UDC 517.5

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CONVERGENCE ANALYSIS OF A POWER SERIES BASED ITERATIVE METHOD HAVING SEVENTH ORDER OF CONVERGENCE

C. D. Sreedeep, I. K. Argyros, K. C. Gopika Dinesh, N. Shrestha, M. Argyros. *Convergence analysis of a power series based iterative method having seventh order of convergence*, Mat. Stud. **64** (2025), 179–193.

In this paper, we propose a new three-point iterative scheme for solving nonlinear equations, which achieves seventh-order convergence. The method begins with a standard Newton iteration, followed by two weighted-Newton steps constructed using power series expansions. The present manuscript enhances the order of convergence by integrating divided difference techniques with power series approaches, leading to an efficient and reliable iterative process. The order of convergence has been established rigorously as seven, and the corresponding error equations are derived to validate the theoretical results. A comprehensive convergence analysis is carried out, encompassing both local and semilocal convergence aspects. The local convergence results are obtained under assumptions involving only the first derivative of the operator, and a computable radius of convergence is derived. Moreover, the uniqueness of the solution within this radius is also discussed in detail. For the semilocal analysis, we employ the majorizing sequence technique, which ensures convergence from a wider range of initial approximations. Extensive numerical experiments are performed to demonstrate the validity and accuracy of the proposed method. The calculated results show excellent agreement with the theoretical predictions, confirming the robustness and efficiency of the new algorithm, particularly when compared in terms of the number of iterations and the approximated computational order of convergence.

1. Introduction. The task of solving nonlinear equations (NLEs), typically expressed as

$$F(u) = 0 \qquad (u \in \mathbb{R} \ \text{ or } \ u \in \mathbb{C}). \tag{1}$$

is a central and challenging aspect of scientific computing, engineering analysis, and applied mathematical modelling. Such equations frequently appear in diverse practical applications and often require specialised numerical approaches for effective resolution. This difficulty arises because researchers across various disciplines often model real-world phenomena using NLEs to gain deeper insight into their behaviour. Numerous examples of real-life situations modelled as nonlinear equations can be found throughout the literature [3, 4, 17, 18, 23].

Finding exact solutions to these nonlinear equations is generally challenging. Consequently, iterative methods are frequently employed to approximate solutions. Among the various iterative methods, Newton's iterative method (NIM, [4, 5, 11, 13, 19–22]) is widely favoured

2020 *Mathematics Subject Classification:* 65H05, 47H99, 65D99, 49M15, 65G99.
Keywords: power series iteration; seventh order of convergence; local convergence; semilocal convergence.
doi:10.30970/ms.64.2.179-193

due to its quadratic convergence, making it a popular choice for solving equation (1). In recent years, significant progress has been made in science and mathematics, leading to the development and application of several advanced iterative techniques for solving nonlinear equations [2, 5, 15, 19–21]. These methods rely on recursion formulas that iteratively refine the solution u^* improving its accuracy with each iteration until the exact or desired solution is reached.

Recent advancements have further led to the discovery and utilisation of various sophisticated iterative techniques for solving nonlinear equations. For example, Chun [10] introduced a power series like expression into the second step of a modified NIM to develop a family of two-point iterative methods (TPIM) with convergence order four. Similarly, Khattri and Abbasbandy ([14]) used a different power series function in the second step of the modified NIM to propose a new class of TPIM. The dynamic behaviour of the generalised methods presented by Khattri and Abbasbandy ([14]) was later studied by Babajee and Khattri ([7]). Further developments include the paper of Ahmad [1], Babajee [8], and Madhu [16], where power series of varying forms were utilised as weight functions in modifying the NIM to construct iterative methods.

Building upon these foundational works, the present manuscript enhances the order of convergence by integrating divided difference techniques with power series approaches. In this study, we propose and analyse a specific three-step iterative method, defined as follows:

$$v_n = u_n - N(u_n), \quad w_n = u_n - N(u_n)\left[\frac{1+[\sum_{i=1}^{k} a_i(p_n)^i]}{1+[\sum_{i=1}^{k} b_i(p_n)^i]}\right],$$
$$u_{n+1} = w_n - \frac{F(w_n)}{[u_n, w_n; F] + [v_n, w_n; F] - [v_n, u_n; F]}, \tag{2}$$

where $N(u_n) = \frac{F(u_n)}{F'(u_n)}$, $p_n = \frac{F(v_n)}{F(u_n)}$ and a_i and b_i are real constants which we will give value later (see Theorem 1) and $[u, v; F]$ is the first order divided difference defined by $[u, v; F](u - v) \approx F(u) - F(v)$ [4]. It is important to note that the first step of this iteration corresponds to the classical NIM, which serves as the foundation for the subsequent improvements.

The structure of this paper is as follows: Section 2 delves into the method (2) and its convergence analysis. Section 3 gives the some challenges associated with our convergence analysis on Section 2 and tries to overcome these challenges by providing Local and Semi-local Convergence without using Taylor series expansion. We have included some numerical examples for validating our results in Section 2 and Section 3 and finally Section 4 concludes with key findings and remarks.

2. Convergence analysis. We begin this section by proving the convergence of the method described in equation (2) through the formal Theorem 1.

