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HAYMAN'S THEOREM FOR ANALYTIC FUNCTIONS IN A COMPLETE REINHARDT DOMAIN

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For functions analytic in a complete multiple circular domain $\mathbb{G} \subset \mathbb{C}^n$ there are established a counterpart of Haymans' Theorem. It specifies that in the definition of boundedness of **L**-index in joint variables the factorials in the denominator can be removed: An analytic function F in $\mathbb{G}$ has bounded **L**-index in joint variables if and only if there exist $p \in \mathbb{Z}_+$ and $c \in \mathbb{R}_+$ such that for each $z \in \mathbb{G}$

$$\max\left\{\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} \colon \|J\| = p+1\right\} \le c \cdot \max\left\{\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} \colon \|K\| \le p\right\},$$

where for $K = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n \colon \ \|K\| = k_1 + \ldots + k_n$,

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K}(z) = \frac{\partial^{k_1+k_2+\ldots+k_n} H}{\partial z_1^{k_1} \partial z_2^{k_2} \ldots \partial z_n^{k_n}}(z_1, z_2, \ldots, z_n), \quad \mathbf{L}^K(z) = l_1^{k_1}(z) \cdot \ldots \cdot l_n^{k_n}(z),$$

and the continuous mapping $\mathbf{L} = (l_1(z), l_2(z), \ldots, l_n(z)) \colon \mathbb{G} \to \mathbb{R}_+^n$ is locally regularly varying in some sense. It allows to apply this statement in study of local properties of analytic solutions for system of linear higher order partial differential equations. Other result concern estimate of sum of first N expressions from the definition by the sum of all next expressions of such form $|F^{(K)}(z)|/(K!\mathbf{L}^K(z))$, where $K! = (k_1, \ldots, k_n)$ for $K = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, and N is the **L**-index in joint variables of the function F.

1. Introduction. The different classes of multivariate analytic functions with bounded L-index in joint variables have sufficiently similar asymptotic and local properties. Especially, if we use same geometric exhaustion of n-dimensional complex space or Cauchy's integral formula and the main difference is their domain holomorphy. The geometric nature of such a domain defines additional conditions by the local behavior of positive continuous vector-valued function L. Mostly, it is sufficient to construct powerful theory of bounded index for this class of functions and to demonstrate its applicability for growth estimates ([1]), value distribution ([5,17]), analytic solutions of ordinary ([22]) and directional differential equations ([6,13,14]). One should observe that method of bouded index is full-fledged supplement to Wiman-Valiron's method ([21,23]).

Recently, there was published papers [2,3] on notion of boundedness of L-index in joint variables for functions which are analytic in a complete Reinhardt domain. Since it is the most general multidimensional holomorphy domain, such an approach allows to overlap all classical domain of holormophy: polydisc ([8]), ball ([4,7,12]). We will need the following

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standard notations from the theory of holomorphic multivariate functions (see, for example, [2, 3, 18]): $\mathbb{R}_+ = [0, +\infty)$, $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$, $\mathbf{1}_j = (0, \dots, 0, 1, 0, \dots, 0)$. For $A = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $B = (b_1, \dots, b_n) \in \mathbb{R}^n$ we use $AB = (a_1 b_1, \dots, a_n b_n)$, $A/B = (a_1/b_1, \dots, a_n/b_n)$, $A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}$. $\|A\| = \sum_{j=1}^n a_j$, and the inequalities $A < B$, $A \leq B$ are understood as coordinate inequalities, for $K = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ $K! = k_1! \cdot \ldots \cdot k_n!$. The arithmetic operations as addition, scalar multiplication, and conjugation for points from the n-dimensional complex space are given component-wise. The open polydisc is $\mathbb{D}^n(z^0, R) = \{z \in \mathbb{C}^n \colon |z_j - z_j^0| < r_j \text{ in all } j \in \{1, \dots, n\}\}$, the polydisc skeleton is $\mathbb{T}^n(z^0, R) = \{z \in \mathbb{C}^n \colon |z_j - z_j^0| = r_j,\ j \in \{1, \dots, n\}\}$, the closed polydisc is $\mathbb{D}^n[z^0, R] = \{z \in \mathbb{C}^n \colon |z_j - z_j^0| \leq r_j,\ j \in \{1, \dots, n\}\}$, while $\mathbb{D}^n = \mathbb{D}^n(\mathbf{0}, \mathbf{1})$, $\mathbb{D} = \mathbb{D}^1$.

