il}^r(a(\tau),\tau)u_i(a_r(\tau),\tau)d\tau + h_l(a(t),t),\ l = 1,2, \tag{46}$$

where $\gamma_{il}^r(a(t),t) = \gamma_{il}^r(a_1(t),a_2(t),t)$, $h_l(a(t),t) = h_l(a_1(t),a_2(t),t)$ are known functions, and $h_1(0,0) \neq h_2(0,0)$.

Let us assume that the following conditions are fulfilled

$$\lambda_i(0,0) - h_l(0,0) > 0,\ \ i = \overline{1,p}, \quad \lambda_i(0,0) - h_l(0,0) < 0,\ \ i = \overline{p+1,n}, \tag{47}$$

i.e., none of the characteristics coming from the point $(0,0)$ falls into the $G_t$.

Consider the following problem: for some $T > 0$, find the functions $a_1, a_2 \in C^1([0,T])$ and in the domain $G_T\colon = \{(x,t)\colon t \in [0,T], a_1(t) < x < a_2(t), a_1(0) = a_2(0) = 0\}$ the solution $u_i \in C^1(\overline{G}_T)$ $(i = \overline{1,n})$ of system (45) such that for all $t \in [0,T]$ conditions (46) and the boundary conditions

$$\sum_{i=1}^{n}\int_{a_1(t)}^{a_2(t)} \alpha_{ki}(y,t)u_i(y,t)dy = H_k(t),\ k = \overline{1,n} \tag{48}$$

are satisfied, where $\alpha_{ki}(y,t)$, $H_k(t)$ are given functions, and $H_k(0) = 0$.

For each $l = 1,2$ we introduce the matrices $\alpha^l(t) = [\alpha_{kj}^l(t)]_{k,j=1}^n$, where
$\alpha_{kj}^l(t) = \alpha_{kj}(a_l(t),t),\ k = \overline{1,n},\ j = \overline{1,p};\ \ \alpha_{kj}^l(t) = \alpha_{kj}(a_{3-l}(t),t),\ k = \overline{1,n},\ j = \overline{p+1,n}.$
It is assumed that

$$\det \alpha^1(0) \neq 0. \tag{49}$$

Let us differentiate (48) with respect to $t$ and set $t = 0$. Then, taking into account (46), we obtain

$$\sum_{i=1}^{n} \alpha_{ki}(0,0)(h_2(0,0) - h_1(0,0))u_i(0,0) = H_k'(0),\ k = \overline{1,n}. \tag{50}$$

Since the condition (49) is satisfied, the system (50) has a unique solution $u_i(0,0)$.

**Definition 1.** The solution of the problem (45)–(48) is the set of functions $a_1, a_2 \in C^1([0,T])$ and the classical solution $u(x,t)$ in the domain $G_T$ of the problem (45), (48) that satisfy the condition (46) for all $t \in [0,T]$.

**Theorem 2.** *Let the following conditions be fulfilled:*

1) *the system* (45) *is hyperbolic, i.e., for some $T > 0$, the functions $\lambda_i(x,t)$ are real-valued, and $\lambda_i \in C^2(\mathbb{R} \times [0,T])$ satisfy* (47)*;*
2) *the functions $f_i \in C^1(\mathbb{R} \times [0,T] \times \mathbb{R}^n)$, $f_{ix}'$ and $f_{iu}'$ satisfy the Lipschitz condition with respect to $u$;*
3) *$\alpha_{ki} \in C^2(\mathbb{R} \times [0,T])$, $H_k \in C^2([0,T])$, $H_k(0) = 0$;*
4) *the functions $\gamma_{il}^r \in C(\mathbb{R}^2 \times [0,T])$, $h_l \in C^1(\mathbb{R}^2 \times [T])$, where $h_1(0,0) \neq h_2(0,0)$ and satisfy the Lipschitz condition by the first variable with constant $L_\gamma$, $L_h$, respectively;*
5) *the condition* (49) *is fulfilled.*

*Then there exists a $\varepsilon \in (0,T]$ such that the problem* (45)–(48) *has a unique solution in $G_\varepsilon$, defined for all $t \in [0,\varepsilon]$.*

*Proof.* Let $[C[0,T]]^2$ be the space of vector functions continuous on $[0,T]$. In $[C[0,T]]^2$ we introduce a set

$Q_T := \{a(t) = (a_1(t), a_2(t)) \colon a \in [C^1[0,T]]^2,\ |a_l(t)| \leqslant T(1+H),\ 0 \leqslant t \leqslant T,\ l = 1,2\}$, (where $H$ is some constant that constrains the continuous functions $h_l(a(t),t)$), with the metric

$$\rho(a^1(t), a^2(t)) = \sum_{l=1}^{2} \max_t |a_l^1(t) - a_l^2(t)|.$$

For each fixed vector function $a \in Q_T$, we have the problem (45)–(48).

Let us introduce the following auxiliary functions

$$u_i(a_1(t),t) = \mu_i(t),\ i = \overline{1,p};\quad u_i(a_2(t),t) = \mu_i(t),\ i = \overline{p+1,n}. \tag{51}$$

Let $\varphi_i(\tau; x, t)$ be the solution of the Cauchy problem $d\xi/d\tau = \lambda_i(\xi,\tau)$, $\xi(t) = x$. It follows from the continuity of $\lambda_i$, $a_l$ and the conditions (46)–(47) that it is always possible to choose $\varepsilon_0 \in (0,T]$ such that at $t \in [0,\varepsilon_0]$ the following holds

$$\lambda_i(a_l(t),t) - a_l(t) > 0,\ i = \overline{1,p},\quad \lambda_i(a_l(t),t) - a_l(t) < 0,\ i = \overline{p+1,n},\ l = 1,2. \tag{52}$$

Assuming that in the system (45) the functions $u_i(x,t)$ are continuously differentiable and integrating (45) along the characteristics, we come to the system of integro-functional equations

$$u_i(x,t) = \mu_i(t_i(x,t)) + \int_{t_i(x,t)}^{t} f_i(\varphi_i(\tau;x,t),\tau;u)d\tau,\ i = \overline{1,n},\ (x,t) \in G_{\varepsilon_0}, \tag{53}$$

where $t_i(x,t) := \min\{\tau \colon (\varphi_i(\tau;x,t),\tau) \in G_{\varepsilon_0}\}$.

In (53), the functions $\mu_i$ are also unknown, which are found as follows. Let us substitute (53) into (48):

$$\sum_{i=1}^{p} \int_{a_1(t)}^{a_2(t)} \alpha_{ki}(x,t)\mu_i(t_i(x,t))dx + \sum_{i=p+1}^{n} \int_{a_1(t)}^{a_2(t)} \alpha_{ki}(x,t)\mu_i(t_i(x,t))dx +$$
$$+ \sum_{i=1}^{p} \int_{a_1(t)}^{a_2(t)} \alpha_{ki}(x,t) \int_{t_i(x,t)}^{t} f_i(\varphi_i(\tau;x,t),\tau;u)d\tau dx +$$
$$+ \sum_{i=p+1}^{n} \int_{a_1(t)}^{a_2(t)} \alpha_{ki}(x,t) \int_{t_i(x,t)}^{t} f_i(\varphi_i(\tau;x,t),\tau;u)d\tau dx = h_k(t),\ k = \overline{1,n},$$

and transform in single integrals the left-hand sides of the obtained equations from the integration variable $x$ to the variable $\tau$ by replacing $\tau = t_i(x,t)$. In double integrals, we change the order of integration and replace the variables $y = \varphi_i(\tau;x,t)$. As a result, we get

$$-\sum_{i=1}^{p} \int_{t_i(a_2(t),t)}^{t} \alpha_{ki}(\rho_i^1(\tau,t),t)\frac{\partial \rho_i^1(\tau,t)}{\partial \tau}\mu_i(\tau) + \sum_{i=p+1}^{p} \int_{t_i(a_1(t),t)}^{t} \alpha_{ki}(\rho_i^2(\tau,t),t)\frac{\partial \rho_i^2(\tau,t)}{\partial \tau}\mu_i(\tau)d\tau +$$
$$+ \sum_{i=1}^{p} \int_{t_i(a_2(t),t)}^{t} d\tau \int_{a_1(\tau)}^{\varphi_i(\tau;a_2(t),t)} \alpha_{ki}(\varphi_i(t;y,\tau),t) f_i(y,\tau;u)\frac{\partial \varphi_i(t;y,\tau)}{\partial y}dy +$$
$$+ \sum_{i=p+1}^{n} \int_{t_i(a_1),t}^{t} d\tau \int_{\varphi_i(\tau;a_1(t),t)}^{a_2(\tau)} \alpha_{ki}(\varphi(t;y,\tau),t) f_i(y,\tau;u)\frac{\partial \varphi_i(t;y,\tau)}{\partial y}dy = H_k(t),\ k = \overline{1,n}, \tag{54}$$

where $\rho_i^l(\tau,t)$ are the same as in Section 2.

By entering the following notation

$$R_{ki}^l(\tau,t) = \alpha_{ki}(\rho_i^l(\tau,t),t)(\lambda_i(a_l(\tau),\tau) - a_l'(\tau)) \exp\left(\int_{\tau}^{t} \lambda_{ix}'(\varphi_i(\sigma;\rho_i^l(\tau,t),t),\sigma)d\sigma\right),$$

where $l=1$ at $i=\overline{1,p}$, $l=2$ at $i=\overline{p+1,n}$,

$$Q_{ki}(y,\tau,t)=\alpha_{ki}(\varphi_i(t;y,\tau),t)\exp\left(\int_\tau^t \lambda'_{iy}(\varphi_i(\sigma;y,t),\sigma)d\sigma\right), \tag{55}$$

rewrite (54) as follows

$$\sum_{i=1}^{p}\int_{t_i(a_2(t),t)}^{t} R^1_{ki}(\tau,t)\mu_i(\tau)d\tau-\sum_{i=p+1}^{n}\int_{t_i(a_1(t),t)}^{t} R^2_{ki}(\tau,t)\mu_i(\tau)d\tau+$$
$$+\sum_{i=1}^{p}\int_{t_i(a_2(t),t)}^{t} d\tau\int_{a_1(\tau)}^{\varphi_i(\tau;a_2(t),t)} Q_{ki}(y,\tau,t)f_i(y,\tau;u)dy+$$
$$+\sum_{i=p+1}^{n}\int_{t_i(a_1(t),t)}^{t} d\tau\int_{\varphi_i(\tau;a_1(t),t)}^{a_2(\tau)} Q_{ki}(y,\tau,t)f_i(y,\tau;u)dy=H_k(t),\ k=\overline{1,n}.$$

Thus, we have a system of Volterra-type equations of the first order. Let us differentiate it in $t$, taking into account that

$$R^l_{ki}(t,t)=\alpha_{ki}(a_l(t),t)(\lambda_i(a_l(t),t)-a'_l(t)),\ l=1,2,\ k=\overline{1,n},\ i=\overline{1,n},$$
$$R^l_{ki}(t_i(a_{3-l}(t),t))\frac{dt_i(a_{3-l}(t),t)}{dt}=\alpha_{ki}(a_l(t),t)(\lambda_i(a_l(t),t)-a'_l(t)).$$