Theorem 1. *Suppose $F\colon D \subset \mathbb{R} \to \mathbb{R}$ is a real valued function that is be sufficiently differentiable in the domain D, such that $u^* \in D$ and $|F'(\cdot)| \neq 0$ in D where u^* is a simple root of F. If u_0 is close to u^* and $k = 3$ then the sequence $(u_j)_{j\geq 0}$ of approximations in $\mathbb{R}$, generated by class of iterative method (2) converges to u^* with convergence order seven when the free parameters a_i and b_i follows the condition $a_1 - b_1 - 1 = 0$, $a_2 - 2 - b_1 - b_2 = 0$, $b_2 + 5 - a_3 + 2b_1 + b_3 = 0$.*

Proof. Let u^* be a simple zero of F and u_n be the n^{th} approximation of u^*, defined by (2). Expanding $F(u_n)$ and $F'(u_n)$ with the help of Taylor's series,

$$F(u_n) = F'(u^*)\big(d_n + c_2 d_n^2 + c_3 d_n^3 + c_4 d_n^4 + c_5 d_n^5 + c_6 d_n^6 + c_7 d_n^7 + c_8 d_n^8\big), \tag{3}$$

$$F'(u_n) = F'(u^*)\big(1+2c_2d_n+3c_3d_n^2+4c_4d_n^3+5c_5d_n^4+6c_6d_n^5+7c_7d_n^6+8c_8d_n^7+9c_9d_n^8+O[d_n^9]\big), \quad (4)$$

where $d_n = u_n - u^*, c_i = \frac{F^{(i)}(u^*)}{F'(u^*)i!}$ and $F(u^*) = 0$.

From equation (3) and (4) we get v_n as

$$v_n = u^* + c_2d_n^2 + (-2c_2^2 + 2c_3)d_n^3 + \ldots + (64c_2^7 - 304c_2^5 + c_3 + 176c_2^4c_4 + 75c_3^2c_4+$$
$$+c_2^3(408c_3^2 - 92c_5) - 31c_4c_5 + 6c_6 + \ldots + 7c_8)d_n^8 + O[d_n]^9.$$

Taylor series expansion of $F(v_n)$ is obtained as

$$F(v_n) = c_2d_n^2 + (-2c_2^2 + 2c_3)d_n^3 + \ldots + (144c_2^7 - 552c_2^5c_3 + \ldots$$
$$\ldots + c_2(-147c_3^3 + 134c_3c_5 - 19c_7) + 7c_8)d_n^8 + O[d_n]^9.$$

Substituting $F(v_n)$, $F(u_n)$, $F'(u_n)$ in (2)

$$w_n = u^* + (c_2 - a_1c_2 + b_1c_2)d_n^2 + (-(2 + a_2 + 4b_1 + b_1^2 - a_1(4 + b_1) - b_2)c_2^2-$$
$$-2(-1 + a_1 - b_1)c_3)d_n^3 + \ldots$$
$$\ldots + (-(-64 + 1289a_3 - 688b_1 + 590a_3b_1 + \cdots + 7c_8 - 7a_1c_8 + 7b_1c_8))d_n^8 + O[d_n]^9.$$

Now substituting above expansion of w_n in (2) and we will obtain

$$u_{n+1} = u^* + (1 - a_1 + b_1)^2c_2^3d_n^4-$$
$$-(-1 + a_1 - b_1)c_2^2(-2(2 + a_2 + 4b_1 + b_1^2 - a_1(4 + b_1) - b_2)c_2^2 + (4 - 5a_1 + 5b_1)c_3)d_n^5 + \ldots$$
$$\ldots + (-2(10 - 12a_3 + 64b_1 - 15a_3b_1 + \ldots - a_1(17 + 18b_1))c_5))d_n^7 + O[d_n]^8.$$

From the above expansion in order to achieve maximum possible order of convergence the coefficients of d_n^4, d_n^5 and d_n^6 must vanish. To achieve this, the following system of equations must be satisfied

$$a_1 = 1 + b_1, \quad a_2 = 2 + b_1 + b_2, \quad a_3 = 5 + b_3 + 2b_1 + b_2. \quad (5)$$

Substituting (5) in u_{n+1} we will obtain

$$u_{n+1} = u^* + c_2^2c_3^2d_n^7 + O[d_n]^8. \quad (6)$$

Thus the iterative scheme defined in (2) is of order seven. □

3. Local and semilocal convergence without Taylor series expansion. The Taylor expansion series approach of iterative methods are frequently employed to show the convergence of iterative methods and consequently of (2) has certain drawbacks which limit their applicability. We begin by listing each problem individually, followed by our corresponding solutions for each.

The problems $(\mathcal{P}_1) - (\mathcal{P}_6)$ are addressed as follows:

$(\mathcal{P}_1)$ According to Theorem 1 the function must be at least nine times differentiable and u^* must be a simple solution of the equation $F(u) = 0$.