The domain $\mathbb{G} \subset \mathbb{C}^n$ is called the complete Reinhardt domain ([19]), if:
(a) $\forall z = (z_1, \dots, z_n) \in \mathbb{G}$ $\forall R = (r_1, \dots, r_n) \in [0, 1]^n$ one has $Rz = (r_1 z_1, \dots, r_n z_n) \in \mathbb{G}$ (*completeness of the domain*);
(b) $\forall (z_1, \dots, z_n) \in \mathbb{G}$, $\forall (\theta_1, \dots, \theta_n) \in [0; 2\pi]^n)$ one has $(z_1 e^{i\theta_1}, \dots, z_n e^{i\theta_n}) \in \mathbb{G}$ (*it is a multiple-circular domain or condition of multiple-circularity*). Denote by $\partial\mathbb{G}$ the boundary of the domain $\mathbb{G}$.

For $J = (j_1, j_2, \dots, j_n) \in \mathbb{Z}_+^n$ we denote the J-th order partial derivative of an analytic function $H \colon \mathbb{G} \to \mathbb{C}^n$ as follows

$$H^{(J)}(z) = \frac{\partial^{\|J\|} H}{\partial z^J}(z) = \frac{\partial^{j_1 + j_2 + \ldots + j_n} H}{\partial z_1^{j_1} \partial z_2^{j_2} \ldots \partial z_n^{j_n}}(z_1, z_2, \dots, z_n).$$

By $\overline{\mathbb{G}}$, we denote the closure of the complete Reinhardt domain $\mathbb{G}$ and $\partial\mathbb{G} = \overline{\mathbb{G}} \setminus \mathbb{G}$. We suppose that an auxiliary mapping $\mathbf{L}(z) = (l_1(z), l_2(z), \dots, l_n(z))$ satisfies the following conditions:

(1) $\forall j \in \{1, 2, \dots, n\}$ the function $l_j \colon \mathbb{G} \to \mathbb{R}_+$ in continuous;
(2) $\forall j \in \{1, 2, \dots, n\}$ $\forall z \in \mathbb{G}$ one has

$$l_j(z) > \frac{\beta}{\inf\limits_{\substack{\widehat{R}_j z \in \partial\mathbb{G}, \\ r > 1}} (r|z_j|) - |z_j|} \tag{1}$$

for some real $\beta > 1$. Here, $\widehat{R}_j = (1, \dots, 1, \underbrace{r}_{j\text{-th item}}, 1 \dots, 1)$.

For simplicity, we also write $\mathcal{B} = (0, \beta]$, $\mathcal{B}^n = (0, \beta]^n$, and $\boldsymbol{\beta} = (\beta, \dots, \beta)$, where the constant β is defined by the mapping $\mathbf{L}$, and $\mathcal{B}^n$ is obtained as the Cartesian product of the left-open interval $\mathcal{B}$.

Below we suppose everywhere that $\mathbb{G} \subset \mathbb{C}^n$ is the complete Reinhardt domain, and we will not repeat this assumption in the following assertions and definitions.

An analytic function $H \colon \mathbb{G} \to \mathbb{C}$ is called a function with *bounded* **L**-*index in joint variables* if there exists $n_0 \in \mathbb{Z}_+$ such that $\forall J \in \mathbb{Z}_+^n$ and $\forall z \in \mathbb{G}$ one has

$$\frac{|H^{(J)}(z)|}{J! \mathbf{L}^J(z)} \leq \max\left\{ \frac{|H^{(K)}(z)|}{K! \mathbf{L}^K(z)} \colon\ K \in \mathbb{Z}_+^n,\ \|K\| \leq n_0 \right\}. \tag{2}$$

The least corresponding number n_0 is the **L**-*index in joint variables* for the function H, and $N(H, \mathbf{L}, \mathbb{G}) = n_0$ stands for the index. If $n = 1$, $\mathbf{L} = l$ then such analytic functions always for given positive continuous function l (see [16]).