As a result, we obtain a system of linear integral equations of the Volterra type of the second order with respect to $\mu_i$:

$$\sum_{i=1}^{p}\alpha_{ki}(a_1(t),t)(\lambda_i(a_1(t),t)-a'_1(t))\mu_i(t)-\sum_{i=p+1}^{n}\alpha_{ki}(a_2(t),t)(\lambda_i(a_2(t),t)-a'_2(t))\mu_i(t)=$$
$$=\sum_{i=1}^{p}\alpha_{ki}(a_2(t),t)(\lambda_i(a_2(t),t)-a'_2(t))\mu_i(t_i(a_2(t),t))-$$
$$-\sum_{i=p+1}^{n}\alpha_{ki}(a_1(t),t)(\lambda_i(a_1(t),t)-a'_1(t))\mu_i(t_i(a_1(t),t))-$$
$$-\sum_{i=1}^{p}\int_{t_i(a_2(t),t)}^{t}(R^1_{ki}(\tau,t))'_t\mu_i(\tau)d\tau+\sum_{i=p+1}^{n}\int_{t_i(a_1(t),t)}^{t}(R^2_{ki}(\tau,t))'_t\mu_i(\tau)d\tau+$$
$$+\sum_{i=1}^{n}\frac{d}{dt}\left\{\int_{\psi_i(t)}^{t}d\tau\int_{\chi^1_i(\tau,t)}^{\chi^2_i(\tau,t)} Q_{ik}(y,\tau,t)f_i(y,\tau;u)dy\right\}+H'_k(t),\ k=\overline{1,n}, \tag{56}$$

where

$$\psi_i(t)=\begin{cases} t_i(a_2(t),t), & i=\overline{1,p};\\ t_i(a_1(t),t), & i=\overline{p+1,n},\end{cases}$$

$$\chi^1_i(\tau,t)=\begin{cases} a_1(\tau), & i=\overline{1,p};\\ \varphi_i(\tau;a_1(t),t), & i=\overline{p+1,n},\end{cases}\quad \chi^2_i(\tau,t)=\begin{cases} \varphi_i(\tau;a_2(t),t), & i=\overline{1,p};\\ a_2(\tau), & i=\overline{p+1,n}.\end{cases}$$

To find the functions $\mu_i$, we rewrite the system (56) in the operator form

$$A(t)\mu(t)=B(t)(P\mu)(t)+(K\mu)(t)+(Lu)(t)+H'(t), \tag{57}$$

where $\mu(t)=\mathrm{col}(\mu_1(t),\dots,\mu_p(t),\mu_{p+1}(t),\dots,\mu_n(t))$; $A(t)=\alpha^1(t)\Lambda_1(t)$; $B(t)=\alpha^2(t)\Lambda_2(t)$; $\Lambda_l(t)=\mathrm{diag}\{\lambda_1(a_l(t),t)-a'_l(t),\dots,\lambda_p(a_l(t),t)-a'_l(t),-(\lambda_{p+1}(a_{3-l}(t),t)-a'_{3-l}(t)),\dots,-(\lambda_n(a_{3-l}(t),t)-a'_{3-l}(t))\}$, $l=1,2$; $K$ is a linear matrix integral operator of Volterra type with continuously differentiable kernels; $L$ is a nonlinear matrix integral operator of the

Volterra type, which elements have continuously differentiable kernels acting on the vector-function $f$ with components $f_i(x,t;u)$; $H'$ — $n$-dimensional column vector with components $H'_k(t)$; $P$ is a shift operator acting according to the formulas:

$$(P\mu)(t) = \mu_i(t_i(a_2(t),t)), \quad i=\overline{1,p}; \quad (P\mu)(t) = \mu_i(t_i(a_1(t),t)), \quad i=\overline{p+1,n}.$$

Since $0 \leqslant t_i(a_l(t),t) \leqslant t$, $l=1,2$, $i=\overline{1,n}$, the operator $P$ translates elements of the space $[C[0,\varepsilon_0]]^n$ into elements of the same space. In particular, from the fact that

$$\max_t |(P\mu)_l(t)| \leqslant \max_t |\mu_l(t)|, \ l=\overline{1,n},$$

it follows that the norm of the operator $P$ is 1 (at $\mu_l(t) \equiv \text{const}$ these inequalities turn into equations).

From the conditions of (47) and (49) it follows that $\det A(t) \neq 0$ for small $t$. Therefore, (57) can be rewritten as $\mu(t) = [A^{-1}B(P\mu)](t) + [A^{-1}K\mu](t) + [A^{-1}Lu](t) + [A^{-1}H'](t)$ and denote by $[M\mu](t) = [A^{-1}B(P\mu)](t)$. Taking into account (47), we obtain the estimate

$$|\Lambda_1^{-1}(0)\Lambda_2(0)| = \left| \frac{(\lambda_1(0,0)-a'_2(0))\cdots(\lambda_p(0,0)-a'_2(0))}{(\lambda_1(0,0)-a'_1(0))\cdots(\lambda_p(0,0)-a'_1(0))} \times \right.$$
$$\left. \times \frac{(a'_1(0)-\lambda_{p+1}(0,0))\cdots(a'_1(0)-\lambda_n(0,0))}{(a'_2(0)-\lambda_{p+1}(0,0))\cdots(a'_2(0)-\lambda_n(0,0))} \right| < 1. \tag{58}$$

Since $\alpha^1(0) = \alpha^2(0)$, $P(0) = I$ and (58) holds, $|M(0)| < 1$. It is obvious that $|M(0)| < 1$ for some $\beta_1 \in (0,\varepsilon_0]$ at $t \in [0,\beta_1]$. So there exists $(I-M)^{-1}$. Then

$$\mu(t) = [(I-M)^{-1}A^{-1}(K\mu + H')](t) + [(I-M)^{-1}A^{-1}Lu](t).$$

Since $K$ is an integral operator of the Volterra type, it follows that its norm for sufficiently small $\beta_1 > 0$ is also arbitrarily small. So,

$$\mu(t) = [L_0Lu](t) + [L_0H'](t), \tag{59}$$

where $L_0 = (I-(I-M)^{-1}A^{-1}K)^{-1}(I-M)^{-1}A^{-1}$.

On the other hand, the system of equations (53) in the operator form is

$$u(x,t) = [\tilde{Q}\mu](x,t) + [L_1u](x,t), \tag{60}$$

where $\tilde{Q}$ is a shift operator similar to the $P$ operator, and $L_1$ is a nonlinear integral operator of the Volterra type. Substitute (59) into (60). We get

$$u(x,t) = [\tilde{Q}L_0H'](x,t) + [\tilde{Q}L_0Lu](x,t) + [L_1u](x,t). \tag{61}$$

Thus, we have an equation that corresponds to the system (53), but no longer contains the functions $\mu_i$.

Let us choose $C$ such that $|\mu(0)| < C$ (i.e., $|(I-M)^{-1}A_{-1}H'(0)| < C$). Then, for a sufficiently small $\beta_2 \in (0,\varepsilon_0]$, the operator $Bu$, defined by the right-hand side of (61), maps the ball $S_{\beta_2} = \{u(x,t)\colon \|u\| = \max_{i,x,t} |u_i(x,t) \leqslant C|\}$ into itself. Indeed, let's take $\beta_2$ so small that $|\tilde{Q}L_0L| + |\tilde{Q}(I-(I-M)^{-1}A^{-1}K)^{-1}| \leqslant 1$ (this is possible because $L$, $L_1$, $K$ are Volterra operators and $\|\tilde{Q}\| = 1$). Then $\|Bu\| \leqslant |\tilde{Q}L_0L + L_1|C + |\tilde{Q}(I-(I-M)^{-1}K)^{-1}|C \leqslant C$. Thus, $BS_{\beta_2} \subset S_{\beta_2}$.

Since the functions $f_i(x,t;u)$ $(i=\overline{1,n})$ in $(\mathbb{R}\times[0,T]\times\mathbb{R}^n)$ satisfy the Lipschitz condition in $u$, and $L$ and $L_1$ are Volterra, then for a sufficiently small $\beta_3 \in (0,T]$ the operator $B$ satisfies the Lipschitz condition in $u$ with an arbitrarily small constant, i.e., it is a contraction. Therefore, according to Banach's theorem, there exists a single fixed point. Thus, for each vector function $a \in Q_{\varepsilon_1}$, where $\varepsilon_1 = \min\{\beta_1,\beta_2,\beta_3\}$, we have found a single continuous solution $u(x,t)$.

Let us prove the existence of a continuously differentiable solution. To do this, we formally differentiate (53) in $x$ and write the result in the operator form

$$\frac{\partial u(x,t)}{\partial x} = [QT^1\mu'](x,t) + \Psi^1(x,t,u) + [F^1u](x,t) + \left[N^1\frac{\partial u}{\partial x}\right](x,t), \tag{62}$$

where $\frac{\partial u(x,t)}{\partial x} = \mathrm{col}(\frac{\partial u_1(x,t)}{\partial x}, \ldots, \frac{\partial u_n(x,t)}{\partial x})$; $T^1$ is a diagonal matrix with bounded elements; $\Psi^1$ is an expression constructed from known continuous vector functions; $F^1$, $N^1$ are nonlinear Volterra-type interior operators.

Let us formally differentiate (54) with respect to $t$, and we obtain

$$\sum_{i=1}^{p}\alpha_{ki}(a_1(t),t)(\lambda_i(a_1(t),t)-a_1'(t))\mu_i'(t) - \sum_{i=p+1}^{n}\alpha_{ki}(a_2(t),t)(\lambda_i(a_2(t),t)-a_2'(t))\mu_i'(t) =$$

$$= \sum_{i=1}^{p}\alpha_{ki}(a_2(t),t)(\lambda_i(a_2(t),t)-a_2'(t))\frac{dt_i(a_2(t),t)}{dt}\mu_i'(t_i(a_2(t),t)) -$$

$$- \sum_{i=p+1}^{n}\alpha_{ki}(a_1(t),t)(\lambda_i(a_1(t),t)-a_1'(t))\frac{dt_i(a_1(t),t)}{dt}\mu_i'(t_i(a_1(t),t)) -$$

$$- \sum_{i=1}^{p}\left(\frac{d}{dt}\left[\alpha_{ki}(a_1(t),t)(\lambda_i(a_1(t),t)-a_1'(t))\right] + \left(\left(R_{ki}^1(\tau,t)\right)_t'\right)|_{\tau=t}\right)\mu_i(t) +$$

$$+ \sum_{i=p+1}^{n}\left(\frac{d}{dt}\left[\alpha_{ki}(a_2(t),t)(\lambda_i(a_2(t),t)-a_2'(t))\right] + \left(\left(R_{ki}^2(\tau,t)\right)_t'\right)|_{\tau=t}\right)\mu_i(t) +$$

$$+ \sum_{i=1}^{p}\left(\frac{d}{dt}\left[\alpha_{ki}(a_2(t),t)(\lambda_i(a_2(t),t)-a_2'(t))\right] + \left(\left(R_{ki}^1(\tau,t)\right)_t'\right)|_{\tau=t_i(a_2(t),t)}\frac{dt_i(a_2(t),t)}{dt}\right)\times$$

$$\times\mu_i(t_i(a_2(t),t)) -$$

$$- \sum_{i=p+1}^{n}\left(\frac{d}{dt}\left[\alpha_{ki}(a_1(t),t)(\lambda_i(a_1(t),t)-a_1'(t))\right] + \left(\left(R_{ki}^2(\tau,t)\right)_t'\right)|_{\tau=t_i(a_1(t),t)}\frac{dt_i(a_1(t),t)}{dt}\right)\times$$

$$\times\mu_i(t_i(a_1(t),t)) -$$

$$- \sum_{i=1}^{p}\int_{t_i(a_2(t),t)}^{t}(R_{ki}^1(\tau,t))_{tt}''\mu_i(\tau)d\tau + \sum_{i=p+1}^{n}\int_{t_i(a_1(t),t)}^{t}(R_{ki}^2(\tau,t))_{tt}''\mu_i(\tau)d\tau +$$

$$+ \sum_{i=1}^{n}\frac{d^2}{dt^2}\left\{\int_{\psi_i(t)}^{t}d\tau\int_{\chi_i^1(\tau,t)}^{\chi_i^2(\tau,t)}Q_{ik}(y,\tau,t)f_i(y,\tau;u)dy\right\} + H_k''(t),\ k=\overline{1,n},$$

and write the results in operator form

$$A(t)\mu'(t) = B(t)T(t)(P\mu')(t) + (M^1\mu)(t) + (L_2u)(t) + H''(t), \tag{63}$$

where

$$T(t) = \mathrm{diag}\left\{\frac{dt_1(a_2(t),t)}{dt}, \ldots, \frac{dt_p(a_2(t),t)}{dt}, \frac{dt_{p+1}(a_1(t),t)}{dt}, \ldots, \frac{dt_n(a_1(t),t)}{dt}\right\}.$$