Let us consider the toy example given for say $D = [-2, 2]$ and function $F\colon D \to (-\infty, \infty)$ defined by

$$F(u) = \begin{cases} \alpha_1u^{10}\log(u) + \alpha_2u^{11} + \alpha_3u^{12}, & u \neq 0;\\ 0, & u = 0, \end{cases}$$

where $\alpha_1 \neq 0$ and $\alpha_2 + \alpha_3 = 0$. It follows by the definition that $u^* = 1 \in D$ solves the equation $F(u) = 0$. But the ninth derivative of the function F is not continuous at $u = 0 \in D$. So, the results of Theorem 1 cannot assure the convergence of $\{u_n\}$ to u^*=1. However, the sequence $\{u_n\}$ converges to $u^* = 1$ if for example $u_0 = 0.9 \in D$, $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = -1$. It is concluded by this observation that the conditions of Theorem 1 can be weakened.

($\mathcal{P}_2$) The selection of u_0 is a «shot in the dark» since the radius of convergence for the method u_n is not given in Theorem 1.

($\mathcal{P}_3$) Let $\epsilon > 0$ denote the desired error tolerance. Then the minimum number K of iterations to be carried out such that $\|u_n - u^*\| < \epsilon$ for each $n \geq K$ is not available in Theorem 1 due to the lack of computable priori estimates on $\|u_0 - u^*\|$.

($\mathcal{P}_4$) The isolation of u^* in a neighbourhood containing is not discussed.

($\mathcal{P}_5$) The results are restricted to hold on the real line.

($\mathcal{P}_6$) The most challenging and important semilocal convergence analysis is not provided.

We next present the solutions corresponding to each of the problems outlined above. However, first let $S = \mathbb{R}$ or $S = \mathbb{C}$. Consider solving the equation $F(u) = 0$ in the more general setting where $F\colon D \subset S \to S$, where D is an open and convex subset of S. Then, the method (2) can be defined as

$$v_n = u_n - F'(u_n)^{-1}F(u_n),\ w_n = u_n - B_n^{-1}A_nF'(u_n)^{-1}F(u_n),\ u_{n+1} = w_n - C_n^{-1}F(w_n),\quad (7)$$

where $P_n = \frac{F(v_n)}{F(u_n)}$, $A_n = 1 + \sum_{i=1}^{3} a_iP_n^i$, $B_n = 1 + \sum_{i=1}^{3} b_iP_n^i$, $a_1 - b_1 = 1$, $a_2 - b_2 = b_1 + 2$, $a_3 - b_3 = 2b_1 + b_2 + 5$ and $C_n = [u_n, w_n; F] + [v_n, w_n; F] - [v_n, u_n; F]$.

($\mathcal{S}_1$) The local convergence is shown using only the functions on the method (7) F, F' and $[\cdot, \cdot; F]$.

($\mathcal{S}_2$) A computable radius of convergence becomes available. So u_0 is picked from a certain ball about u^*.

($\mathcal{S}_3$) A priori computable error bounds on $\|u^* - u_n\|$ are provided, so K is known in advance (see Theorem 2).

($\mathcal{S}_4$) The uniqueness of u^* in a certain domain is discussed.

($\mathcal{S}_5$) The results are provided on the more general setting of the complex plane S.

($\mathcal{S}_6$) The semilocal convergence analysis of the sequence u_n is presented using majorizing sequence ([3, 5, 6, 17, 22]).

The items ($\mathcal{S}_1$)–($\mathcal{S}_6$) constitute the novelty of this section.

3.1. Local convergence. Let us introduce some functions which play a vital role in showing the local convergence of method (7). Let $L(D, S)$ denote set of all linear operators from D to S and $E(u, \alpha)$ denote the open ball centered at $u \in D$ and $\alpha > 0$. Let T=[0,+∞).

The following assumptions are used for local convergence analysis:

($\mathcal{H}_1$) There exists a continuous and nondecreasing function $f_0\colon T \to T$ such that the function $1 - f_0(t)$ has a smallest positive zero in the interval T which is denoted by s_0. Set $T_0 = [0, s_0)$.

($\mathcal{H}_2$) There exists a continuous and nondecreasing function $f\colon T_0 \to T$ such that for $g_1\colon T_0 \to T$ defined by

$$g_1(t) = \frac{\int_0^1 f((1-\theta)t)d\theta}{1-f_0(t)},$$

the function $1-g_1(t)$ has a smallest positive zero in the interval T_0 which is denoted by μ_1.

($\mathcal{H}_3$) For functions $h^{(1)}\colon T_0 \to T$, $h^{(2)}\colon T_0 \to T$, $\bar{P}\colon T_0 \to T$ and $\phi^{(1)}\colon T_0 \to T$ defined by

$$h^{(1)}(t) = \int_0^1 f_0(\theta t)d\theta, \quad \bar{P}(t) = \frac{(1+\int_0^1 f_0(\theta g_1(t)t)d\theta)g_1(t)}{1-h^{(1)}}, \quad h^{(2)}(t) = \sum_{i=1}^{3} |b_i|\bar{P}(t)^i$$

and

$$\phi^{(1)}(t) = (1+|b_1+2|\bar{P}(t)+|2b_1+b_2+5|\bar{P}^2(t))\bar{P}(t)$$

the functions $1-h^{(1)}(t)$ and $1-h^{(2)}(t)$ have smallest positive zeros in the interval T_0 which are denoted by s_1, s_2 respectively . Set $\bar{s} = \min\{s_1, s_2\}$ and $T_1 = [0, \bar{s})$.