Suggest that the following expression

$$\lambda_j(R) = \sup_{z, w \in \mathbb{G}} \left\{ \frac{l_j(z)}{l_j(w)} \colon |z_k - w_k| \leq \frac{r_k}{\min\{l_k(z), l_k(w)\}},\ k \in \{1, \dots, n\} \right\} \tag{3}$$

is finite for some $R \in \mathcal{B}^n$. The class of these mapping $\mathbf{L}\colon \mathbb{G} \to \mathbb{R}^n_+$ satisfying (1) and (3) is denoted by $Q(\mathbb{G})$. It is easy to see that a validity of inequality (3) for some $R \in \mathcal{B}^n$ yields the validity of the same inequality for all $R \in \mathcal{B}^n$. We will use the notation $\Lambda(R) = (\lambda_1(R), \dots, \lambda_n(R))$.

2. Auxiliary propositions. To prove an analog of Hayman's Theorem we need some theorems. Let $\widetilde{\mathbf{L}}(z) = (\widetilde{l}_1(z), \dots, \widetilde{l}_n(z)) : \mathbb{G} \to \mathbb{C}^n$. The notation $\mathbf{L} \asymp \widetilde{\mathbf{L}}$ means that there exist $\Theta_1 = (\theta_{1,j}, \dots, \theta_{1,n})$, $\Theta_2 = (\theta_{2,j}, \dots, \theta_{2,n}) \in \mathbb{R}^n_+$ such that $\theta_{1,j}\widetilde{l}_j(z) \le l_j(z) \le \theta_{2,j}\widetilde{l}_j(z)$ $\forall j \in \{1, 2, 3, \dots, n\}$ and $\forall z \in \mathbb{G}$. For an analytic in $\mathbb{G}$ function F we put $M(r, z^0, F) = \max\{|F(z)|\colon z \in \mathbb{T}^n(z^0, r)\}$, where $z^0 \in \mathbb{G}$, $r \in \mathbb{R}^n_+$. Then $M(R, z^0, F) = \max\{|F(z)|\colon z \in \mathbb{D}^n[z^0, R]\}$.

Theorem 1 ([2])**.** *Let $\mathbf{L} \in Q(\mathbb{G})$, $\mathbf{L} \asymp \widetilde{\mathbf{L}}$, $\beta\Theta_1 > \mathbf{1}$. A function H belonging to the class $\mathcal{A}^n(\mathbb{G})$ of analytic functions has bounded $\widetilde{\mathbf{L}}$-index in joint variables if and only if the function is of finite joint $\mathbf{L}$-index.*

Theorem 2 ([3])**.** *Let $\mathbf{L} \in Q(\mathbb{G})$, $F\colon \mathbb{G} \to \mathbb{C}$ be an analytic function. If there exist R', $R'' \in \mathcal{B}^n$, $R' < R''$, and $p_1 \ge 1$ such that for every $z^0 \in \mathbb{G}$*

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) \le p_1 M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right) \tag{4}$$

then F is of bounded $\mathbf{L}$-index in joint variables.

Theorem 3 ([3])**.** *Let $\mathbf{L} \in Q(\mathbb{G})$. If an analytic in $\mathbb{G}$ function F has bounded $\mathbf{L}$-index in joint variables then for any R', $R'' \in \mathcal{B}^n$, $R' < R''$, there exists a number $p_1 = p_1(R', R'') \ge 1$ such that for every $z^0 \in \mathbb{G}$ inequality (4) holds.*

3. Analog of Theorem of Hayman for analytic in a ball function of bounded L-index in joint variables. The following theorem was firstly deduced by W. Hayman ([17]). There are known its many analogs for various classes of analytic functions of several complex variables (see more references in [8,12,13]). It admits an easy application to partial differential equations ([6,14]), their systems ([11]) and for composition of analytic functions ([9,15]). Similar estimates to (11) was recently obtained in [10] for multivariate functions of exponential type. For random entire functions such problems are not considered yet ([20]).