Given (47) and the fact that

$$\frac{dt_i(a_{3-l}(t),t)}{dt} = \frac{(\lambda_i(a_{3-l}(t),t)-a_l'(t))\exp\left(-\int_{t_i(a_{3-l}(t),t)}^{t}\lambda_{ix}'(\varphi_i(\sigma;a_{3-l}(t),t),\sigma)d\sigma\right)}{\lambda_i(a_l(t_i(a_{3-l}(t),t)),t_i(a_{3-l}(t),t)) - a_l'(t_i(a_{3-l}(t),t))},$$

$$\text{where } \ l = 1 \text{ at } \ i = \overline{1, p}, \quad l = 2 \text{ at } \ i = \overline{p+1, n},$$

obtain

$$0 < \frac{dt_i(a_{3-l}(t), t)}{dt} \Big|_{(0,0)} = \frac{\lambda_i(0,0) - a'_{3-l}(0)}{\lambda_i(0,0) - a'_l(0)} < 1.$$

From this and from the continuity of the functions $\frac{\partial u(x,t)}{\partial x}$ it follows that for sufficiently small $\varepsilon_2 \in (0, T]$ the inequalities

$$0 < \frac{dt_i(a_2(t), t)}{dt} < 1, \ \ i = \overline{1, p}; \quad 0 < \frac{dt_i(a_1(t), t)}{dt} < 1, \ \ i = \overline{p+1, n}, \ t \in [0; \varepsilon_2].$$

Thus, $|A^{-1}(t)B(t)T(t)| < 1$ at $t \in [0, \varepsilon_2]$, so (63) is rewritten as follows

$$\mu'(t) = (I - A^{-1}BTP)^{-1} \left[ (M^1\mu)(t) + (L_2 u)(t) + H''(t) \right].$$

We substitute this result into (62).

The resulting operator equation with respect to $\frac{\partial u(x,t)}{\partial x}$ is similar to (61). By similar considerations, there exists a single continuous vector function $\frac{\partial u(x,t)}{\partial x}$ in the domain $\overline{G}_{\varepsilon_2}$. Taking this into account and equation (45), we also obtain that $\frac{\partial u}{\partial t} \in [C(\overline{G}_{\varepsilon_2})]^n$.

Thus, for each vector function $a \in Q_{\varepsilon_3}$, where $\varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2\}$, we have found the corresponding classical solution to the problem (45)–(47) in the domain $\overline{G}_{\varepsilon_3}$. We denote this solution by $U(x, t; a)$. It remains only to choose the one for which the conditions (48) are satisfied from the whole set of admissible vector functions $a(t)$.

Following the same reasoning as in Section 2, we obtain that for every $i = \overline{1, n}$, the dependence $U_i(a(t), t; a)$ in the metric of uniform deviation from $a$ as an element of $[C[0, \varepsilon_4]]^2$, $\varepsilon_4 \in (0, T]$ satisfies the Lipschitz condition: $\exists L_u \geqslant 0$, $\forall a^1, a^2 \in Q_{\varepsilon_4}$:

$$\max_{0 \leqslant t \leqslant \varepsilon_4} \left| U_i(a^1_r(t), t; a^1) - U_i(a^2_r(t), t; a^2) \right| \leqslant L_u \max_{0 \leqslant t \leqslant \varepsilon_4} \left| a^1(t) - a^2(t) \right|, \ r = 1, 2. \tag{64}$$

Let us choose $\varepsilon_4 > 0$ so small that the following conditions

$$\varepsilon_4 < \min \left\{ \frac{1}{n\Gamma U_0}, \frac{1}{2n\Gamma L_u + 2nU_0 L_\gamma}, \frac{1}{2L_h + 1} \right\} \tag{65}$$

are satisfied, here $\Gamma$ is the constant that limits the continuous function $\gamma^r_{li}(a, t)$ at $a \in Q_{\varepsilon_3}$; $U_0$ is the constant for $U_i(x, t; a)$ from Subsection 2.3.

Consider on $Q_{\varepsilon_4}$ the operator $D\colon a \to Da$, which acts according to the formula

$$(Da)_l(t) = \sum_{r=1}^{2} \sum_{i=1}^{n} \int_0^t d\tau \int_0^\tau \gamma^r_{il}(a(\eta), \eta) U_i(a_r(\eta), \eta; a) d\eta + \int_0^t h_l(a(\tau), \tau) d\tau, \ l = 1, 2. \tag{66}$$

The operator $D$ maps $Q_{\varepsilon_4}$ into itself and in the metric $[C[0, \varepsilon_4]]^2$ is a compression. Indeed, from the conditions of the theorem on the functions $\gamma^r_{il}(z, t)$, $h_k(z, t)$ and the fact that $U_i(x, t; a)$ are continuous, it follows that $Da \in [C[0, \varepsilon_4]]^2$. In addition, given (65), we have

$$|(Da_l(t))| \leqslant \sum_{r=1}^{2} \sum_{i=1}^{n} \int_0^t d\tau \int_0^\tau |\gamma^r_{il}(a(\eta), \eta) U_i(a_r(\eta), \eta; a)| d\eta + \int_0^t |h_l(a(\tau), \tau)| d\tau \leqslant$$
$$\leqslant \varepsilon_4^2 n\Gamma U_0 + \varepsilon_4 H \leqslant \varepsilon_4(1 + H).$$

Thus, $DQ_{\varepsilon_4} \subset Q_{\varepsilon_4}$. Let us show that the operator given by (66) is compressible. Since

$$\max_{0 \leqslant t \leqslant \varepsilon_4} |(Da^1)_l(t) - (Da^2)_l(t)| \leqslant \max_{0 \leqslant t \leqslant \varepsilon_4} \sum_{r=1}^{2} \sum_{i=1}^{n} \int_0^t \int_0^\tau |\gamma^r_{il}(a^1(\eta), \eta) U_i(a^1_r(\eta), \eta; a^1) -$$
$$- \gamma^r_{il}(a^2(\eta), \eta) U_i(a^r_2(\eta), \eta; a^2)| d\eta d\tau + \max_{0 \leqslant t \leqslant \varepsilon_4} \int_0^t |h_l(a^1(\tau), \tau) - h_l(a^2(\tau), \tau)| d\tau \leqslant$$

$$\leqslant \max_{0\leqslant t\leqslant \varepsilon_4} \sum_{r=1}^{2}\sum_{i=1}^{n}\int_0^t\int_0^\tau \left(\left|\gamma_{il}^r(a^1(\eta),\eta)-\gamma_{il}^r(a^2(\eta),\eta)\right|\cdot\left|U_i(a_r^1(\eta),\eta;a^1)\right|+\right.$$
$$\left.+\left|\gamma_{il}^r(a^2(\eta),\eta)\right|\cdot\left|U_i(a_r^2(\eta),\eta;a^2)\right|\right)d\eta d\tau+$$
$$+\varepsilon_4 L_h\rho(a^1(t),a^2(t))\leqslant \varepsilon_4^2 n(U_0L_\gamma\rho(a^1(t),a^2(t))+\Gamma L_u\rho(a^1(t),a^2(t)))+$$
$$+\varepsilon_4 L_h\rho(a^1(t),a^2(t))\leqslant \varepsilon_4\left(\varepsilon_4 n\left(U_0L_\gamma+\Gamma L_u\right)+L_h\right)\rho(a^1(t),a^2(t)),$$

then, given (65), we have $\rho((Da^1)(t),(Da^2)(t))\leqslant \varepsilon_4\left(2\varepsilon n\left(U_0L_\gamma+\Gamma L_u\right)+2L_h\right)\rho(a^1(t),a^2(t)) < \varepsilon_4\left(2L_h+1\right)\rho(a^1(t),a^2(t)) < \rho(a^1(t),a^2(t))$. Therefore, the Banach theorem implies the existence and uniqueness of the fixed point of the operator, i.e. $a(t)$. Next, we take $\varepsilon=\min\{\varepsilon_3,\varepsilon_4\}$ and, using the already known vector function $a(t)$, we choose the corresponding unique solution $u(x,t)=U(x,t;a)$ from the domain $G_\varepsilon$. □

**3.1. Commentary.** In the case when $q$ characteristics starting from the point $(0,0)$ fall into the domain $G_t$, i.e.

$$\begin{gathered}\lambda_i(0,0)-a_1'(0)>0,\quad i=\overline{1,p+q},\quad \lambda_i(0,0)-a_1'(0)<0,\quad i=\overline{p+q+1,n},\\ \lambda_i(0,0)-a_2'(0)>0,\quad i=\overline{1,p},\quad \lambda_i(0,0)-a_2'(0)<0,\quad i=\overline{p+1,n},\end{gathered}\tag{67}$$

a problem with integral conditions of the type (48) (there must be $n+q$ conditions) is incorrect, because $\det\alpha^1(0)=0$ always holds. Indeed:

$$\alpha^1(0)=\left.\begin{pmatrix}\alpha_{11} & \dots & \alpha_{1,p} & \alpha_{1,p+1} & \dots & \alpha_{1,p+q} & \alpha_{1,p+1} & \dots & \alpha_{1,n}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \alpha_{n+q,1} & \dots & \alpha_{n+q,p} & \alpha_{n+q,p+1} & \dots & \alpha_{n+q,p+q} & \alpha_{n+q,p+1} & \dots & \alpha_{n+q,n}\end{pmatrix}\right|_{(0,0)},$$

hence, $\det\alpha^1(0)=0$. Now let us show by an example that if the condition 5) of Theorem 3.1 is not fulfilled, then the correctness of the problem (45)–(48) is violated.

Let $Q$ be the sector in the plane $x0t$ bounded by the rays $a_1(t)=k_1t$ and $a_2(t)=k_2t$, where $-1<k_1<1$, $-1<k_2<1$, $k_1<k_2$. Let us consider the system in $Q$

$$\begin{cases}\frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial x}=0,\\ \frac{\partial u_2}{\partial t}-\frac{\partial u_2}{\partial x}=0,\end{cases}\tag{68}$$

with the following conditions

$$\begin{cases}\int_{k_1t}^{k_2t}[(x-1)u_1(x,t)+u_2(x,t)]dx=-t,\\ \int_{k_1t}^{k_2t}[(x+t+1)u_1(x,t)+(t-1)u_2(x,t)]dx=t.\end{cases}\tag{69}$$

This problem is derived from problems (45), (47), (48) if $n=2$, $p=1$, $a_k(t)$ are known, and the coefficients and free terms are equal to the corresponding constants and functions. Here, all the assumptions of Theorem 3.1 are fulfilled except for the condition (49), since

$$\det\alpha^1(0)=\begin{vmatrix}-1 & 1\\ 1 & -1\end{vmatrix}=0.$$

Let us put

$$\begin{cases}u_1(k_1t,t)=\mu_1(t),\\ u_2(k_2t,t)=\nu_2(t).\end{cases}$$

Using the method of characteristics, we find

$$\begin{cases}u_1(x,t)=\mu_1(\frac{x-t}{k_1-1}),\quad k_1t\leqslant x\leqslant k_2t;\\ u_2(x,t)=\nu_2(\frac{x+t}{k_2+1}),\quad k_1t\leqslant x\leqslant k_2t\end{cases}$$

and substitute into the conditions of (69). After replacing the variables and differentiating the resulting system of equations in $t$, we obtain

$$\begin{cases} (k_2-1)(k_2t-1)\mu_1(\frac{k_2-1}{k_1-1}t)-(k_1-1)(k_1t-1)\mu_1(t)+(k_2+1)\nu_2(t)- \\ -(k_1+1)\nu_2(\frac{k_1+1}{k_2+1}t)=(1-k_1)\int_t^{\frac{k_2-1}{k_1-1}t}\mu_1(\tau)d\tau-1; \\ (k_2-1)(k_2t+t-1)\mu_1(\frac{k_2-1}{k_1-1}t)-(k_1-1)(k_1t+t-1)\mu_1(t)+(k_2+1)(t-1)\times \\ \times\nu_2(t)-(k_1+1)(t-1)\nu_2(\frac{k_1+1}{k_2+1}t)=2(1-k_1)\int_t^{\frac{k_2-1}{k_1-1}t}\mu_1(\tau)d\tau- \\ -(k_2+1)\int_{\frac{k_1+1}{k_2+1}t}^{t}\nu_2(\tau)d\tau+1. \end{cases}$$

This implies, in particular, that at the point (0,0) this system:

$$\begin{cases} (k_1-k_2)\mu_1(0)+(k_2-k_1)\nu_2(0)=-1, \\ (k_2-k_1)\mu_1(0)+(k_1-k_2)\nu_2(0)=1 \end{cases}$$

has many solutions. Therefore, the problem (68)–(69) is incorrect.

