UDC 517.5

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SHARP CONSTANT OF APPROXIMATION OF PERIODIC FUNCTIONS BY CESÀRO MEANS

O. Rovenska. *Sharp constant of approximation of periodic functions by Cesàro means*, Mat. Stud. **64** (2025), 161–169.

In the paper, we present new findings concerning Cesàro-type operators. Special attention is given to the Cesàro (C,α) summation operators, which form a widely used family of summation methods in Fourier analysis. We establish sharp inequality for the upper bound of uniform deviations of Cesàro (C,α) summation operators of the second order for the class of continuous periodic functions.

Let $n\in\mathbb{N}$ and $x_k^{(n)}=x_{k-1}^{(n)}+2\pi/(2n+1)$, $k\in\{0,\pm1,\ldots,\pm(n-1),n\}$, be the points from the interval $[-\pi,\pi]$ such that $-\pi\le x_{-n}^{(n)}<x_{-n+1}^{(n)}<\ldots<x_{-1}^{(n)}<x_0^{(n)}<x_1^{(n)}<\ldots<x_n^{(n)}\le\pi$; the set of points $\{x_k^{(n)}\}$ is uniquely determined by the value of $x_0^{(n)}$. The Cesàro (C,α) summation operators are defined by

$$\sigma_n^{(\alpha)}[f]\left(\left\{x_k^{(n)}\right\};x\right)=\frac{2}{2n+1}\sum_{k=-n}^{n}f\left(x_k^{(n)}\right)K_n^{\alpha}\left(x-x_k^{(n)}\right),\ K_n^{(\alpha)}(t)=\frac{1}{A_n^{\alpha}}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}D_\nu(t),$$

where $D_\nu(t)=\frac{\sin((\nu+1/2)t)}{2\sin(t/2)}$ is the Dirichlet kernel, and $A_n^{\alpha}=\frac{(\alpha+1)\ldots(\alpha+n)}{n!}$ $(n\in\mathbb{N})$, $A_0^{\alpha}=1$, are the Cesàro numbers, $\alpha>-1$. Let $\mathbb{T}=[-\pi,\pi]$ and $\mathrm{C}(\mathbb{T})$ be the space of continuouson $\mathbb{T}$ functions with the norm $\|f\|_{\mathrm{C}}=\max\{|f(t)|\colon t\in\mathbb{T}\}$.

The main result is contained in the following statement (Theorem 1): Let $f\in\mathrm{C}(\mathbb{T})$. Then the inequality

$$\left\|f-\sigma_n^{(2)}[f]\left(\left\{x_k^{(n)}\right\}\right)\right\|_{\mathrm{C}}\le\frac{11}{9\ln 2}\cdot\ln(n+1)\cdot\omega\left(f;2\pi/(2n+1)\right),\ n\in\mathbb{N},$$

holds. The constant $11/(9\ln 2)$ in this inequality is sharp.

1. Introduction.

Let $\mathrm{C}(\mathbb{T})$, where $\mathbb{T}=[-\pi,\pi]$ be the space of continuous, 2π-periodic functions with the norm

$$\|f\|_{\mathrm{C}}=\max\{|f(t)|\colon t\in\mathbb{T}\},$$

and let $\omega(f;\delta)$ be the modulus of continuity of functions $f\in\mathrm{C}(\mathbb{T})$

$$\omega(f;\delta)=\sup\{|f(t_1)-f(t_2)|\colon |t_1-t_2|\le\delta\}.$$

Cesàro (C,α) operators on $\mathrm{C}(\mathbb{T})$ are defined by the relation [32, Ch. 3]

$$\sigma_n^{(\alpha)}[f](x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)K_n^{(\alpha)}(t-x)\,dt,\quad K_n^{(\alpha)}(t)=\frac{1}{A_n^{\alpha}}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}D_\nu(t),$$

2020 *Mathematics Subject Classification:* 42A10.
Keywords: Cesàro operator; Fourier series; approximation; periodic functions; uniform norm; sharp constants.
doi:10.30970/ms.64.2.161-169

where $D_\nu(t)=\frac{\sin((\nu+1/2)t)}{2\sin(t/2)}$ is the Dirichlet kernel, and A_n^α are the Cesàro numbers

$$A_0^\alpha=1,\quad A_n^\alpha=\frac{(\alpha+1)\dots(\alpha+n)}{n!},\quad n\in\mathbb{N},\quad \alpha>-1.$$