($\mathcal{H}_4$) Define $g_2\colon T_1 \to T$ defined by

$$g_2(t) = \frac{\int_0^1 f((1-\theta)t)d\theta}{1-f_0(t)} + \frac{\phi^{(1)}(t)(1+\int_0^1 f_0(\theta t)d\theta)}{(1-f_0(t))(1-h^{(2)}(t))},$$

such that the function $1-g_2(t)$ has a smallest positive zero in the interval T_1 which is denoted by μ_2.

($\mathcal{H}_5$) There exists continuous and nondecreasing symmetric functions $q_0\colon T_1\times T_1 \to T$, $q_1\colon T_1\times T_1 \to T$ and $q_2\colon T_1\times T_1\times T_1 \to T$ such that functions $1-f_0(g_2(t)t)$, $1-h^{(3)}(t)$, where $h^{(3)}(t) = q_0(t, g_2(t)t) + q_2(t, g_1(t)t), g_2(t)t)$ have smallest positive zeros in the interval T_1, which are denoted by s_3, s_4 respectively. Set $s = \min\{s_3, s_4\}$ and $T_2 = [0, s)$. Define the function

$$\phi^{(2)}(t) = q_1(t, g_2(t)t) + q_2(t, g_1(t)t, g_2(t)t)$$

and

$$g_3(t) = \left(\frac{\int_0^1 f((1-\theta)g_2(t)t)d\theta}{1-f_0(g_2(t)t)} + \frac{\phi^{(2)}(t)(1+\int_0^1 f_0(\theta g_2(t)t)d\theta)}{(1-f_0(g_2(t)t))(1-h^{(3)}(t))}\right) g_2(t).$$

($\mathcal{H}_6$) The function $1-g_3(t)$ has a smallest positive zero in the interval T_2 which is denoted by μ_3.

Define the parameter μ and interval T^* by

$$\mu = \min\{\mu_m\}, \quad m = 1,2,3 \quad \text{and } T^* = [0, \mu). \tag{8}$$

The parameter μ is shown to be a radius of convergence for the method (7) in Theorem 2.

It follows by these definition that for each $t \in T^*$

$$0 \le f_0(t) < 1, \tag{9}$$

$$0 \le h^{(1)}(t) < 1, \tag{10}$$

$$0 \le h^{(2)}(t) < 1, \tag{11}$$

$$0 \le h^{(3)}(t) < 1, \tag{12}$$
$$0 \le f_0(g_2(t)t) < 1 \tag{13}$$

and

$$0 \le g_m(t) < 1. \tag{14}$$

Let us relate the functions f_0, f, q_0, q_1 and q_2 to the ones on the method (7).

($\mathcal{H}_7$) There exists a solution $u^* \in D$ and a linear operator $M \in L(S,S)$ which is invertible such that for each $w \in D$

$$\|M^{-1}(F'(w) - M)\| \le f_0(\|w - u^*\|).$$

Set $D_0 = E(u^*, s_0) \cap D$.

($\mathcal{H}_8$)

$$\|M^{-1}(F'(v) - F'(w))\| \le f(\|v - w\|), \quad w, v \in D_0,$$
$$\|M^{-1}([u, w; F] - M)\| \le q_0(\|u - u^*\|, \|\|w - u^*\|), u, w \in D_0,$$
$$\|M^{-1}([u, w; F] - F'(w))\| \le q_1(\|u - u^*\|, \|w - u^*\|),$$
$$\|M^{-1}([v, w; F] - [v, u; F])\| \le q_2(\|u - u^*\|, \|v - u^*\|, \|w - u^*\|), u, v, w \in D_0.$$

($\mathcal{H}_9$) $E(u^*, \mu) \subseteq D$.

Remark 1. Possible choices for $M = I$ or $M = F'(\bar{u})$, where $\bar{u} \in D$ is an auxiliary point other than u^* or $M = F'(u^*)$. Under the last case it follows that u^* is a simple solution of the equation $F(x) = 0$. But notice however that such assumption is not made or implied by the conditions ($\mathcal{H}_1$)–($\mathcal{H}_9$).

The main results follows for the local convergence analysis of method (7) based on the conditions ($\mathcal{H}_1$)–($\mathcal{H}_9$). Set $E_0 = E(u^*, \mu) - \{u^*\}$.

Theorem 2. *Suppose that the conditions ($\mathcal{H}_1$)–($\mathcal{H}_9$) hold and the initial point $u_0 \in E_0$. Then, the following assertions hold for sequence $\{u_n\}$ generated by the method (7)*

$$\{u_n\} \subset E(u^*, \mu), \tag{15}$$
$$\|v_n - u^*\| \le g_1(\|u_n - u^*\|)\|u_n - u^*\| \le \|u_n - u^*\| < r, \tag{16}$$
$$\|w_n - u^*\| \le g_2(\|u_n - u^*\|)\|u_n - u^*\| \le \|u_n - u^*\|, \tag{17}$$
$$\|u_{n+1} - u^*\| \le g_3(\|u_n - u^*\|)\|u_n - u^*\| \le \|u_n - u^*\| \tag{18}$$

and the sequence $\{u_n\}$ is convergent to u^* so that

$$\|u_n - u^*\| \le c^n \|u_0 - u^*\|, \tag{19}$$

where $c = g_3(\|u_0 - u^*\|) \in [0, 1)$.