**4. The generalized hyperbolic Darboux-Stefan problem.** Consider the system (45) in $G_{u,T} := \{(x,t)\colon 0<t\leqslant T,\ a_{u,1}(t)<x<a_{u,2}(t),\ a_{u,1}(0)=a_{u,2}(0)=0\}$ with unknown boundaries $a_{u,k}(t)$, $k=1.2$, which are subject to the following conditions

$$\frac{da_{u,k}}{dt}=h_k(t;u),\quad u=\mathrm{col}(u_1,\dots,u_n),\quad k=1,2, \tag{70}$$

where the right-hand sides in (45) and (70) are Volterra-type functionals.

Let's define the behavior of the characteristics of the system (45) passing through the vertex of the sector $G_{u,T}$:

$$\lambda_i(0,0)-a'_{u,1}(0)>0,\quad i=\overline{1,p+q};\quad \lambda_i(0,0)-a'_{u,1}(0)<0,\quad i=\overline{p+q+1,n},$$
$$\lambda_i(0,0)-a'_{u,2}(0)>0,\quad i=\overline{1,p};\quad \lambda_i(0,0)-a'_{u,2}(0)<0,\quad i=\overline{p+1,n},\ p,q\in[0,n], \tag{71}$$

i.e., $q$ characteristics that start from the point $(0,0)$ fall into the sector. Let us denote

$$I_k^+=\{i\colon \lambda_i(0,0)>a'_{u,k}(0)\},\quad I_k^-=\{i\colon \lambda_i(0,0)<a'_{u,k}(0)\},\quad k=1,2.$$

The boundary conditions

$$u_i(a_{u,1}(t),t)=g_{i1}(t;\{u_{i'}(a_{u,1}(t),t)\}),\quad i\in I_1^+,\ i'\in I_1^-,$$
$$u_i(a_{u,2}(t),t)=g_{i2}(t;\{u_{i'}(a_{u,2}(t),t)\}),\quad i\in I_2^-,\ i'\in I_2^+,\ t\in[0,T] \tag{72}$$

are set, i.e., the right-hand sides of (72) contain only those sets of indices $i'$ that are not present in the left-hand sides; $g_{i1}\colon [0,T]\times\mathbb{R}^{n-p-q}\to\mathbb{R}$, $g_{i2}\colon [0,T]\times\mathbb{R}^p\to\mathbb{R}$.

Let us introduce the metric space $S_T$ of "sets"$v=\{u_i,a_{u,k}\}$ $(i=\overline{1,n},\ k=1,2)$, where $u_i\in C(\overline{G}_{u,T})$, $a_{u,k}\in C^1([0,T])$, $a_{u,1}(t)\leqslant a_{u,2}(t)$, with the metric

$$\rho(v^1,v^2)=\max\{\max_{k,t}|a_{u^1,k}(t)-a_{u^2,k}(t)|,\max_{i,x,t}|\overline{u}_i^1(x,t)-\overline{u}_i^2(x,t)|\},$$

hereinafter, for any $r\colon [a,b]\to\mathbb{R}$, the notation $\overline{r}$ means the extension of $r$ by $\mathbb{R}$ by the formulas $\overline{r}(x)=r(a)$ $(x<a)$, $\overline{r}(x)=r(b)$ $(x>b)$.

Let us find the values of the functions $u_i(0,0)\equiv u_{i,0}$ $(i=\overline{1,n})$ from the boundary conditions (72), namely, from the system

$$u_i(0,0)=g_{i1}(0;u_{p+q+1,0},\dots,u_{n,0}),\ i=\overline{1,p+q};\ u_i(0,0)=g_{i2}(0;u_{1,0},\dots,u_{p,0}),\ i=\overline{p+1,n}. \tag{73}$$

To do this, we solve the system of nonlinear equations

$$\begin{cases} u_{1,0}=g_{11}\left(0;g_{p+q+1,2}(0;u_{1,0},\dots,u_{p,0}),\dots,g_{n,2}(0;u_{1,0},\dots,u_{p,0})\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ u_{p,0}=g_{p1}\left(0;g_{p+q+1,2}(0;u_{1,0},\dots,u_{p,0}),\dots,g_{n,2}(0;u_{1,0},\dots,u_{p,0})\right). \end{cases} \tag{74}$$

Let us introduce vector functions

$$G_1(x_{p+q+1},\dots,x_n)=\operatorname{col}\left(g_{11}(0;x_{p+q+1},\dots,x_n),\dots,g_{p1}(0;x_{p+q+1},\dots,x_n)\right),$$
$$G_2(x_1,\dots,x_p)=\operatorname{col}\left(g_{p+q+1,2}(0;x_1,\dots,x_p),\dots,g_{n,2}(0;x_1,\dots,x_p)\right).$$

Here, $G_1\colon \mathbb{R}^{n-p-q}\to\mathbb{R}^p$, and $G_2\colon \mathbb{R}\to\mathbb{R}^{n-p-q}$, then the composition of these vector functions is $G_1\circ G_2\colon \mathbb{R}^p\to\mathbb{R}^p$. If in the space $\mathbb{R}^p$ with the metric $d(x,y)=\max\limits_{1\leqslant k\leqslant p}|x_k-y_k|$ $\forall x,y\in\mathbb{R}^p$ the following compression condition

$$g^1g^2<1 \tag{75}$$

is satisfied, where $g^1$ is the Lipschitz constant for $G_1$ and $g^2$ is the Lipschitz constant for $G_2$, then there exists a unique solution of (74) that can be found by simple iteration.

Having the solution of the system (74), we find

$$\begin{cases} u_{p+1,0}=g_{p+1,2}(0;u_{1,0},\dots,u_{p,0}),\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\\ u_{p+q,0}=g_{p+q,2}(0;u_{1,0},\dots,u_{p,0}),\\ u_{p+q+1,0}=g_{p+q+1,2}(0;u_{1,0},\dots,u_{p,0}),\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\\ u_{n,0}=g_{n,2}(0;u_{1,0},\dots,u_{p,0}).\end{cases} \tag{76}$$

Therefore, if the condition (75) and the consistency conditions

$$\begin{cases} g_{p+1,1}(0;u_{p+q+1,0},\dots,u_{n,0})=g_{p+1,2}(0;u_{1,0},\dots,u_{p,0}),\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ g_{p+q,1}(0;u_{p+q+1,0},\dots,u_{n,0})=g_{p+q,2}(0;u_{1,0},\dots,u_{p,0})\end{cases} \tag{77}$$

are satisfied, then $u_{i,0}$ is determined in the only way from the boundary conditions (72). Hence, it follows from conditions (70) and (71) that

$$\lambda_i(0,0)\neq h_k(0;u_0),\ \ u_0=\operatorname{col}(u_{1,0},\dots,u_{n,0}),\ k=1,2,\ i=\overline{1,n}. \tag{78}$$

Let us denote by $\varphi_i(\tau;x,t)$ the solution of the Cauchy problem $d\xi/d\tau=\lambda_i(\xi,\tau)$, $\xi(t)=x$ $(x,t)\in\overline{G}_{u,T}$, $i=\overline{1,n}$. Let $L_i(x,t)$ be the corresponding integral curves, and

$$t_i(x,t;u):=\min\{\tau\colon (\varphi_i(\tau;x,t),\tau)\in G_{u,T}\}.$$

It is obvious that $0\leqslant t_i(x,t;u)\leqslant t$. If $t_i(x,t;u)>0$, then $\varphi_i(t_i(x,t;u);x,t)=a_{u,1}(t_i(x,t;u))$ or $\varphi_i(t_i(x,t;u);x,t)=a_{u,2}(t_i(x,t;u))$. It follows that the characteristic $L_i(0,0)$, defined only in the case $a'_{u,1}(0)<\lambda_i(0,0)<a'_{u,2}(0)$ (i.e. $i=\overline{p+1,p+q}$) divides $G_{u,T}$ into two components $G^{i1}_{u,T}$ and $G^{i2}_{u,T}$. It is obvious that at $\lambda_i(0,0)<a'_{u,1}(0)$ $(\lambda_i(0,0)>a'_{u,2}(0))$ we have $G^{i1}_{u,T}=\varnothing$, $G^{i2}_{u,T}=G_{u,T}$ (respectively $G^{i1}_{u,T}=G_{u,T}$, $G^{i2}_{u,T}=\varnothing$). Then, for $t>0$, the condition $\varphi_i(t_i(x,t;u);x,t)=a_{u,1}(t_i(x,t;u))$ $(\varphi_i(t_i(x,t;u);x,t)=a_{u,2}(t_i(x,t;u)))$ is equivalent to the fact that $(x,t)\in G^{i1}_{u,T}$ (respectively $(x,t)\in G^{i2}_{u,T}$).

Assuming that in the system (45) the functions $u_i$ are continuously differentiable and

$$\lambda_i(a_{u,k}(t),t)\neq h_k(t;u),\ \ k=1,2,\ i=\overline{1,n} \tag{79}$$

is fulfilled, and integrating (45) over the characteristics, we come to the system of integro-functional equations

$$u_i(x,t)=\omega_i(x,t;u)+\int_{t_i(x,t;u)}^{t} f_i(\varphi_i(\tau;x,t),\tau;u)d\tau,\ \ i=\overline{1,n},\ (x,t)\in G_{u,T}, \tag{80}$$

$$\omega_i(x,t;u)=\begin{cases} g_{i1}(t_i(x,t;u);u_{p+q+1}(a_{u,1}(t_i(x,t;u)),t_i(x,t;u))\\ \dots,u_n(a_{u,1}(t_i(x,t;u)),t_i(x,t;u))),\ \text{at } (x,t)\in G^{i1}_{u,T},\\ g_{i2}(t_i(x,t;u);u_1(a_{u,2}(t_i(x,t;u)),t_i(x,t;u))\\ \dots,u_p(a_{u,2}(t_i(x,t;u)),t_i(x,t;u))),\ \text{at } (x,t)\in G^{i2}_{u,T}.\end{cases}$$

**Definition 2.** The generalized solution of the problem (45), (70)–(72) is the set $v \in S_T$ of functions that satisfy the condition (79) and the system of equations (70), (80).