For $\alpha=1$, the Cesàro (C,α)-means are the Fejér $(C,1)$-means, denoted by $\sigma_n[f]$:

$$\sigma_n[f](x)=\frac{2}{\pi(n+1)}\int_{-\pi}^{\pi} f(t)\Big(\frac{\sin((n+1)(t-x)/2)}{\sin((t-x)/2)}\Big)^2\,dt.$$

Many prominent authors — including Fejér ([10]), Riesz ([20]), Zygmund ([31]), Leindler ([13, 14]), Móricz and Shi ([17]), Totik ([26, 27]), and others — have contributed to the study of convergence and approximation properties of Cesàro (C,α) means in various function spaces. Their ideas continue to develop in numerous works (see, for example, [1, 2, 6, 7, 8, 9, 15, 23, 24, 28, 30] and others). The concept of Cesàro means is presented in monographs [5, 12, 25].

In numerical settings, it is often necessary to replace the integral operator by a discrete counterpart. In this work, we consider summation analogues of Cesàro (C,α) operators, which are defined as follows.

Let $n\in\mathbb{N}$. Denote by $\mathcal{A}_n$ the family of sets of $2n+1$ points $\{x_k^{(n)}\}$, $k=0,\pm1,\pm2,\dots,\pm n$, from the interval $[-\pi,\pi]$ such that

$$-\pi\le x_{-n}^{(n)}<x_{-n+1}^{(n)}<\dots<x_{-1}^{(n)}<x_0^{(n)}<x_1^{(n)}<\dots<x_n^{(n)}\le\pi,$$

and

$$x_k^{(n)}=x_{k-1}^{(n)}+\frac{2\pi}{2n+1},\quad k=0,\pm1,\dots,\pm(n-1),n.$$

It is clear that

$$-\frac{\pi}{2n+1}\le x_0^{(n)}\le\frac{\pi}{2n+1},$$

and the set of points $\{x_k^{(n)}\}\in\mathcal{A}_n$ is uniquely determined by the value of $x_0^{(n)}$.

The Cesàro (C,α) summation operators are defined by

$$\sigma_n^{(\alpha)}[f]\left(\left\{x_k^{(n)}\right\};x\right)=\frac{2}{2n+1}\sum_{k=-n}^{n} f\left(x_k^{(n)}\right)K_n^\alpha\left(x-x_k^{(n)}\right).$$

We consider the problem of computing the minimal value of the constant $A(\alpha)$ in the inequality

$$\left\|f-\sigma_n^{(\alpha)}[f]\left(\left\{x_k^{(n)}\right\}\right)\right\|_{\mathrm{C}}\le A(\alpha)\ln(n+1)\omega\left(f;\frac{2\pi}{2n+1}\right),$$

which is determined by the relation

$$A^*(\alpha)=\sup\Big\{\sup\Big\{\sup\Big\{\big\|f-\sigma_n^{(\alpha)}[f]\big(\{x_k^{(n)}\}\big)\big\|_{\mathrm{C}}\Big/\big(\ln(n+1)\,\omega\big(f;2\pi/(2n+1)\big)\big):\; f\in\mathrm{C}(\mathbb{T}),f\not\equiv\mathrm{const}\Big\}:\{x_k^{(n)}\}\in\mathcal{A}_n\Big\}:n\in\mathbb{N}\Big\}.$$

Our goal is to obtain the sharp constant for the approximation of periodic functions by second-order Cesàro (C,α) summation operators. This study continues the author's earlier studies reported in [18, 19], in which sharp constants of approximation were obtained for

Cesàro integral operators of the second and third order acting on classes of periodic Lipschitz functions. Sharp constants of approximation have been obtained for Fejér summation operators in [16], and for some integral operators in [3, 4, 11, 16, 21, 22, 29].