Proof. Assertions (15)–(18) are established using induction on j. Assertion (15) holds if $j = 0$, since by hypothesis $u_0 \in E_0 \subseteq E$. Let $w \in E_0$ be an arbitrary number. Then, it follows by conditions ($\mathcal{H}_1$), ($\mathcal{H}_7$), (8) and (9) that

$$\|M^{-1}(F'(w) - M)\| \le f_0(\|w - u^*\|) \le f_0(\mu) < 1. \tag{20}$$

By (20) and Banach lemma on invertible functions ([3, 4, 17]) we conclude that $F'(w)^{-1}$ exists and

$$\|F'(w)^{-1}M\| \leq \frac{1}{1 - f_0(\|w - u^*\|)}. \tag{21}$$

In particular, estimate (21) holds if $w = u_0$. So, the first iterate v_0 exists by the first substep of the method (8) and we can write in turn

$$v_0 - u^* = u_0 - u^* - F'(u_0)^{-1}F(u_0) =$$
$$= \left(F'(u_0)^{-1}M\right)\left(M^{-1}\int_0^1 \left[F'(u_0) - F'(u^* + \theta(u_0 - u^*))\right] d\theta\right)(u_0 - u^*).$$

We can use the conditions $(\mathcal{H}_8)$, (8), (14) for $m = 3$, (20) for $w = u_0$ in the identity (21) to obtain

$$\|v_0 - u^*\| \leq \frac{\int_0^1 f((1-\theta)\|u_0 - u^*\|)d\theta\|u_0 - u^*\|}{1 - f_0(\|u_0 - u^*\|)} \leq g_1(\|u_0 - u^*\|)\|u_0 - u^*\| \leq$$
$$\leq \|u_0 - u^*\| < \mu. \tag{22}$$

Thus, the iterate $v_0 \in E$ and the assertion (17) holds if $j = 0$.

We suppose $u_0 \neq u^*$, since otherwise we have found the solution. Then, by (8), (9) and $(\mathcal{H}_7)$ we have

$$\left\|(M(u_0 - u^*))^{-1}\left[F(u_0) - F(u^*) - M(u_0 - u^*)\right]\right\| \leq \int_0^1 f_0\left(\theta\|u_0 - u^*\|\right) d\theta = h_0^{(1)} < 1, \tag{23}$$

where we also used the estimates

$$F(u_0) = F(u_0) - F(u^*) = \int_0^1 F'(u^* + \theta(u_0 - u^*))d\theta(u_0 - u^*).$$

Hence, by (23) and since $F(u_0) \neq 0$,

$$\|F(u_0)^{-1}M\| \leq \frac{1}{\|u_0 - u^*\|(1 - h_0^{(1)})}$$

and B_0 is well defined. Moreover, we can write by (8) and (11)

$$\|B_0 - I\| = \left\|\sum_{i=1}^{3} b_i P_0^i\right\| \leq \sum_{i=1}^{3} |b_i|\bar{P}^i = h_0^{(2)} < 1. \tag{24}$$

So,

$$\|B_0\|^{-1} \leq \frac{1}{1 - h_0^{(2)}}, \tag{25}$$

where, we have used

$$F(v_0) = F(v_0) - F(u^*) = \int_0^1 F'(u^* + \theta(v_0 - u^*))d\theta(v_0 - u^*)$$

and

$$\|M^{-1}F(v_0)\| = \left\|M^{-1}\int_0^1 (F'(u^*+\theta(v_0-u^*)) - M + M)\,d\theta(v_0-u^*)\right\| \leq$$
$$\leq \left(1 + \left\|\int_0^1 M^{-1}(F'(u^*+\theta(v_0-u^*)) - M)d\theta\right\|\right)\|v_0-u^*\| \leq$$
$$\leq \left(1 + \int_0^1 f_0(\theta\|v_0-u^*\|)d\theta\right)\|v_0-u^*\|, \tag{26}$$

to obtain by (25). Also we have used

$$\|P_0\| = \frac{\|M^{-1}F(v_0)\|}{\|M^{-1}F(u_0)\|} \leq \frac{(1+\int_0^1 f_0(\theta\|v_0-u^*\|)d\theta)\|v_0-u^*\|}{\|u_0-u^*\|(1-h_0^{(1)})} \leq$$
$$\leq \frac{(1+\int_0^1 f_0(\theta g_1(\|u_0-u^*\|)\|u_0-u^*\|)d\theta)g_1(\|u_0-u^*\|)\|u_0-u^*\|}{\|u_0-u^*\|(1-h_0^{(1)})} = \bar{P}_0. \tag{27}$$

Furthermore, by the definition of A_0 and B_0, we have in turn

$$B_0 - A_0 = \sum_{i=1}^{3}(b_i - a_i)P_0^i = -P_0 - (b_1+2)P_0^2 - (2b_1+b_2+5)P_0^3,$$

so by (27)

$$\|B_0 - A_0\| \leq (1 + |b_1+2|\bar{P}_0 + |2b_1+b_2+5|\bar{P}_0^{\,2})\bar{P}_0 = \phi_0^{(1)}.$$