**Theorem 3.** *Let the following assumptions hold:*

a) *all the functions* $\lambda_i(x,t)$ $(i=\overline{1,n})$ *are real, except* $\lambda_i \in C^1(\mathbb{R}\times[0,T])$;

b) *each functional* $f_i(x,t;u)$ *and continuous under* $v\in S_T$, $(x,t)\in G_{u,T}$, *and accordingly* $h_k(t;u)$ *is defined and continuous under* $[0,T]\times S_T$, *moreover, in some neighborhood of an arbitrary point of* $S_T$ *these functional satisfy the Lipschitz condition in* $u$ *and there exists* $M>0$ *such that if* $u(.,.)$ *satisfies the Lipschitz condition with constant* $L$ *in* $x$, *then* $f_i(.,.;u)$ *satisfies the Lipschitz condition with constant* $ML$ *in* $x$;

c) *the functions* $g_{i1}$ *are defined on* $[0,T]\times\mathbb{R}^{n-p-q}$, $g_{i2}$ — *on* $[0,T]\times\mathbb{R}^p$ *and satisfy the Lipschitz condition with constant* $Tg_0(T)$ *for all variables, where* $g_0(T)$ *is bounded at* $T\in(0,\infty)$; *and the functions* $g_{ik}(0;\{u_{i',0}\})$ *satisfy the condition* (75) *and the matching condition* (77).

*Then, for some* $\varepsilon>0$, *there exists a unique generalized solution of the problem* (45), (70)–77 *at* $t\in[0,\varepsilon]$.

*Proof.* Let us choose any $\varepsilon\in(0,T]$, $\alpha>0$, $\delta$, $\sigma\geqslant 1$ and denote by $S=S_{\varepsilon,\alpha,\delta,\sigma}$ a subset of $S_\varepsilon$ consisting of sets $v=\{u_i,a_{u,k}\}$ for which the following conditions hold:

1) the functions $(a_{u,k}(t)-h_k(0;u_0)t)$ satisfy the Lipschitz condition with constant $\alpha$;

2) functions $u_i$ satisfy the Lipschitz condition in $x$ with constant $\delta$;

3) if $(x_j,t_j)\in G_{u,\varepsilon}$, $j=1,2$, and $t_1\neq t_2$ and
$$\left|\frac{x_2-x_1}{t_2-t_1}-h_k(0;u_0)\right|\leqslant\alpha,\quad |x_j-h_k(0;u_0)t_j|\leqslant\alpha t_j,\quad j=1,2, \tag{81}$$
and $i\in I_1^-\cup I_2^+$, then $|u_i(x_2,t_2)-u_i(x_1,t_1)|\leqslant\sigma|t_2-t_1|$.

The set $S$ is not empty if $\delta$ and $\sigma$ are sufficiently large. Indeed, let us denote by $u_{i,0\varepsilon}(x,t)=u_{i,0}$, $a_{u_{0\varepsilon},k}(t)=h_k(0;u_0)t$, $t\in[0,\varepsilon]$, $v_{0\varepsilon}=\{u_{i,0\varepsilon},a_{u_{0\varepsilon},k}\}$, then $v_{0\varepsilon}\in S$ (this can be verified by directly checking conditions 1)–3)). In addition, the set $S$ is closed in $S_\varepsilon$, because if $\lim\limits_{m\to\infty}v^m=v$, where the sequence of sets is $v^m\in S$, then conditions 1)–3) are also satisfied for $v$, i.e. $v\in S$.

On $S$, we define the operator $A$ as follows: let $v\in S$, then $Av=\{A_iu,a_{Au,k}\}$, where $A_iu\colon G_{Au,\varepsilon}\to\mathbb{R}$, and $G_{Au,\varepsilon}$ is bounded on the sides by the lines
$$a_{Au,k}(t):=\int_0^t h_k(\tau;u)d\tau,\quad k=1,2,\ 0\leqslant t\leqslant\varepsilon, \tag{82}$$
and the values of the function $A_iu$ are given by the formula
$$(A_iu)(x,t)=\omega_i(x,t;\tilde{A}u)+\int_{t_i(x,t;\tilde{A}u)}^t f_i(\varphi_i(\tau;x,t),\tau;\tilde{A}u)d\tau,\quad i=\overline{1,n},\ (x,t)\in G_{Au,\varepsilon}, \tag{83}$$
where $\tilde{A}u=\{\tilde{A}_iu\}$, and $\tilde{A}_iu$ is the restriction of the function $\overline{u}_i$ by $G_{Au,\varepsilon}$. It is obvious that $t_i(x,t;\tilde{A}u)=t_i(x,t;Au)$.

It follows from the continuity of the given functions and functionalities that when $\varepsilon_0$, $\alpha_0$ $(>0)$ are sufficiently small, then when $0<\varepsilon\leqslant\varepsilon_0$, $0<\alpha\leqslant\alpha_0$ in $S$, the relation (79) is satisfied and therefore the definition of the operator $A$ makes sense. Furthermore, for $v\in S$, $(x,t)\in G_{u,\varepsilon}$, uniform estimates hold: $|f_i(x,t;u)|\leqslant F$, $|h_ki(t;u)|\leqslant H$, $|\lambda_i(x,t)|\leqslant\Lambda$, $|\lambda_i(a_{Au,k}(t),t)-h_k(t;u)|>\gamma>0$, and, given the conditions of Theorem 3. 1, the functionals

$f_i$ satisfy with respect to $u$ and $x$, the functionals $h_k$ satisfy with respect to $u$, the functions $\lambda_i$ satisfy with respect to $x$, and the functions $g_{ik}$ satisfy the Lipschitz condition with constant $f_0$, $Mp$, $h_0$, $\lambda_0$, and $\varepsilon g_0(\varepsilon)$, respectively, for all arguments.

Next, we will use the consequence of the Lipschitz condition for $\lambda_i$:

$$|\varphi_i(\tau;x_1,t)-\varphi_i(\tau;x_2,t)|\leqslant|x_1-x_2|e^{\lambda_0|t-\tau|},\quad i=\overline{1,n}. \tag{84}$$

Indeed, from the definition of the functions $\varphi_i(\tau;x_j,t)$ we have

$$\varphi_i(\tau;x_j,t)=\varphi_i(t;x_j,t)+\int_t^\tau\lambda_i(\varphi_i(\eta;x_j,t),\eta)d\eta,\quad j=1,2.$$

Then

$$|\varphi_i(\tau;x_1,t)-\varphi_i(\tau;x_2,t)|\leqslant|x_1-x_2|+\int_\tau^t|\lambda_i(\varphi_i(\eta;x_1,t),\eta)-\varphi_i(\eta;x_2,t)|d\eta\leqslant$$

$$\leqslant|x_1-x_2|+\int_\tau^t\lambda_0|\varphi_i(\eta;x_1,t)-\varphi_i(\eta;x_2,t)|d\eta.$$

Using Gronwall's lemma, we obtain (84).

Let us establish the conditions under which the operator $A$ reflects $S$ into itself, that is, when $v\in S$ for $Av$, 1)–3) are true.

**1.** Since condition 1) is satisfied for $v$, then $\forall t_j\in[0,\varepsilon]$ $(j=1,2)$, we have

$$|a_{Au,k}(t_2)-a_{Au,k}(t_1)-h_k(0;u_0)(t_2-t_1)|=$$

$$=\left|\int_0^{t_2}h_k(\tau;u)d\tau-\int_0^{t_1}h_k(\tau;u)d\tau-h_k(0;u_0)(t_2-t_1)\right|=$$

$$=\left|\int_{t_1}^{t_2}h_k(\tau;u)d\tau-h_k(0;u_0)(t_2-t_1)\right|=\left|\int_{t_1}^{t_2}\frac{da_{u,k}(\tau)}{d\tau}d\tau-h_k(0;u_0)(t_2-t_1)\right|=$$

$$=|a_{u,k}(t_2)-a_{u,k}(t_1)-h_k(0;u_0)(t_2-t_1)|\leqslant\alpha|t_2-t_1|.$$

Thus, condition 1) for $Av$ is satisfied.

**2.** Consider different cases:

**a)** when $(x_j,t)\in G^{i1}_{u,\varepsilon}$ or $G^{i2}_{u,\varepsilon}$ simultaneously $(j=1,2)$. Let $(x_j,t)\in G^{i1}_{u,\varepsilon}$ be $(i=\overline{1,p+q})$ and $x_1<x_2$ for certainty, then $t_i(x_1,t;u)>t_i(x_2,t;u)$. Using (83) and the notation $\Delta\Phi_j=\Phi_1-\Phi_2$, we have

$$|\Delta(A_iu)(x_j,t)|\leqslant\int\limits_{t_i(x_1,t;Au)}^{t}|\Delta f_i(\varphi_i(\tau;x_j,t),\tau;\tilde{A}u)|d\tau+\int\limits_{t_i(x_2,t;Au)}^{t_i(x_1,t;Au)}|f_i(\varphi_i;x_1,t),\tau;\tilde{A}u|d\tau+$$

$$+|\Delta g_{i1}(t_i(x_j,t;Au);(\tilde{A}_{p+q+1}u)(a_{Au,1}(t_i(x_j,t;Au)),t_i(x_j,t;Au)),\ldots,$$

$$(\tilde{A}_nu)a_{Au,1}(t_i(x_j,t;Au)))|\leqslant\varepsilon M\delta|\Delta\varphi(\tau;x_j,t)|+F|\Delta t_i(x_j,t;Au)|+\varepsilon g_0(\varepsilon)\sigma|\Delta t_i(x_j,t;Au)|.$$

Since

$$|\Delta t_i(x_j,t;Au)|=|t_i(x_2,t;Au)-t_i(x_1,t;Au)|\leqslant\left|\frac{\partial t_i(\theta,t;Au)}{\partial\theta}\right||x_2-x_1|=$$

$$=\left|\frac{\exp\left(\int_{t_i(\theta,t;Au)}^t\lambda'_{i\theta}(\varphi_i(\sigma;\theta,t),\sigma)d\sigma\right)}{\lambda_i(a_{Au,1}(t_i(\theta,t;Au)),t_i(\theta,t;Au))-a'_{Au,1}(t_i(\theta,t;Au))}\right||\Delta x_j|\leqslant\frac{1}{\gamma}e^{\lambda_0}|\Delta x_j|, \tag{85}$$

then, given (84), we obtain $|\Delta(A_iu)(x_j,t)|\leqslant(\varepsilon M\delta+(F+\varepsilon g_0(\varepsilon)\sigma)\frac{1}{\gamma})e^{\lambda_0\varepsilon}|\Delta x_j|$. Thus, if one has

$$\left(\varepsilon_0M\delta+(F+\varepsilon_0g_0(\varepsilon_0)\sigma)\frac{1}{\gamma}\right)e^{\lambda_0\varepsilon_0}\leqslant\delta, \tag{86}$$

then condition 2) is satisfied for $Av$.

A similar result is obtained when $(x,t) \in G^{i2}_{u,\varepsilon}$, $i=\overline{p+1,n}$.