Theorem 4. *Let $\mathbf{L} \in Q(\mathbb{G})$. An analytic function F in $\mathbb{G}$ has bounded $\mathbf{L}$-index in joint variables if and only if there exist $p \in \mathbb{Z}_+$ and $c \in \mathbb{R}_+$ such that for each $z \in \mathbb{G}$*

$$\max\left\{\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)}\colon \|J\| = p+1\right\} \le c \cdot \max\left\{\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)}\colon \|K\| \le p\right\}. \tag{5}$$

Proof. Let $N = N(F, \mathbf{L}, \mathbb{G}) < +\infty$. The definition of the boundedness of $\mathbf{L}$-index in joint variables yields the necessity with $p = N$ and $c = ((N+1)!)^n$.

We prove the sufficiency. For $F \equiv 0$ the theorem is obvious. Thus, we suppose that $F \not\equiv 0$.

Assume that (5) holds, $z^0 \in \mathbb{G}$, $z \in \mathbb{D}^n[z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}]$. For all $J \in \mathbb{Z}^n_+$, $\|J\| \le p+1$, one has

$$\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z^0)} \le \Lambda^J(\boldsymbol{\beta})\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} \le c \cdot \Lambda^J(\boldsymbol{\beta}) \max\left\{\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)}\colon \|K\| \le p\right\} \le$$
$$\le c \cdot \Lambda^J(\boldsymbol{\beta}) \max\left\{\Lambda^K(\boldsymbol{\beta})\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z^0)}\colon \|K\| \le p\right\} \le BG(z), \tag{6}$$

where $B = c\cdot\max\{\Lambda^K(\boldsymbol{\beta})\colon \|K\| = p+1\}\max\{\Lambda^K(\boldsymbol{\beta})\colon \|K\| \le p\}$, and $G(z) = \max\big\{\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z^0)}\colon \|K\| \le p\big\}$. We choose $z^{(1)} = (z_1^{(1)}, \dots, z_n^{(1)}) \in \mathbb{T}^n(z^0, \frac{\mathbf{1}}{2\beta\mathbf{L}(z^0)})$ and $z^{(2)} = (z_1^{(2)}, \dots, z_n^{(2)}) \in \mathbb{T}^n(z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)})$ such that $F(z^{(1)}) \ne 0$ and

$$|F(z^{(2)})| = M\left(\frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}, z^0, F\right) \ne 0. \tag{7}$$

These points exist, otherwise if $F(z) \equiv 0$ on skeleton $\mathbb{T}^n(z^0, \frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)})$ or $\mathbb{T}^n(z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)})$ then by the uniqueness theorem $F \equiv 0$ in $\mathbb{G}$. We connect the points $z^{(1)}$ and $z^{(2)}$ with plane

$$\alpha = \begin{cases} z_2 = k_2 z_1 + c_2, \\ z_3 = k_3 z_1 + c_3, \\ \dots \\ z_n = k_n z_1 + c_n, \end{cases}$$

where $k_i = \frac{z_i^{(2)} - z_i^{(1)}}{z_1^{(2)} - z_1^{(1)}}$, $c_i = \frac{z_i^{(1)} z_1^{(2)} - z_i^{(2)} z_1^{(1)}}{z_1^{(2)} - z_1^{(1)}}$, $i \in \{2, \dots, n\}$. It is easy to check that $z^{(1)} \in \alpha$ and $z^{(2)} \in \alpha$. Let $\widetilde{G}(z_1) = G(z)|_\alpha$ be a restriction of the function G onto α.