2. Result. Our main result is contained in the following theorem.

Theorem 1. *Let* $f \in \mathrm{C}(\mathbb{T})$, $n \in \mathbb{N}$. *Then the inequality*

$$\left\| f - \sigma_n^{(2)}[f]\left(\left\{x_k^{(n)}\right\}\right)\right\|_{\mathrm{C}} \le \frac{11}{9\ln 2}\cdot \ln(n+1)\cdot \omega\left(f; 2\pi/(2n+1)\right) \qquad (1)$$

holds. The constant $11/(9\ln 2)$ *is sharp.*

Proof. It is known that the equalities

$$\sum_{k=-n}^{n} \cos m x_k^{(n)} = 0, \qquad \sum_{k=-n}^{n} \sin m x_k^{(n)} = 0, \quad m, n \in \mathbb{N} \qquad (2)$$

hold for every set of points $\{x_k^{(n)}\} \in \mathcal{A}_n$ ([33, Ch. 1]).

Based on (2), one obtains the identity

$$\sigma_n^{(2)}[1]\left(\left\{x_k^{(n)}\right\}; x\right) \equiv 1, \quad n \in \mathbb{N}, \qquad (3)$$

which will be essential in what follows.

In view of (2) and the properties of the modulus of continuity $\omega(f;\delta)$, it follows that for each set $\{x_k^{(n)}\} \in \mathcal{A}_n$ and each function $f \in \mathrm{C}(\mathbb{T})$, there exists a constant $A > 0$ such that the inequality

$$\left\| f - \sigma_n^{(2)}[f]\left(\left\{x_k^{(n)}\right\}\right)\right\|_{\mathrm{C}} \le A\ln(n+1)\cdot \omega\left(f; 2\pi/(2n+1)\right), \quad n \in \mathbb{N} \qquad (4)$$

holds.

As in [4] (see also [16]), for a similar quantity, we obtain

$$\sup_{\substack{f\in \mathrm{C}(\mathbb{T})\\ f\not\equiv \mathrm{const}}} \frac{\left\| f - \sigma_n^{(2)}[f]\left(\left\{x_k^{(n)}\right\}\right)\right\|_{\mathrm{C}}}{\omega\left(f; \frac{2\pi}{2n+1}\right)} = \sigma_n^{(2)}[f_n]\left(\left\{x_k^{(n)}\right\}; 0\right), \qquad (5)$$

where $f_n(x)$ is an even 2π-periodic function defined on $[0,\pi]$ by the relation

$$f_n(x) = \begin{cases} \left[\frac{(2n+1)x}{2\pi}\right] + 1, & 2k\pi/(2n+1) < x < \frac{2(k+1)\pi}{2n+1}, \\ \left[\frac{(2n+1)x}{2\pi}\right], & x = 2k\pi/(2n+1), \quad k = 0, 1, \dots, n. \end{cases} \qquad (6)$$

Here, $[a]$ denotes the floor function (integer part of a real number a).

Denote

$$A^* = \sup_{n\in\mathbb{N}} \sup_{\{x_k^{(n)}\}\in\mathcal{A}_n} \frac{1}{\ln(n+1)} \sigma_n^{(2)}[f_n]\left(\left\{x_k^{(n)}\right\}; 0\right), \qquad (7)$$

and

$$\lambda_n\left(\left\{x_k^{(n)}\right\}\right) = \frac{1}{\ln(n+1)} \sigma_n^{(2)}[f_n]\left(\left\{x_k^{(n)}\right\}; 0\right). \qquad (8)$$

First, we derive a set $\{x_k^{(n)}\} \in \mathcal{A}_n$ such that $x_0^{(n)} = 0$. We have

$$\lambda_n\left(\{2k\pi/(2n+1)\}\right) = \frac{1}{\ln(n+1)}\frac{2}{2n+1}\sum_{k=-n}^{n} f_n\left(x_k^{(n)}\right) K_n^{(2)}\left(x_k^{(n)}\right) =$$
$$= \frac{2}{\ln(n+1)(2n+1)}\sum_{k=-n}^{n} f_n\left(2k\pi/(2n+1)\right) K_n^{(2)}\left(2k\pi/(2n+1)\right) =$$
$$= \frac{4}{(2n+1)\ln(n+1)}\sum_{k=1}^{n} kK_n^{(2)}\left(2k\pi/(2n+1)\right). \tag{9}$$