**b)** if $(x_1,t) \in G^{i1}_{u,\varepsilon}$, $(x_2,t) \in G^{i2}_{u,\varepsilon}$, $i=\overline{p+1,p+q}$ (for other $i$ this case coincides with case a)). Let us assume for certainty $x_1 < x_2$, $t_i(x_1,t;Au) > t_i(x_2,t;Au)$, then (see a)):

$$|\Delta(A_iu)(x_j,t)| \leqslant \left(\varepsilon M\delta + F\frac{1}{\gamma}\right) e^{\lambda_0\varepsilon}|\Delta x_j| +$$

$$+\Big|g_{i1}(t_i(x_1,t;Au);(\tilde{A}_{p+q+1}u)(a_{Au,1}(t_i(x_1,t;Au)),t_i(x_1,t;Au)),\dots$$

$$\dots,(\tilde{A}_nu)(a_{Au,1}(t_i(x_1,t;Au)),t_i(x_1,t;Au)))-$$

$$-g_{i2}(t_i(x_2,t;Au);(\tilde{A}_1u)(a_{Au,2}(t_i(x_2,t;Au)),t_i(x_2,t;Au)),\dots$$

$$\dots,(\tilde{A}_pu)(a_{Au,2}(t_i(x_2,t;Au)),t_i(x_2,t;Au)))\Big| \leqslant \left(\varepsilon M\delta + F\frac{1}{\gamma}\right) e^{\lambda_0\varepsilon}|\Delta x_j| +$$

$$+\Big|g_{i1}(t_i(x_1,t;Au);(\tilde{A}_{p+q+1}u)(a_{Au,1}(t_i(x_1,t;Au)),t_i(x_1,t;Au)),\dots$$

$$\dots,(\tilde{A}_nu)(a_{Au,1}(t_i(x_1,t;Au)),t_i(x_2,t;Au)))-$$

$$-g_{i1}(t_i(x_3,t;Au);(\tilde{A}_{p+q+1}u)(a_{Au,1}(t_i(x_3,t;Au)),t_i(x_3,t;Au),t_i(x_3,t;Au)),\dots$$

$$\dots,(\tilde{A}_nu)(a_{Au,1}(t_i(x_3,t;Au)),t_i(x_3,t;Au)))\Big| +$$

$$+\Big|g_{i2}(t_i(x_3,t;Au);(\tilde{A}_1u)(a_{Au,2}(t_i(x_3,t;Au)),t_i(x_3,t;Au)),\dots$$

$$\dots,(\tilde{A}_pu)(a_{Au,2}(t_i(x_3,t;Au)),t_i(x_3,t;Au)))-$$

$$-g_{i2}(t_i(x_2,t;Au);(\tilde{A}_1u)(a_{Au,2}(t_i(x_2,t;Au)),t_i(x_2,t;Au)),\dots$$

$$\dots,(\tilde{A}_pu)(a_{Au,2}(t_i(x_2,t;Au)),t_i(x_2,t;Au)))\Big| \leqslant$$

(here $(x_3,t_3) \in L_i(0,0)$, then $(a_{Au,j}(t_i(x_3,t;Au)),t_i(x_3,t;Au)) = (0,0)$)

$$\leqslant \left(\varepsilon M\delta + F\frac{1}{\gamma}\right) e^{\lambda_0\varepsilon}|\Delta x_j| + \varepsilon g_0(\varepsilon)\sigma(|t_i(x_1,t;Au) - t_i(x_3,t;Au)| +$$

$$+|t_i(x_3,t;Au) - t_i(x_2,t;Au)|) \leqslant \left(\varepsilon M\delta + F\frac{1}{\gamma}\right) e^{\lambda_0\varepsilon}|\Delta x_j| +$$

$$+\varepsilon g_0(\varepsilon)\sigma\frac{1}{\gamma}e^{\lambda_0\varepsilon}(|x_1-x_3|+|x_3-x_2|) \leqslant \left(\varepsilon M\delta + (F+\varepsilon g_0(\varepsilon)\sigma)\frac{1}{\gamma}\right) e^{\lambda_0\varepsilon}|\Delta x_j|.$$

Thus, the fulfillment of condition 2) for $Av$ is ensured by the condition (86).

**3.** Let $(x_j,t_j) \in G_{Au,\varepsilon}$, $i=1,2$, with $t_1 < t_2$ and the conditions of (81) are satisfied, and $i \in I_1^- \cup I_2^+$. Let us take $k=2$, $i \in I_2^+$ for certainty, then we get

$$|(A_iu)(x_1,t_1) - (A_iu)(x_2,t_2)| \leqslant$$

$$\leqslant \left| \int_{t_i(x_1,t_1;Au)}^{t_1} |f_i(\varphi_i(\tau;x_1,t_1),\tau;\tilde{A}u)|d\tau - \int_{t_i(x_2,t_2;Au)}^{t_2} |f_i(\varphi_i(\tau;x_2,t_2),\tau;\tilde{A}u)|d\tau \right| +$$

$$+\Big|g_{i1}(t_i(x_1,t_1;Au);(\tilde{A}_{p+q+1}u)(a_{Au,1}(t_i(x_1,t_1;Au)),t_i(x_1,t_1;Au)),\dots$$

$$\dots,(\tilde{A}_nu)(a_{Au,1}(t_i(x_1,t_1;Au)),t_i(x_1,t_1;Au)))-$$

$$g_{i1}(t_i(x_2,t_2;Au);(\tilde{A}_{p+q+1}u)(a_{Au,1}(t_i(x_2,t_2;Au)),t_i(x_2,t_2;Au)),\dots$$

$$\dots,(\tilde{A}_nu)(a_{Au,1}(t_i(x_2,t_2;Au)),t_i(x_2,t_2;Au)))\Big| \leqslant F|t_2-t_1| +$$

$$+\varepsilon M\delta \max_{0\leqslant\tau\leqslant t_1} |\varphi_i(\tau;x_1,t_1) - \varphi_i(\tau;x_2,t_2)| + (F+\varepsilon g_0(\varepsilon)\sigma)|t_i(x_2,t_2;Au) - t_i(x_1,t_1;Au)|.$$

Taking into account (81), we have $|x_2 - x_1| \leqslant (\alpha + H)|t_2 - t_1|$;

$$|t_i(x_2,t_2;Au) - t_i(x_1,t_1;Au)| \leqslant |t_i(x_2,t_2;Au) - t_i(x_2,t_1;Au)| +$$
$$+|t_i(x_2,t_1;Au) - t_i(x_1,t_1;Au)| \leqslant \left|\frac{\partial t_i(x_1\eta;Au)}{\partial \eta}\right| |t_2 - t_1| + \left|\frac{\partial t_i(\theta,t;Au)}{\partial \theta}\right| |x_2 - x_1| \leqslant$$
$$\leqslant \left|\frac{\lambda_i(x_1,\eta)\exp\left(\int_{t_i(x_1,\eta;Au)}^{\eta} \lambda'_{ix}(\varphi_i(\sigma;x_1,\eta),\sigma)d\sigma\right)}{\lambda_i(a_l(t_i(x_1,\eta;Au)),t_i(x_1,\eta;Au)) - a'_l(t_i(x_1,\eta;Au))}\right| |t_2 - t_1| +$$
$$+\left|\frac{\exp\left(\int_{t_i(\theta,t_2;Au)}^{t_2} \lambda'_{ix}(\varphi_i(\sigma;\theta,t_2),\sigma)d\sigma\right)}{\lambda_i(a_l(t_i(\theta,t_2;Au)),t_i(\theta,t_2;Au)) - a'_l(t_i(\theta,t_2;Au))}\right| |x_2 - x_1| \leqslant$$
$$\leqslant \frac{1}{\gamma} e^{\lambda_0 \varepsilon}(\Lambda + \alpha + H)|t_2 - t_1|;$$
$$(l = 1 \text{ at } (x_1,\eta) \in G^{i1}_{Au,\varepsilon},\ l = 2 \text{ at } (x_1,\eta) \in G^{i2}_{Au,\varepsilon}, \text{ for } (\theta,t_2) \text{ similarly});$$
$$|\varphi_i(\tau;x_2,t_2) - \varphi_i(\tau;x_1,t_1)| \leqslant |\varphi_i(\tau;x_2,t_2) - \varphi_i(\tau;x_2,t_1)| + |\varphi_i(\tau;x_2,t_1) - \varphi_i(\tau;x_1,t_1)| \leqslant$$
$$\leqslant \left|\frac{\partial \varphi_i(\tau;x_2,\eta)}{\partial \eta}\right| |t_2 - t_1| + \left|\frac{\partial \varphi_i(\tau;\theta,t_1)}{\partial \theta}\right| |x_2 - x_1| \leqslant$$
$$\leqslant \left|\lambda_i(x_2,\eta)\exp\left(\int_\tau^\eta \lambda'_{ix}(\varphi_i(\sigma;x_2,\eta),\sigma)d\sigma\right)\right| |t_2 - t_1| +$$
$$+\left|\exp\left(\int_\tau^{t_1} \lambda'_{ix}(\varphi_i(\sigma;\theta,t_1),\sigma)d\sigma\right)\right| |x_2 - x_1| \leqslant e^{\lambda_0\varepsilon}(\Lambda + \alpha + H)|t_2 - t_1|.$$

Thus,

$$|(A_iu)(x_1,t_1) - (A_iu)(x_2,t_2)| \leqslant \left(F + \left(\frac{1}{\gamma}(F + \varepsilon g_0(\varepsilon)\sigma) + \varepsilon M\delta\right)(\Lambda + \alpha + H)e^{\lambda_0\varepsilon}\right)|t_2 - t_1|.$$

So, if

$$F + \left(\frac{1}{\gamma}(F + \varepsilon_0 g_0(\varepsilon_0)\sigma) + \varepsilon_0 M\delta\right)(\Lambda + \alpha_0 + H)e^{\lambda_0\varepsilon_0} \leqslant \sigma, \tag{87}$$

then the operator $Av$ has the property 3).

Suppose that all above conditions providing the inclusion of $AS \subset S$ are met. Let us determine when the mapping $A$ is compressible. Let $v^j \in S$ $(j = 1,2)$ and $\rho(v^1,v^2) = \rho$.

Then

$$|\Delta a_{Au^j,k}(t)| \leqslant \int_0^t |\Delta h_k(\tau;u^j)|d\tau \leqslant \varepsilon h_0\rho, \quad 0 \leqslant t \leqslant \varepsilon.$$

Let us also define

$$\rho(\tilde{A}u^1, \tilde{A}u^2) = \max\{\max_{k,t} |a_{Au^1,k}(t) - a_{Au^2,k}(t)|, \max_{i,x,t} |\overline{\tilde{A}_iu^1}(x,t) - \overline{\tilde{A}_iu^2}(x,t)|\}.$$

There are various options for the relative positioning of the domains $G_{u,\varepsilon}$ and $G_{Au,\varepsilon}$, which define $\overline{\tilde{A}_iu}(x,t)$ in different ways.

**I)** Thus, with $a_{u,1} < a_{Au,1} < a_{Au,2} < a_{u,2}$, we have

$$\overline{\tilde{A}_iu}(x,t) = \begin{cases} u_i(a_{Au,1}(t),t), & x \leqslant a_{Au,1}(t); \\ u_i(x,t), & a_{Au,1}(t) < x < a_{Au,2}(t); \\ u_i(a_{Au,2}(t),t), & x \geqslant a_{Au,2}(t). \end{cases}$$

**II)** In the case of $a_{Au,1} < a_{u,1} < a_{u,2} < a_{Au,2}$, we obtain

$$\overline{\tilde{A}_i u}(x,t) = \begin{cases} u_i(a_{u,1}(t),t), & x \leqslant a_{u,1}(t); \\ u_i(x,t), & a_{u,1}(t) < x < a_{u,2}(t); \\ u_i(a_{u,2}(t),t), & x \geqslant a_{u,2}(t). \end{cases}$$

**III)** If $a_{u,1} < a_{Au,1} < a_{u,2} < a_{Au,2}$, then

$$\overline{\tilde{A}_i u}(x,t) = \begin{cases} u_i(a_{Au,1}(t),t), & x \leqslant a_{Au,1}(t); \\ u_i(x,t), & a_{Au,1}(t) < x < a_{u,2}(t); \\ u_i(a_{u,2}(t),t), & x \geqslant a_{u,2}(t). \end{cases}$$

**IV)** If $a_{Au,1} < a_{u,1} < a_{Au,2} < a_{u,2}$, then

$$\overline{\tilde{A}_i u}(x,t) = \begin{cases} u_i(a_{u,1}(t),t), & x \leqslant a_{u,1}(t); \\ u_i(x,t), & a_{u,1}(t) < x < a_{Au,2}(t); \\ u_i(a_{Au,2}(t),t), & x \geqslant a_{Au,2}(t). \end{cases}$$