Fo every $K \in \mathbb{Z}_+^n$ the function $F^{(K)}(z)\big|_\alpha$ is analytic function of variable z_1 and $\tilde{G}(z_1^{(1)}) = G(z^{(1)})\big|_\alpha \ne 0$ because $F(z^{(1)}) \ne 0$. Hence, all zeros of the function $F^{(K)}(z)\big|_\alpha$ are isolated as zeros of a function of one variable. Thus, zeros of the function $\tilde{G}(z_1)$ are isolated too. Therefore, we can choose piecewise analytic curve γ onto α as following $z = z(t) = (z_1(t), k_2 z_1(t) + c_2, \dots, k_n z_1(t) + c_n)$, $t \in [0, 1]$, which connect the points $z^{(1)}$, $z^{(2)}$ and such that $G(z(t)) \ne 0$ and $\int_0^1 |z_1'(t)|dt \le \frac{2\beta}{\sqrt{n}l_1(z_1^0)}$. For a construction of the curve we connect $z_1^{(1)}$ and $z_1^{(2)}$ by a line $z_1^*(t) = (z_1^{(2)} - z_1^{(1)})t + z_1^{(1)}$, $t \in [0, 1]$. The curve γ can cross points z_1 at which the function $\widetilde{G}(z_1) = 0$. The number of such points $m = m(z^{(1)}, z^{(2)})$ is finite. Let $(z_{1,k}^*)$ be a sequence of these points in ascending order of the value $|z_1^{(1)} - z_{1,k}^*|$, $k \in \{1, 2, ..., m\}$. We choose

$$r < \min_{1 \le k \le m-1}\Big\{|z_{1,k}^* - z_{1,k+1}^*|, |z_{1,1}^* - z_1^{(1)}|, |z_{1,m}^* - z_1^{(2)}|, \frac{2\beta^2 - 1}{2\pi\sqrt{n}\beta l_1(z^0)}\Big\}.$$

Now we construct circles with centers at the points $z_{1,k}^*$ and corresponding radii $r_k' < \frac{r}{2^k}$ such that $\widetilde{G}(z_1) \ne 0$ for all z_1 on the circles. It is possible, because $F \not\equiv 0$.

Every such circle is divided onto two semicircles by the line $z_1^*(t)$. The required piecewise-analytic curve consists with arcs of the constructed semicircles and segments of line $z_1^*(t)$, which connect the arcs in series between themselves or with the points $z_1^{(1)}, z_1^{(2)}$. The length of $z_1(t)$ in $\mathbb{C}$ (but not $z(t)$ in $\mathbb{C}^n$!) is lesser than $\frac{\beta/\sqrt{n}}{l_1(z^0)} + \frac{1}{2\sqrt{n}\beta l_1(z^0)} + \pi r \le \frac{2\beta}{\sqrt{n}l_1(z^0)}$. Then

$$\int_0^1 |z_s'(t)|dt = |k_s|\int_0^1 |z_1'(t)|dt \le \frac{|z_s^{(2)} - z_s^{(1)}|}{|z_1^{(2)} - z_1^{(1)}|}\frac{2\beta}{\sqrt{n}l_1(z^0)} \le$$
$$\le \frac{2\beta^2 + 1}{2\sqrt{n}\beta l_s(z^0)}\frac{2\sqrt{n}\beta l_1(z^0)}{2\beta^2 - 1}\frac{2\beta}{\sqrt{n}l_1(z^0)} \le \frac{2\beta(2\beta^2 + 1)}{(2\beta^2 - 1)\sqrt{n}l_s(z^0)},\ s \in \{2, \dots, n\}.$$

Hence,

$$\int_0^1 \sum_{s=1}^n l_s(z^0)|z_s'(t)|dt \le \frac{2\beta(2\beta^2 + 1)\sqrt{n}}{2\beta^2 - 1} = S. \tag{8}$$

Since the function $z = z(t)$ is piece-wise analytic on $[0, 1]$, then for arbitrary $K \in \mathbb{Z}^n_+$, $J \in \mathbb{Z}^n_+$, $\|K\| \le p$, either

$$\frac{|F^{(K)}(z(t))|}{\mathbf{L}^K(z^0)} \equiv \frac{|F(J)(z(t))|}{\mathbf{L}^J(z^0)}, \tag{9}$$

$$\text{or} \quad \frac{|F^{(K)}(z(t))|}{\mathbf{L}^K(z^0)} = \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)} \tag{10}$$

holds only for a finite set of points $t_k \in [0; 1]$.