Next, we obtain a convenient representation for the kernel $K_n^{(2)}(t)$. One can see that

$$K_n^{(2)}(t) = \frac{2}{(n+1)(n+2)}\sum_{\nu=0}^{n}(n-\nu+1)\frac{\sin\left((\nu+1/2)\,t\right)}{2\sin\left(t/2\right)} =$$
$$= \frac{1}{(n+1)(n+2)\sin\left(t/2\right)}\sum_{\nu=0}^{n}(n-\nu+1)\sin\big((\nu+1/2)t\big), \tag{10}$$

and

$$\sum_{\nu=0}^{n}(n-\nu+1)\sin\left((\nu+1/2)\,t\right) = \sum_{\nu=0}^{n}(n-\nu)\sin\left((\nu+1/2)\,t\right) + \sum_{\nu=0}^{n}\sin\left((\nu+1/2)\,t\right) =$$
$$= \frac{1}{4}\left(\frac{2n}{\sin\left(t/2\right)} - \frac{\sin\left((2n+3)t/2\right)}{\sin^2\left(t/2\right)} + \frac{3}{\sin\left(t/2\right)}\right) = \frac{1}{4}\left(\frac{2n+3}{\sin\left(t/2\right)} - \frac{\sin\left((2n+3)t/2\right)}{\sin^2\left(t/2\right)}\right). \tag{11}$$

Combining (10) and (11), the following formula is obtained

$$K_n^{(2)}(t) = \frac{1}{4(n+1)(n+2)}\left(\frac{2n+3}{\sin^2\left(t/2\right)} - \frac{\sin((2n+3)t/2)}{\sin^3\left(t/2\right)}\right), \quad n \in \mathbb{N}. \tag{12}$$

From (12), it follows that

$$K_n^{(2)}\left(2k\pi/(2n+1)\right) = \frac{(2n+3) + 2(-1)^{k+1}\cos(k\pi/(2n+1))}{4(n+1)(n+2)\sin^2(k\pi/(2n+1))}.$$

Applying Jordan's inequality $\pi|\sin t| \geq 2t$, $t \in \left[0, \frac{\pi}{2}\right]$, we obtain

$$K_n^{(2)}\left(2k\pi/(2n+1)\right) \leq \frac{(2n+3) + 2(-1)^{k+1}\cos(k\pi/(2n+1))}{4(n+1)(n+2)}\left(\frac{\pi}{2}\right)^2\left(\frac{2n+1}{k\pi}\right)^2.$$

Then

$$K_n^{(2)}\left(2k\pi/(2n+1)\right) \leq \frac{\big((2n+3) + 2(-1)^{k+1}\cos(k\pi/(2n+1))\big)(2n+1)^2}{16(n+1)(n+2)}\frac{1}{k^2} \leq$$
$$\leq \frac{\left((2n+3)+2\right)(2n+1)^2}{16(n+1)(n+2)}\frac{1}{k^2} = \frac{(2n+5)\,(2n+1)^2}{16(n+1)(n+2)}\frac{1}{k^2}. \tag{13}$$

Combining (9) and (13), we obtain

$$\lambda_n\left(\{2k\pi/(2n+1)\}\right) \leq \frac{4}{(2n+1)\ln(n+1)}\sum_{k=1}^{n} k\frac{(2n+5)\,(2n+1)^2}{16(n+1)(n+2)}\frac{1}{k^2} = $$

$$= \frac{(2n+1)(2n+5)}{4(n+1)(n+2)\ln(n+1)}\sum_{k=1}^{n}\frac{1}{k}. \tag{14}$$

Combining the inequality ([16])

$$\frac{(2n+1)}{2(n+1)\ln(n+1)}\sum_{k=1}^{n}\frac{1}{k} < \frac{\ln(2n+1)}{\ln(n+1)}, \quad n \geq 2,$$

and (9), (14), we have

$$\lambda_n\left(\{2k\pi/(2n+1)\}\right) \leq \frac{(2n+5)\ln(2n+1)}{2(n+2)\ln(n+1)}, \quad n \geq 2.$$

In particular, from the last inequality we obtain

$$\lambda_n\left(\{2k\pi/(2n+1)\}\right) \leq \frac{9\ln 5}{8\ln 3}, \quad n \geq 2. \tag{15}$$