To find $\rho(\tilde{A}u^1, \tilde{A}u^2)$, we also consider different variants of the placement of the domains $G_{u^1,\varepsilon}$, $G_{Au^1,\varepsilon}$, $G_{u^2,\varepsilon}$ and $G_{Au^2,\varepsilon}$:

**a)** at $a_{Au^1,1} < a_{u^1,1} < a_{u^2,1} < a_{Au^2,1} < a_{u^2,2} < a_{Au^1,2} < a_{u^1,2} < a_{Au^2,2}$, we have

$$\max_{i,x,t} |\overline{\tilde{A}_i u^1}(x,t) - \overline{\tilde{A}_i u^2}(x,t)| = \begin{cases} \max\limits_{i,x,t} |u_i^1(a_{u^1,1}(t),t) - u_i^2(a_{Au^2,1}(t),t)|, & x \leqslant a_{u^1,1}(t); \\ \max\limits_{i,x,t} |u_i^1(x,t) - u_i^2(a_{Au^2,1}(t),t)|, & a_{u^1,1}(t) < x \leqslant a_{Au^2,1}(t); \\ \max\limits_{i,x,t} |u_i^1(x,t) - u_i^2(x,t)|, & a_{Au^2,1}(t) < x \leqslant a_{u^2,2}(t); \\ \max\limits_{i,x,t} |u_i^1(x,t) - u_i^2(a_{u^2,2}(t),t)|, & a_{u^2,2}(t) \leqslant x < a_{Au^1,2}(t); \\ \max\limits_{i,x,t} |u_i^1(a_{Au^1,2}(t),t) - u_i^2(a_{u^2,2}(t),t)|, & x \geqslant a_{Au^1,2}(t); \end{cases}$$

**b)** if $a_{Au^2,1} < a_{u^2,1} < a_{u^1,1} < a_{Au^1,1} < a_{Au^1,2} < a_{u^1,2} < a_{u^2,2} < a_{Au^2,2}$, then we get

$$\max_{i,x,t} |\overline{\tilde{A}_i u^1}(x,t) - \overline{\tilde{A}_i u^2}(x,t)| = \begin{cases} \max\limits_{i,x,t} |u_i^1(a_{Au^1,1}(t),t) - u_i^2(a_{u^2,1}(t),t)|, & x \leqslant a_{u^2,1}(t); \\ \max\limits_{i,x,t} |u_i^1(a_{Au^1,1}(t),t) - u_i^2(x,t)|, & a_{u^2,1}(t) < x \leqslant a_{Au^1,1}(t); \\ \max\limits_{i,x,t} |u_i^1(x,t) - u_i^2(x,t)|, & a_{Au^1,1}(t) < x < a_{Au^1,2}(t); \\ \max\limits_{i,x,t} |u_i^1(a_{Au^1,2}(t),t) - u_i^2(x,t)|, & a_{Au^1,2}(t) \leqslant x < a_{u^2,2}(t); \\ \max\limits_{i,x,t} |u_i^1(a_{Au^1,2}(t),t) - u_i^2(a_{u^2,2}(t),t)|, & x \geqslant a_{u^2,2}(t); \end{cases}$$

**c)** in the case of $a_{Au^2,1} < a_{Au^1,1} < a_{u^2,1} < a_{u^1,1} < a_{u^1,2} < a_{u^2,2} < a_{Au^1,2} < a_{Au^2,2}$, we have

$$\max_{i,x,t} |\overline{\tilde{A}_i u^1}(x,t) - \overline{\tilde{A}_i u^2}(x,t)| = \begin{cases} \max\limits_{i,x,t} |u_i^1(a_{u^1,1}(t),t) - u_i^2(a_{u^2,1}(t),t)|, & x \leqslant a_{u^2,1}(t); \\ \max\limits_{i,x,t} |u_i^1(a_{u^1,1}(t),t) - u_i^2(x,t)|, & a_{u^2,1}(t) < x \leqslant a_{u^1,1}(t); \\ \max\limits_{i,x,t} |u_i^1(x,t) - u_i^2(x,t)|, & a_{u^1,1}(t) < x < a_{u^1,2}(t); \\ \max\limits_{i,x,t} |u_i^1(a_{u^1,2}(t),t) - u_i^2(x,t)|, & a_{u^1,2}(t) \leqslant x < a_{u^2,2}(t); \\ \max\limits_{i,x,t} |u_i^1(a_{u^1,2}(t),t) - u_i^2(a_{u^2,2}(t),t)|, & x \geqslant a_{u^2,2}(t); \end{cases}$$

**d)** if $a_{Au^1,1} < a_{u^1,1} < a_{u^2,1} < a_{Au^2,1} < a_{Au^2,2} < a_{u^2,2} < a_{Au^1,2} < a_{u^1,2}$, then

$$\max_{i,x,t}|\overline{\tilde{A}_i u^1}(x,t)-\overline{\tilde{A}_i u^2}(x,t)|=\begin{cases}\max\limits_{i,x,t}|u_i^1(a_{u^1,1}(t),t)-u_i^2(a_{Au^2,1}(t),t)|,\ x\leqslant a_{u^1,1}(t);\\ \max\limits_{i,x,t}|u_i^1(x,t)-u_i^2(a_{Au^2,1}(t),t)|,\ a_{u^1,1}(t)<x\leqslant a_{Au^2,1}(t);\\ \max\limits_{i,x,t}|u_i^1(x,t)-u_i^2(x,t)|,\ a_{Au^2,1}(t)<x<a_{Au^2,2}(t);\\ \max\limits_{i,x,t}|u_i^1(x,t)-u_i^2(a_{Au^2,2}(t),t)|,\ a_{Au^2,2}(t)\leqslant x<a_{Au^1,2}(t);\\ \max\limits_{i,x,t}|u_i^1(a_{Au^1,2}(t),t)-u_i^2(a_{Au^2,2}(t),t)|,\ x\geqslant a_{Au^1,2}(t);\end{cases}$$

**e)** if $a_{u^2,1}<a_{Au^2,1}<a_{u^1,1}<a_{Au^1,1}<a_{Au^1,2}<a_{u^1,2}<a_{Au^2,2}<a_{u^2,2}$, then

$$\max_{i,x,t}|\overline{\tilde{A}_i u^1}(x,t)-\overline{\tilde{A}_i u^2}(x,t)|=\begin{cases}\max\limits_{i,x,t}|u_i^1(a_{Au^1,1}(t),t)-u_i^2(a_{Au^2,1}(t),t)|,\ x\leqslant a_{Au^2,1}(t);\\ \max\limits_{i,x,t}|u_i^1(a_{Au^1,1}(t),t)-u_i^2(x,t)|,\ a_{Au^2,1}(t)<x\leqslant a_{Au^1,1}(t);\\ \max\limits_{i,x,t}|u_i^1(x,t)-u_i^2(x,t)|,\ a_{Au^1,1}(t)<x<a_{Au^1,2}(t);\\ \max\limits_{i,x,t}|u_i^1(a_{Au^1,2}(t),t)-u_i^2(x,t)|,\ a_{Au^1,2}(t)\leqslant x<a_{Au^2,2}(t);\\ \max\limits_{i,x,t}|u_i^1(a_{Au^1,2}(t),t)-u_i^2(a_{Au^2,2}(t),t)|,\ x\geqslant a_{Au^2,2}(t);\end{cases}$$

**f)** in the case of $a_{Au^1,1}<a_{u^1,1}<a_{Au^2,1}<a_{u^2,1}<a_{u^2,2}<a_{Au^2,2}<a_{Au^1,2}<a_{u^1,2}$, we obtain

$$\max_{i,x,t}|\overline{\tilde{A}_i u^1}(x,t)-\overline{\tilde{A}_i u^2}(x,t)|=\begin{cases}\max\limits_{i,x,t}|u_i^1(a_{u^1,1}(t),t)-u_i^2(a_{u^2,1}(t),t)|,\ \ x\leqslant a_{u^1,1}(t);\\ \max\limits_{i,x,t}|u_i^1(x,t)-u_i^2(a_{u^2,1}(t),t)|,\ \ a_{u^1,1}(t)<x\leqslant a_{u^2,1}(t);\\ \max\limits_{i,x,t}|u_i^1(x,t)-u_i^2(x,t)|,\ \ a_{u^2,1}(t)<x<a_{u^2,2}(t);\\ \max\limits_{i,x,t}|u_i^1(x,t)-u_i^2(a_{u^2,2}(t),t)|,\ \ a_{u^2,2}(t)\leqslant x<a_{Au^1,2}(t);\\ \max\limits_{i,x,t}|u_i^1(a_{Au^1,2}(t),t)-u_i^2(a_{u^2,2}(t),t)|,\ \ x\geqslant a_{Au^1,2}(t).\end{cases}$$

Since

$$\max_{i,x,t}|u_i^1(a_{u^1,1}(t),t)-u_i^2(a_{Au^2,1}(t),t)|\leqslant\max_{i,x,t}\{|u_i^1(a_{u^1,1}(t),t)-u_i^2(a_{u^1,1}(t),t)|+$$
$$+|u_i^2(a_{u^1,1}(t),t)-u_i^2(a_{Au^2,1}(t),t)|\}\leqslant\rho+\delta\max_t|a_{u^1,1}(t)-a_{Au^2,1}(t)|\leqslant$$
$$\leqslant\rho+\delta\max_t|a_{Au^1,1}(t)-a_{Au^2,1}(t)|\leqslant\rho(1+\delta\varepsilon h_0);$$
$$\max_{i,x,t}|u_i^1(x,t)-u_i^2(a_{Au^2,1}(t),t)|\leqslant\max_{i,x,t}\{|u_i^1(x,t)-u_i^2(x,t)|+$$
$$+|u_i^2(x,t)-u_i^2(a_{Au^2,1}(t),t)|\}\leqslant\rho+\delta\max_{x,t}|x-a_{Au^2,1}(t)|\leqslant$$
$$\leqslant\rho+\delta\max_t|a_{u^1,1}(t)-a_{Au^2,1}(t)|\leqslant\rho(1+\delta\varepsilon h_0);$$
$$\max_{i,x,t}|u_i^1(x,t)-u_i^2(x,t)|\leqslant\rho;$$
$$\max_{i,x,t}|u_i^1(x,t)-u_i^2(a_{u^2,k}(t),t)|\leqslant\max_{i,x,t}\{|u_i^1(x,t)-u_i^2(x,t)|+$$
$$+|u_i^2(x,t)-u_i^2(a_{u^2,k}(t),t)|\}\leqslant\rho+\delta\max_{x,t}|x-a_{u^2,k}(t)|\leqslant$$
$$\leqslant\rho+\delta\max_t|a_{Au^1,k}(t)-a_{u^2,k}(t)|\leqslant\rho+\delta\max_t|a_{u^1,k}(t)-a_{u^2,k}(t)|\leqslant\rho(1+\delta);$$
$$\max_{i,x,t}|u_i^1(a_{Au^1,2}(t),t)-u_i^2(a_{u^2,2}(t),t)|\leqslant\rho+\delta\max_t|a_{Au^1,2}(t)-a_{u^2,2}(t)|\leqslant$$
$$\leqslant\rho+\delta\max_t|a_{u^1,2}(t)-a_{u^2,2}(t)|\leqslant\rho(1+\delta);$$

$$\max_{i,x,t} |u_i^1(a_{Au^1,k}(t),t) - u_i^2(a_{u^2,k}(t),t)| \leqslant \rho + \delta \max_t |a_{Au^1,k}(t) - a_{Au^2,k}(t)| \leqslant \rho(1+\delta\varepsilon h_0);$$