It follows by (21) (for $w = u_0$), (24) and (25) that iterate w_0 is well defined by the second substep of the method (8) and we can write in turn

$$w_0 - u^* = u_0 - u^* - F(u_0)^{-1}F(u_0) + (I - B_0^{-1}A_0)F'(u_0)^{-1}F(u_0) =$$
$$= u_0 - u^* - F'(u_0)^{-1}F(u_0) + B_0^{-1}(B_0 - A_0)F'(u_0)^{-1}F(u_0). \tag{28}$$

In view of (8), (14) (for $m = 2$), (21) (for $w = u_0$), (22) and (24)–(28) identity (28) can give

$$\|w_0 - u^*\| \leq \left[\frac{\int_0^1 f((1-\theta)\|u_0-u^*\|)\,d\theta}{1 - f_0(\|u_0-u^*\|)} + \frac{\phi_0^{(1)}\left(1+\int_0^1 f_0(\theta\|u_0-u^*\|)\,d\theta\right)}{(1-h_0^{(2)})(1-f_0(\|u_0-u^*\|))}\right]\|u_0-u^*\| \leq$$
$$\leq g_2(\|u_0-u^*\|)\|u_0-u^*\| \leq \|u_0-u^*\|.$$

Thus, the iterate $w_0 \in E_0$ and the assertion (18) holds if $j = 0$.

Next, we need to show that C_0^{-1} exists. It follows by (8), (12), (H_8) and the definition of C_0 that

$$\|M^{-1}(C_0 - M)\| \leq \|M^{-1}([u_0, w_0; F] - M)\| + \|M^{-1}([v_0, w_0; F] - [v_0, u_0; F])\| \leq$$
$$\leq q_0(\|u_0-u^*\|, \|w_0-u^*\|) + q_2(\|u_0-u^*\|, \|v_0-u^*\|, \|w_0-u^*\|) \leq h_0^{(3)} < 1.$$

So, C_0^{-1} exists and

$$\|C_0^{-1}M\| \leq \frac{1}{1-h_0^{(3)}}. \tag{29}$$

Consequently, the iterate u_1 exists by the third substep of the method (2) and we can write in turn that

$$u_1 - u^* = w_0 - u^* - F'(w_0)^{-1}F(w_0) + (F'(w_0)^{-1} - C_0^{-1})F(w_0) =$$
$$= w_0 - u^* - F'(w_0)^{-1}F(w_0) + F'(w_0)^{-1}(C_0 - F'(w_0))C_0^{-1}F(w_0). \tag{30}$$

We need some estimates for

$$C_0 - F'(w_0) = ([u_0, w_0; F] - F'(w_0)) + ([v_0, w_0; F] - [v_0, u_0; F]),$$

which can give

$$\|M^{-1}(C_0 - F'(w_0))\| \leq q_1(\|u_0 - u^*\|, \|w_0 - u^*\|) + q_2(\|u_0 - u^*\|, \|v_0 - u^*\|, \|w_0 - u^*\|) = \phi_0^{(2)},$$

as well as

$$F(w_0) = F(w_0) - F(u^*) = \int_0^1 F'(u^* + \theta(w_0 - u^*))d\theta(w_0 - u^*),$$

implying, as in (26)(for $v_0 = w_0$)

$$\|M^{-1}(F(w_0))\| \leq (1 + \int_0^1 f_0(\theta\|w_0 - u^*\|)d\theta)\|w_0 - u^*\|. \tag{31}$$

It follows by (8), (14) (for $m = 3$), (21) (for $w = w_0$), (22)(for $v_0 = u_0$) (29) and (31) that (30) can also give

$$\|u_1 - u^*\| \leq \frac{\int_0^1 f((1-\theta)\|w_0 - u^*\|)\, d\theta}{1 - f_0(\|w_0 - u^*\|)}\|w_0 - u^*\| +$$
$$+\frac{\phi_0^{(2)}\left(1 + \int_0^1 f_0(\theta\|w_0 - u^*\|)\, d\theta\right)}{(1 - f_0(\|w_0 - u^*\|))(1 - h_0^{(3)})}\|w_0 - u^*\| \leq g_3(\|u_0 - u^*\|)\|u_0 - u^*\| \leq \|u_0 - u^*\|.$$

Hence, the iterate $u_1 \in E$ and the assertion (18) holds if $j = 0$. Furthermore, simply exchange u_0, v_0, w_0, u_1 by u_n, v_n, w_n, u_{n+1} respectively in the preceding calculations to complete the induction for the assertions (16)–(18). Then, assertion (18) also implies (19) from which it follows that $\lim_{n\to\infty} u_n = u^*$. □

A domain is established next that contains only u^* as a solution of the equation $F(x) = 0$.

Proposition 1. *Suppose the condition ($\mathcal{H}_7$) holds in the ball $E(u^*, d_1)$ for some $d_1 > 0$ and there exists $d_2 \geq d_1$ such that*

$$\int_0^1 f_0(\theta d_2)d\theta < 1. \tag{32}$$

Define the domain $D_1 = E[u^*, \mu] \cap D$. Then, u^* is the only solution of the equation $F(x) = 0$ in the domain D_1.