It follows directly from (9) that

$$\lambda_1\left(\left\{\frac{2k\pi}{3}\right\}\right) = \frac{4}{3\ln 2}\left(1/2 + \frac{1}{3}\cos\frac{2\pi}{3}\right) = \frac{4}{9\ln 2} < \frac{9\ln 5}{8\ln 3}. \tag{16}$$

Next, let $x_0^{(n)} \neq 0$, $0 < x_0^{(n)} \leq \frac{\pi}{2n+1}$ (the case $-\frac{\pi}{2n+1} \leq x_0^{(n)} < 0$ can be seen analogously). Using (8) and (6), we have

$$\lambda_1\left(\left\{x_k^{(1)}\right\}\right) = \frac{2}{3\ln 2}\sum_{k=-1}^{1} f_1\left(x_k^{(1)}\right) K_1^{(2)}\left(x_k^{(1)}\right) =$$

$$= \frac{2}{3\ln 2}\left(K_1^{(2)}\left(x_{-1}^{(1)}\right) + K_1^{(2)}\left(x_0^{(1)}\right) + 2K_1^{(2)}\left(x_1^{(1)}\right)\right) =$$

$$= \frac{2}{3\ln 2}\left(2 + \frac{1}{3}\left(\cos x_{-1}^{(1)} + \cos x_0^{(1)} + 2\cos x_1^{(1)}\right)\right).$$

Taking into account the definition of set $\left\{x_k^{(n)}\right\}$, we have

$$\lambda_1\left(\left\{x_k^{(1)}\right\}\right) = \frac{2}{3\ln 2}\Big(2 + \frac{1}{3}\Big(- 1/2\cos x_0^{(1)} - \frac{\sqrt{2}}{3}\sin x_0^{(1)}\Big)\Big) =$$

$$= \frac{2}{3\ln 2}\Big(2 - \frac{1}{3}\cos\left(x_0^{(1)} - \frac{\pi}{3}\right)\Big).$$

Since $0 < x_0^{(1)} \leq \frac{\pi}{3}$, then

$$\sup_{\{x_k^{(1)}\}\in\mathcal{A}_1} \lambda_1\left(\left\{x_k^{(1)}\right\}\right) = \lambda_1\left(\left\{(2k+1)\frac{\pi}{3}\right\}\right) = \frac{11}{9\ln 2}. \tag{17}$$

We now consider the quantities $\lambda_n(\{x_k^{(n)}\})$ for $n \geq 2$. Using the inequalities

$$\frac{2k\pi}{2n+1} < x_k^{(n)} \leq \frac{(2k+1)\pi}{2n+1}, \tag{18}$$

$$\frac{(2k-1)\pi}{2n+1} \leq \left|x_{-k}^{(n)}\right| < \frac{2k\pi}{2n+1}, \quad k = 1, 2, \ldots, n, \tag{19}$$

equality (3), and, according to the evenness of the function $f_n(x)$, we have

$$\lambda_n\left(\left\{x_k^{(n)}\right\}\right) = \frac{2}{(2n+1)\ln(n+1)}\left(\sum_{k=1}^{n} k\, K_n^{(2)}\left(x_{-k}^{(n)}\right) + \sum_{k=0}^{n}(k+1)\, K_n^{(2)}\left(x_k^{(n)}\right)\right) =$$

$$= \frac{2}{(2n+1)\ln(n+1)}\left(\sum_{k=1}^{n}(k-1)\, K_n^{(2)}\left(x_{-k}^{(n)}\right) + \sum_{k=1}^{n} K_n^{(2)}\left(x_{-k}^{(n)}\right) + \right.$$

$$\left. + \sum_{k=0}^{n} k\, K_n^{(2)}\left(x_k^{(n)}\right) + \sum_{k=0}^{n} K_n^{(2)}\left(x_k^{(n)}\right)\right) =$$

$$= \frac{1}{\ln(n+1)}\left(1 + \frac{2}{2n+1}\sum_{k=2}^{n}(k-1)\, K_n^{(2)}\left(x_{-k}^{(n)}\right) + \sum_{k=1}^{n} k\, K_n^{(2)}\left(x_k^{(n)}\right)\right). \tag{20}$$