$$\max_{i,x,t} |u_i^1(a_{Au^1,1}(t),t) - u_i^2(x,t)| \leqslant \rho + \delta \max_{x,t} |x - a_{Au^1,1}(t)| \leqslant$$
$$\leqslant \rho + \delta \max_t |a_{Au^1,1}(t) - a_{Au^2,1}(t)| \leqslant \rho(1+\delta\varepsilon h_0);$$

$$\max_{i,x,t} |u_i^1(a_{Au^1,2}(t),t) - u_i^2(x,t)| \leqslant \rho + \delta \max_{x,t} |x - a_{Au^1,2}(t)| \leqslant \rho(1+\delta\varepsilon h_0);$$

$$\max_{i,x,t} |u_i^1(a_{u^1,k}(t),t) - u_i^2(a_{u^2,k}(t),t)| \leqslant \rho + \delta \max_t |a_{u^1,k}(t) - a_{u^2,k}(t)| \leqslant \rho(1+\delta);$$

$$\max_{i,x,t} |u_i^1(x,t) - u_i^2(a_{Au^2,k}(t),t)| \leqslant \rho + \delta \max_{x,t} |x - a_{Au^2,k}(t)| \leqslant$$
$$\leqslant \rho + \delta \max_t |a_{Au^1,k}(t) - a_{Au^2,k}(t)| \leqslant \rho(1+\delta\varepsilon h_0);$$

$$\max_{i,x,t} |u_i^1(a_{Au^1,k}(t),t) - u_i^2(a_{Au^2,k}(t),t)| \leqslant \rho + \delta \max_t |a_{Au^1,k}(t) - a_{Au^2,k}(t)| \leqslant \rho(1+\delta\varepsilon h_0),$$

then

$$\rho(\tilde{A}u^1, \tilde{A}u^2) \leqslant \max\{\varepsilon h_0 \rho, \rho(1+\delta), \rho(1+\delta\varepsilon h_0)\} = \max\{1+\delta, 1+\delta\varepsilon h_0\}. \tag{88}$$

Taking into account the definition of the metric in $S_T$, we obtain

$$\rho(Av^1, Av^2) = \max\Big\{\varepsilon h_0 \rho, \max_{i,x,t} \Big|\Delta \overline{A_i u^j}(x,t)\Big|\Big\}.$$

If $(x,t) \in G_{Au^1,\varepsilon} \cap G_{Au^2,\varepsilon}$ and for certainty $t_i(x,t;Au^1) < t_i(x,t;Au^2)$, $k=1$, then

$$|\Delta(\overline{A_i u^j})(x,t)| = |\Delta(A - iu^j)(x,t)| \leqslant \left|\Delta \int_{t_i(x,tlAu^j)}^t f_i(\varphi(\tau;x,t),\tau;\tilde{A}u^j)d\tau\right| +$$
$$+\Big|\Delta g_{i1}(t_i(x,t;Au^j);(\tilde{A}_{p+q+1}u^j)(a_{Au^j,1}(t_i(x,t;Au^j)),t_i(x,t;Au^j)),\ldots$$
$$\ldots,(\tilde{A}_n u^j)(a_{Au^j,1}(t_i(x,t;Au^j)),t_i(x,t;Au^j)))\Big| \leqslant$$
$$\leqslant F|t_i(x,t;Au^2) - t_i(x,t;Au^1)| + \varepsilon f_0 \rho(\tilde{A}u^1,\tilde{A}u^2) +$$
$$+\Big|\Delta g_{i1}(t_i(x,t;Au^1);(\tilde{A}_{p+q+1}u^1)(a_{Au^j,1}(t_i(x,t;Au^1)),t_i(x,t;Au^1)),\ldots$$
$$\ldots,(\tilde{A}_n u^1)(a_{Au^j,1}(t_i(x,t;Au^1)),t_i(x,t;Au^1)))\Big| +$$
$$+\Big|\Delta g_{i1}(t_i(x,t;Au^j);(\tilde{A}_{p+q+1}u^j)(a_{Au^2,1}(t_i(x,t;Au^j)),t_i(x,t;Au^j)),\ldots$$
$$\ldots,(\tilde{A}_n u^j)(a_{Au^2,1}(t_i(x,t;Au^j)),t_i(x,t;Au^j)))\Big| \leqslant$$
$$\leqslant F|t_i(x,t;Au^2) - t_i(x,t;Au^1)| + \varepsilon f_0 \rho(\tilde{A}u^1,\tilde{A}u^2) +$$
$$+\varepsilon g_0(\varepsilon)\delta\Big|a_{Au^1,1}(t_i(x,t;Au^1)) - a_{Au^2,1}(t_i(x,t;Au^1))\Big| +$$
$$+\varepsilon g_0(\varepsilon)\max\Big\{\max_{i,x,t}|\Delta t_i(x,t;Au^j)|;\ \max_{\substack{i,x,t,\\ p+q<l\leqslant n}} |\Delta(\tilde{A}_l u^j)(a_{Au^2,1}(t_i(x,t;Au^j)),t_i(x,t;Au^j))|\Big\}. \tag{89}$$

From the following

$$\lambda_i(a_{Au^1,1}(t),t) = \frac{a_{Au^1,1}(t_i(x,t;Au^1)) - a_{Au^2,1}(t_i(x,t;Au^2))}{t_i(x,t;Au^1) - t_i(x,t;Au^2)},$$

the estimate

$$|\lambda_i(a_{Au^1,1}(t),t)(t_i(x,t;Au^1) - t_i(x,t;Au^2))| \leqslant |a_{Au^1,1}(t_i(x,t;Au^1)) - a_{Au^2,1}(t_i(x,t;Au^1))| +$$
$$+|a_{Au^2,1}(t_i(x,t;Au^1)) - a_{Au^2,1}(t_i(x,t;Au^2))| \leqslant \varepsilon h_0 \rho + h_1(t;u)|t_i(x,t;Au^1) - t_i(x,t;Au^2)|$$

follows. Hence, we get

$$|t_i(x,t;Au^1)-t_i(x,t;Au^2)|\leqslant\frac{\varepsilon h_0\rho}{|\lambda_i(a_{Au^1,1}(t),t)-h_1(t;u)|}\leqslant\frac{1}{\gamma}\varepsilon h_0\rho.$$

Taking into account (89) and (88), we obtain

$$\begin{aligned}|\Delta(\overline{A_iu^j})(x,t)|\leqslant F\frac{1}{\gamma}\varepsilon h_0\rho+\varepsilon f_0\rho\max\{1+\delta,1+\delta\varepsilon h_0\}+\varepsilon h_0\rho+\\
+\varepsilon g_0(\varepsilon)\max\Big\{\frac{1}{\gamma}\varepsilon h_0\rho;\ \max_{\substack{i,x,t,\\ p+q<l\leqslant n}}\Big(|\Delta(\tilde{A}_lu^1)(a_{Au^2,1}(t_i(x,t;Au^j)),t_i(x,t;Au^j))|+\\
+|\Delta(\tilde{A}_lu^j)(a_{Au^2,1}(t_i(x,t;Au^2)),t_i(x,t;Au^2)\Big))\Big\}\leqslant\\
\leqslant\left(\frac{F}{\gamma}+\varepsilon g_0(\varepsilon)\delta\right)\varepsilon h_0\rho+\varepsilon f_0\rho\max\{1+\delta,1+\delta\varepsilon h_0\}+\varepsilon g_0(\varepsilon)\times\\
\times\max\Big\{\frac{1}{\gamma}\varepsilon h_0\rho;\sigma\max_{i,x,t}|t_i(x,t;Au^1)-t_i(x,t;Au^2)|+\rho(\tilde{A}_lu^1,\tilde{A}_lu^2)\Big\}\leqslant\\
\leqslant\Bigg(\Big(\frac{F}{\gamma}+\varepsilon g_0(\varepsilon)\delta\Big)\varepsilon h_0+\varepsilon f_0\max\{1+\delta,1+\delta\varepsilon h_0\}+\varepsilon g_0(\varepsilon)\max\Big\{\frac{1}{\gamma}\varepsilon h_0;\\
\frac{1}{\gamma}\sigma\varepsilon h_0+\max\{1+\delta,1+\delta\varepsilon h_0\}\Big\}\Bigg)\rho\leqslant\\
\leqslant\Bigg(\Big(\frac{F+\sigma\varepsilon g_0(\varepsilon)}{\gamma}+\varepsilon g_0(\varepsilon)\delta\Big)\varepsilon h_0+(\varepsilon f_0+\varepsilon g_0(\varepsilon))\max\{1+\delta,1+\delta\varepsilon h_0\}\Bigg)\rho.\end{aligned}\tag{90}$$

Consider the case when $(x,t)\in G_{Au^2,\varepsilon}$, but $(x,t)\notin G_{Au^1,\varepsilon}$. Then, assuming $k=1$ for certainty, we obtain

$$\begin{aligned}|\Delta(\overline{A_iu^j})(x,t)|\leqslant|(A_iu^2)(x,t)-(A_iu^1)(a_{Au^1,2}(t),t)|\leqslant\\
\leqslant|(A_iu^2)(x,t)-(A_iu^2)(a_{Au^1,2}(t),t)|+|(A_iu^2)(a_{Au^1,2}(t),t)-(A_iu^1)(a_{Au^1,2}(t),t)|\leqslant\\
\leqslant\delta|x-a_{Au^1,2}(t)|+|\Delta(A_iu^j)(a_{Au^1,2}(t),t)|,\end{aligned}$$

with $|x-a_{Au^1,2}(t)|\leqslant|a_{Au^2,2}(t)-a_{Au^1,2}(t)|\leqslant\varepsilon h_0\rho$, and $(a_{Au^1,2}(t),t)\in G_{Au^1,\varepsilon}\cap G_{Au^2,\varepsilon}$, that is, we have come to a case that has already been analyzed. Therefore

$$\begin{aligned}|\Delta(\overline{A_iu^j})(x,t)|\leqslant\\
\leqslant\delta\varepsilon h_0\rho+\left(\left(\frac{F+\sigma\varepsilon g_0(\varepsilon)}{\gamma}+\varepsilon g_0(\varepsilon)\delta\right)\varepsilon h_0+(\varepsilon f_0+\varepsilon g_0(\varepsilon))\max\{1+\delta,1+\delta\varepsilon h_0\}\right)\rho.\end{aligned}$$

So, the condition for the compression of the operator $A\colon S\to S$ is as follows

$$\varepsilon_0\left(h_0\left(\delta+\varepsilon_0g_0(\varepsilon_0)\left(\delta+\frac{\sigma}{\gamma}\right)+\frac{F}{\gamma}\right)+(f_0+g_0(\varepsilon_0))\max\{1+\delta,1+\delta\varepsilon_0h_0\}\right)<1.\tag{91}$$

Now let's show that the set of all these conditions is compatible. Indeed, let us first choose $\varepsilon_0$, $\alpha_0$ as we said at the beginning of the proof. Then we choose $\sigma>F+F(\Lambda+\alpha_0+H)/\gamma$ and $\delta>\frac{F}{\gamma}$ (note that $\sigma$ and $\delta$ can be considered as large as desired). Next, for fixed $\alpha_0$, $\sigma$, $\delta$, we reduce $\varepsilon_0$ so that the following ratios (86), (87), and (91) are fulfilled. Then, according to the Banach theorem on compressible mappings in $S$, there exists a unique solution of the system of equations (70), (80) satisfying the condition (79), and therefore $v$ is a generalized solution of the problem (45), (70)–(72). □

**Remark 1.** From the proof of the theorem and equations (80), it follows that $u_i(x,t)$ satisfies the Lipschitz condition in all variables.

**Remark 2.** If it is known a priori that the boundaries of the sector $G_{u,T}$ are linear functions, and $f_i$, $h_k$, $g_{i,k}$ are also linear in $u$, then Theorem 4.1 holds for all $t \in [0,T] \subset [0,\infty)$.

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