Proof. Suppose there exists $w^* \in D_1$ solving the equation $F(u) = 0$ such that $w^* \neq u^*$. Define the linear function $L = \int_0^1 F'(u^* + \theta(w^* - u^*))d\theta$. Then, the application of the condition ($\mathcal{H}_7$) and (32) can give $\|M^{-1}(L - M)\| \leq \int_0^1 f_0(\theta\|w^* - u^*\|)d\theta \leq \int_0^1 f_0(\theta d_2)d\theta < 1$. Consequently, L^{-1} exists and from the identity $w^* - u^* = L^{-1}(F(w^*) - F(u^*)) = L^{-1}(0) = 0$, we deduce that $w^* = u^*$. □

Remark 2. 1) The radius μ in $(\mathcal{H}_9)$ can be replaced by s_0 given in $(\mathcal{H}_1)$.

2) Under all conditions $(\mathcal{H}_1)$–$(\mathcal{H}_9)$, we can set $d_1 = \mu$ in the Proposition 1.

3.2. Semi local convergence. The formulas and calculations are the same as in the local analysis of the method (7). But the role of u^*, f_0, f, q_0 and q_2 is exchanged by $u_0, \tilde{f}_0, \tilde{f}, \tilde{q}_0$ and $\tilde{q}_2$, respectively. The following assumptions are used for semilocal convergence analysis

$(\mathcal{C}_1)$ There exists a continuous and nondecreasing function $\tilde{f}_0 \colon T \to T$ such that the function $1 - \tilde{f}_0(t)$ has a smallest positive zero in the interval T which is denoted by $\tilde{\delta}$. Set $T_3 = [0, \tilde{\delta})$.

$(\mathcal{C}_2)$ There exists continuous and nondecreasing functions $\tilde{f} \colon T_3 \to T$, $\tilde{q}_0 \colon T_3 \times T_3 \to T$ and $\tilde{q}_2 \colon T_3 \times T_3 \times T_3 \to T$.

Define the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ for $\alpha_0 = 0$, some $\beta_0 \geq 0$ and each $n \in \{0, 1, 2, \ldots\}$ by

$$\tilde{P}_n = \frac{\int_0^1 \tilde{f}(\theta(\beta_n - \alpha_n))d\theta}{(1 - \tilde{f}_0(\alpha_n))}, \quad \tilde{h}_n^{(1)} = \sum_{i=1}^{3} |b_i| \tilde{P}_n^i,$$

$$\psi_n = (1 + |b_1 + 2|\tilde{P}_n + |2b_1 + b_2 + 5|\tilde{P}_n^2)\tilde{P}_n, \quad \gamma_n = \beta_n + \frac{\psi_n(\beta_n - \alpha_n)}{1 - \tilde{h}_n^{(1)}},$$

$$\lambda_n = \int_0^1 \tilde{f}((\theta)(\gamma_n - \alpha_n))d\theta(\gamma_n - \alpha_n) + (1 + \tilde{f}_0(\alpha_n))(\gamma_n - \beta_n),$$

$$\tilde{h}_n^{(2)} = \tilde{q}_0(\alpha_n, \gamma_n) + \tilde{q}_2(\alpha_n, \beta_n, \gamma_n), \quad \alpha_{n+1} = \gamma_n + \frac{\lambda_n}{1 - \tilde{h}_n^{(2)}},$$

$$\mu_{n+1} = \int_0^1 \tilde{f}((\theta)(\alpha_{n+1} - \alpha_n))d\theta(\alpha_{n+1} - \alpha_n) + (1 + \tilde{f}_0(\alpha_n))(\alpha_{n+1} - \beta_n)$$

and

$$\beta_{n+1} = \alpha_{n+1} + \frac{\mu_{n+1}}{1 - \tilde{f}_0(\alpha_{n+1})}.$$

It is shown in Theorem 3 that the sequence α_n is majorizing for $\{u_n\}$. But first a general convergence condition for the sequence $\{\alpha_n\}$ is needed.

$(\mathcal{C}_3)$ There exists $\delta \in [0, \tilde{\delta})$ such that for each $n = 0, 1, 2, \ldots$. $\tilde{h}_n^{(1)} < 1, \tilde{h}_n^{(2)} < 1, \tilde{f}_0(\alpha_n) < 1$ and $\alpha_n \leq \delta$. This condition and the definition of the sequence $\{\alpha_n\}$ imply by induction that $0 \leq \alpha_n \leq \beta_n \leq \gamma_n \leq \alpha_{n+1} < \delta$ and there exists $\alpha^* \in [0, \delta]$ such that

$$\lim_{n \to \infty} \alpha_n = \alpha^*.$$

It is known that α^* is the unique least upper bound of the sequence $\{\alpha_n\}$. As in the local analysis $\tilde{f}_0, \tilde{f}, \tilde{q}_0$ and $\tilde{q}_2$ relate to the functions on the method (7).