Using (12) and Jordan's inequality, we obtain

$$\left|K_n^{(2)}\left(x_k^{(n)}\right)\right| \leq \frac{1}{4(n+1)(n+2)}\left|\frac{2n+3}{\sin^2\left(x_k^{(n)}/2\right)} - \frac{\sin\frac{2n+3}{2}x_k^{(n)}}{\sin^3\left(x_k^{(n)}/2\right)}\right| \leq$$

$$\frac{1}{4(n+1)(n+2)}\left((2n+3)\sin^{-2}\left(x_k^{(n)}/2\right) + \left|\sin^{-3}\left(x_k^{(n)}/2\right)\right|\right) \leq$$

$$\leq \frac{1}{4(n+1)(n+2)}\left((2n+3)\left(\pi/x_k^{(n)}\right)^2 + \left(\pi/x_k^{(n)}\right)^3\right). \tag{21}$$

Combining (18) and (21), we have

$$\left|K_n^{(2)}\left(x_k^{(n)}\right)\right| \leq \frac{1}{4(n+1)(n+2)}\left(\frac{(2n+3)(2n+1)^2}{4k^2} + \frac{(2n+1)^3}{8k^3}\right). \tag{22}$$

Similarly, we have

$$\left|K_n^{(2)}\left(x_{-k}^{(n)}\right)\right| \leq \frac{1}{4(n+1)(n+2)}\left((2n+3)\left(\pi/x_{-k}^{(n)}\right)^2 + \left|\pi/x_{-k}^{(n)}\right|^3\right). \tag{23}$$

From (19) and (23), it follows that

$$\left|K_n^{(2)}\left(x_{-k}^{(n)}\right)\right| \leq \frac{1}{4(n+1)(n+2)}\left(\frac{(2n+3)(2n+1)^2}{(2k-1)^2} + \frac{(2n+1)^3}{(2k-1)^3}\right). \tag{24}$$

Combining (20), (22), (24), we obtain

$$\lambda_n\left(\left\{x_k^{(n)}\right\}\right) \leq \frac{1}{\ln(n+1)}\left(1 + \frac{2}{2n+1}\left(\sum_{k=2}^{n}\frac{k-1}{4(n+1)(n+2)}\left(\frac{(2n+3)(2n+1)^2}{(2k-1)^2} + \right.\right.\right.$$

$$+\frac{(2n+1)^3}{(2k-1)^3}\Bigg) + \sum_{k=1}^{n} \frac{k}{4(n+1)(n+2)} \left(\frac{(2n+3)(2n+1)^2}{4k^2} + \frac{(2n+1)^3}{8k^3} \right) \Bigg) \Bigg) =$$

$$= \frac{1}{\ln(n+1)} \Bigg(1 + \frac{(2n+1)(2n+3)}{8(n+1)(n+2)} \sum_{k=1}^{n} \frac{1}{k} + \frac{(2n+3)(2n+1)}{2(n+1)(n+2)} \sum_{k=2}^{n} \frac{k-1}{(2k-1)^2} +$$

$$+ \frac{(2n+1)^2}{16(n+1)(n+2)} \sum_{k=1}^{n} \frac{1}{k^2} + \frac{(2n+1)^2}{2(n+1)(n+2)} \sum_{k=2}^{n} \frac{k-1}{(2k-1)^3} \Bigg). \tag{25}$$

Let us note estimations:

$$\sum_{k=1}^{n} \frac{1}{k} < \sum_{k=1}^{n} \int_{2k-1}^{2k+1} \frac{dx}{x} = \sum_{k=1}^{n} (\ln(2k+1) - \ln(2k-1)) = \ln(2n+1), \quad n \geq 2;$$

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \sum_{k=n+1}^{\infty} \frac{1}{k^2} < \frac{\pi^2}{6};$$

$$\sum_{k=2}^{n} \frac{k-1}{(2k-1)^2} < \sum_{k=1}^{n-1} \frac{k}{(2k)^2} = \frac{1}{4} \sum_{k=1}^{n-1} \frac{1}{k} < \frac{1}{4} \ln(2n-1), \quad n \geq 3;$$

$$\sum_{k=2}^{n} \frac{k-1}{(2k-1)^3} = \sum_{k=1}^{\infty} \frac{k}{(2k+1)^3} - \sum_{k=n}^{\infty} \frac{k}{(2k+1)^3} < \frac{\pi^2}{16} - \frac{7}{16} \xi(3).$$