Then for function $G(z(t))$ as maximum of such expressions $\frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ by all $\|J\| \le p$ two cases are possible:

1. In some interval of analyticity of the curve γ the function $G(z(t))$ identically equals simultaneously to some derivatives, that is (9) holds. It means that $G(z(t)) \equiv \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ for some J, $\|J\| \le p$. Clearly, the function $F^{(J)}(z(t))$ is analytic. Then $|F^{(J)}(z(t))|$ is continuously differentiable function on the interval of analyticity except points where this partial derivative equals zero $|F^{(j_1,j_2)}(z_1(t), z_2(t))| = 0$. However, there are not the points, because in the opposite case $G(z(t)) = 0$. But it contradicts the construction of the curve γ.
2. In some interval of analyticity of the curve γ the function $G(z(t))$ equals simultaneously to some derivatives at a finite number of points t_k, that is (10) holds. Then the points t_k divide interval of analyticity onto a finite number of segments, in which of them $G(z(t))$ equals to one from the partial derivatives, i. e. $G(z(t)) \equiv \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ for some J, $\|J\| \le p$. As above, in each from these segments the functions $|F^{(J)}(z(t))|$, and $G(z(t))$ are continuously differentiable except the points t_k.

The inequality $\frac{d}{dt}|f(t)| \le |\frac{df(t)}{dt}|$ holds for complex-valued functions of real argument outside a countable set of points. In view of this fact and (6) we have

$$\frac{d}{dt}G(z(t)) \le \max\Big\{\frac{1}{\mathbf{L}^J(z^0)}\Big|\frac{d}{dt}F^{(J)}(z(t))\Big| \colon \|J\| \le p\Big\} \le$$
$$\le \max\Big\{\sum_{s=1}^n \Big|\frac{\partial^{\|J\|+1}F}{\partial z_1^{j_1}\dots\partial z_s^{j_s+1}\dots\partial z_n^{j_n}}(z(t))\Big|\frac{|z_s'(t)|}{\mathbf{L}^j(z^0)} \colon \|J\| \le p\Big\} \le$$
$$\le \max\Big\{\sum_{s=1}^n \Big|\frac{\partial^{\|J\|+1}F}{\partial z_1^{j_1}\dots\partial z_s^{j_s+1}\dots\partial z_n^{j_n}}(z(t))\Big|\frac{l_s(z^0)|z_s'(t)|}{l_1^{j_1}(z^0)\dots l_s^{j_1+1}(z^0)\dots l_n^{j_n}(z^0)} \colon \|J\| \le p\Big\} \le$$
$$\le \Big(\sum_{s=1}^n l_s(z^0)|z_s'(t)|\Big)\max\Big\{\frac{|F^{(j)}(z(t))|}{\mathbf{L}^J(z^0)} \colon \|J\| \le p+1\Big\} \le \Big(\sum_{s=1}^n l_s(z^0)|z_s'(t)|\Big)BG(z(t)).$$

Therefore, (8) yields

$$\Big|\ln\frac{G(z^{(2)})}{G(z^{(1)})}\Big| = \Big|\int_0^1 \frac{1}{G(z(t))}\frac{d}{dt}G(z(t))dt\Big| \le B\int_0^1 \sum_{s=1}^n l_s(z^0)|z_s'(t)|dt \le S \cdot B.$$