Returning to (25), for $n \geq 3$, we have

$$\lambda_n \left(\left\{ x_k^{(n)} \right\} \right) \leq \frac{1}{\ln(n+1)} + \frac{(2n+1)(2n+3)}{8(n+1)(n+2)} \frac{\ln(2n-1) + \ln(2n+1)}{\ln(n+1)} +$$

$$+ \frac{(2n+1)^2}{32(n+1)(n+2)\ln(n+1)} \left(\frac{4\pi^2}{3} - 7\xi(3) \right). \tag{26}$$

Show that the sequence

$$\gamma_n := \frac{1}{\ln(n+1)} + \frac{\ln(2n-1) + \ln(2n+1)}{\ln(n+1)} +$$

$$+ \frac{(2n+1)^2}{32(n+1)(n+2)\ln(n+1)} \left(\frac{4\pi^2}{3} - 7\xi(3) \right) := \varphi_n^{(1)} + \varphi_n^{(2)} + \varphi_n^{(3)}$$

is monotone decreasing for $n \geq 3$. Obviously, the sequences $\varphi_n^{(1)}$ and $\varphi_n^{(3)}$ are decreasing.

Let us prove that the sequence $\varphi_n^{(2)}$ is also decreasing. Consider a function $\varphi(x)$ whose values for $x \geq 3$ at integer points coincide with the elements of the sequence $\varphi_n^{(2)}$. We have

$$\varphi'(x) = \left(\frac{\ln(2x-1)}{\ln(x+1)} \right)' + \left(\frac{\ln(2x+1)}{\ln(x+1)} \right)' =$$

$$= \frac{1}{\ln^2(x+1)} \left(\frac{\ln(x+1)}{2x-1} - \frac{\ln(2x-1)}{x+1} + \frac{\ln(x+1)}{2x+1} - \frac{\ln(2x+1)}{x+1} \right).$$

Since the function $f(x) = x \ln x$ is increasing for $x \geq 3$, then

$$\frac{\ln(x+1)}{2x-1} - \frac{\ln(2x-1)}{x+1} < 0, \quad \frac{\ln(x+1)}{2x-1} - \frac{\ln(2x-1)}{x+1} < 0.$$

The above inequalities imply that $\varphi'(x) < 0$ for $x \geq 3$, and therefore the function $\varphi(x)$ is monotonically decreasing on this interval. Consequently, the sequence $\varphi_n^{(2)}$ (and hence γ_n) also monotonically decreases for $n \geq 3$.

Using inequality

$$\frac{(2n+1)(2n+3)}{8(n+1)(n+2)} \leq \frac{1}{2}, \quad n \in \mathbb{N},$$

for $n \geq 16$ we have

$$\lambda_n\left(\left\{x_k^{(n)}\right\}\right) \leq \varphi_n^{(1)} + 1/2\varphi_n^{(2)} + \varphi_n^{(3)} < \left(\varphi_n^{(1)} + 1/2\varphi_n^{(2)} + \varphi_n^{(3)}\right)_{n=15} < \frac{11}{9\ln 2}. \tag{27}$$

In view of (27), applying inequality (25) for $n = 2, 3, \ldots, 6$ and inequality (26) for $n = 8, 9, \ldots, 15$, we conclude that

$$\lambda_n\left(\left\{x_k^{(n)}\right\}\right) < \frac{11}{9\ln 2}, \quad n \geq 2. \tag{28}$$

Comparing (15)–(17) and (28), we have

$$A^* = \sup_{n\in\mathbb{N}} \sup_{\{x_k^{(n)}\}\in\mathcal{A}_n} \lambda_n\left(\left\{x_k^{(n)}\right\}\right) = \lambda_1\left(\left\{(2k+1)\frac{\pi}{3}\right\}\right) = \frac{11}{9\ln 2}.$$

□

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Received 31.05.2025
Revised 27.11.2025