Using (7), we deduce $M(\frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}, z^0, F) \le G(z^{(2)}) \le G(z^{(1)})e^{SB}$. Since $z^{(1)} \in \mathbb{T}^n(z^0, \frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)})$, the Cauchy inequality holds $\frac{|F^{(J)}(z^{(1)})|}{\mathbf{L}^J(z^0)} \le J!(2\beta\sqrt{n})^{\|J\|}M(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F)$ for all $J \in \mathbb{Z}^n_+$. Therefore, for $\|J\| \le p$ we obtain $G(z^{(1)}) \le (p!)^n(2\beta\sqrt{n})^pM(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F)$,

$$M\left(\frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}, z^0, F\right) \leq e^{SB}(p!)^n(2\beta\sqrt{n})^p M\left(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F\right).$$

Hence, by Theorem 2 the function F has bounded $\mathbf{L}$-index in joint variables. □

Theorem 5. *Let $\mathbf{L} \in Q(\mathbb{G})$. An analytic function F in $\mathbb{G}$ has bounded $\mathbf{L}$-index in joint variables if and only if $\exists c \in (0; +\infty)$ $\exists N \in \mathbb{N}$ $\forall z \in \mathbb{G}$ one has*

$$\sum_{\|K\|=0}^{N} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \geq c \sum_{\|K\|=N+1}^{\infty} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}. \tag{11}$$

Proof. Let $\frac{1}{\beta} < \theta_j < 1$, $j \in \{1, \ldots, n\}$, $\Theta = (\theta_1, \ldots, \theta_n)$. If the function F has bounded $\mathbf{L}$-index in joint variables then by Theorem 1 F has bounded $\widetilde{\mathbf{L}}$-index in joint variables, where $\widetilde{\mathbf{L}} = (\widetilde{l}_1(z), \ldots, \widetilde{l}_n(z))$, $\widetilde{l}_j(z) = \theta_j l_j(z)$. Let $\widetilde{N} = N(F, \widetilde{L}, \mathbb{G})$. Therefore, for all $J \geq \mathbf{0}$

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \leq \widetilde{N}\right\} = \max\left\{\frac{\Theta^K |F^{(K)}(z)|}{K!\widetilde{\mathbf{L}}^K(z)} \colon \|K\| \leq \widetilde{N}\right\} \geq$$

$$\geq \prod_{s=1}^{n} \theta_s^{\widetilde{N}} \max\left\{\frac{|F^{(K)}(z)|}{K!\widetilde{\mathbf{L}}^K(z)} \colon \|K\| \leq \widetilde{N}\right\} \geq \prod_{s=1}^{n} \theta_s^{\widetilde{N}} \frac{|F^{(J)}(z)|}{J!\widetilde{\mathbf{L}}^J(z)} = \prod_{s=1}^{n} \theta_s^{\widetilde{N}-j_s} \frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \text{ and}$$

$$\sum_{\|J\|=\widetilde{N}+1}^{\infty} \frac{|F^{(J)}(z)|}{J!\mathbf{L}^j(z)} \leq \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \leq \widetilde{N}\right\} \sum_{\|J\|=\widetilde{N}+1}^{\infty} \theta_s^{j_s-\widetilde{N}} =$$

$$= \prod_{i=1}^{n} \frac{\theta_s}{1-\theta_s} \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \leq \widetilde{N}\right\} \leq \prod_{i=1}^{n} \frac{\theta_s}{1-\theta_s} \sum_{\|K\|=0}^{\widetilde{N}} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}.$$

Hence, we obtain (11) with $N = \widetilde{N}$ and $c = \prod_{i=1}^{n} \frac{\theta_s}{1-\theta_s}$. On the contrary, (11) implies

$$\max\left\{\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \colon \|J\| = N+1\right\} \leq \sum_{\|K\|=N+1}^{\infty} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \leq \frac{1}{c} \sum_{\|K\|=0}^{N} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \leq$$

$$\leq \frac{1}{c} \sum_{i=0}^{N} C_{n+i-1}^{i} \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \leq N\right\}$$

and by Theorem 4 F is of bounded $\mathbf{L}$-index in joint variables. □

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