$(\mathcal{C}_4)$ There exists $u_0 \in D$ and an invertible linear function M such that for each $u \in D$ $\|M^{-1}(F'(u) - M)\| \leq \tilde{f}_0(\|u - u_0\|)$. It follows by this condition that if $u = u_0$, $\|M^{-1}(F'(u) - M)\| \leq \tilde{f}_0(0) < 1$. So, $F'(u_0)^{-1}$ exists and we can take

$$\beta_0 \geq \|F'(u_0)^{-1}F(u_0)\|.$$

$(\mathcal{C}_5)$ Define the domain $D_2 = E(u_0, \delta) \cap D$ and for each $w, v \in D_2$

$$\|M^{-1}(F'(v) - F'(u))\| \leq \tilde{f}(\|v - u\|),$$

$$\|M^{-1}([u, w; F] - M)\| \leq \tilde{q}_0(\|u - u_0\|, \|w - u_0\|),$$

$$\|M^{-1}([v, w; F] - [v, u; F])\| \leq \tilde{q}_2(\|u - u_0\|, \|v - u_0\|, \|w - u_0\|)$$

and

$(\mathcal{C}_6)$ $E[u_0, \alpha^*] \subseteq D_0$.

Remark 3. As in the local analysis , one can choose $M = I$ or $M = F'(\tilde{u})$ for some auxilary point $\tilde{u} \in D$ with $\tilde{u} \neq u_0$ or $M = F'(u_0)$ or $M = [\bar{u}, \tilde{u}; F]$ for $\bar{u}, \tilde{u} \in D$ or some other choice.

The main semilocal analysis of this subsection follows for the method (7).

Theorem 3. *Suppose that the conditions* $(\mathcal{C}_1)$–$(\mathcal{C}_6)$ *hold. Then, the sequence* $\{u_n\}$ *generated by the method (7) converges to a solution* $u^* \in E(u_0, \alpha^*)$ *of the equation* $F(x) = 0$*. Moreover, the following assertion holds for each* $n = 0, 1, 2, \ldots$

$$\|u^* - u_n\| \leq \alpha^* - \alpha_n. \tag{33}$$

Proof. The following assertions are established using induction on j

$$\{u_j\} \subset E(u_0, \alpha^*), \tag{34}$$

$$\|v_j - u_j\| \leq \beta_j - \alpha_j, \tag{35}$$

$$\|w_j - v_j\| \leq \gamma_j - \beta_j \tag{36}$$

and

$$\|u_{j+1} - w_j\| \leq \alpha_{j+1} - \gamma_j. \tag{37}$$

Assertion (34) holds for $j = 0$, since $u_0 \in E(u_0, \alpha^*)$. By the definition of the parameter β_0 in the condition $\mathcal{C}_4$, (8), and the first substep of the method (7), we have

$$\|v_0 - u_0\| = \|F'(u_0)^{-1}F(u_0)\| \leq \beta_0 - \alpha_0 = \beta_0 < \alpha^*.$$

Thus, $v_0 \in E(u_0, \alpha^*)$, and assertion (35) holds for $n = 0$. We need some estimates

$$F(v_j) = F(v_j) - F(u_j) - F'(u_j)(v_j - u_j) = \int_0^1 (F'(u_j + \theta(v_j - u_j)) - F'(u_j))\, d\theta (v_j - u_j),$$

$$\|M^{-1}F(v_j)\| \leq \int_0^1 \tilde{f}\,(\theta\|v_j - u_j\|)\, d\theta \|v_j - u_j\| \leq \int_0^1 \tilde{f}\,(\theta(\beta_j - \alpha_j))\, d\theta(\beta_j - \alpha_j) = \lambda_j$$

and

$$\begin{aligned} &\|(M(v_j - u_j))^{-1}(F(u_j) - M(v_j - u_j))\| = \\ &= \|(M(v_j - u_j))^{-1}(F'(u_j)(v_j - u_j) - M(v_j - u_j))\| \leq \tilde{f}_0(\alpha_j). \end{aligned}$$

By Banach lemma on invertible operators

$$\|F(u_j)^{-1}M\| \leq \frac{1}{(1 - \tilde{f}_0(\alpha_j))(\beta_j - \alpha_j)}.$$

So,

$$\|P_j\| = \frac{\|M^{-1}F(v_j)\|}{\|M^{-1}F(u_j)\|} \leq \frac{\int_0^1 \tilde{f}(\theta(\beta_j - \alpha_j))\, d\theta}{1 - \tilde{f}_0(\alpha_j)} = \tilde{P}_j,$$

$$\psi_j = \left(1 + |b_1 + 2|\tilde{P}_j + |2b_1 + b_2 + 5|\tilde{P}_j^2\right)\tilde{P}_j, \quad \tilde{h}_j^{(1)} = \sum_{i=1}^{3} |b_i|\tilde{P}_j^i < 1.$$

Thus,

$$\|B_j^{-1}M\| \leq \frac{1}{1-\tilde{h}_j^{(1)}}.$$

Using these estimates and by subtracting the first from the second substep of the method, we obtain

$$w_j - v_j = (I - B_j^{-1}A_j)F'(u_j)^{-1}F(u_j) = B_j^{-1}(B_j - A_j)F'(u_j)^{-1}F(u_j),$$

which

$$\|w_j - v_j\| \leq \frac{\psi_j\|v_j - u_j\|}{1-\tilde{h}_j^{(1)}} \leq \frac{\psi_j(\beta_j - \alpha_j)}{1-\tilde{h}_j^{(1)}} = \gamma_j - \beta_j$$

and

$$\|w_j - u_0\| \leq \|w_j - v_j\| + \|v_j - u_0\| \leq \gamma_j - \beta_j + \beta_j - \alpha_0 = \gamma_j < \alpha^*.$$

Thus the iterate, $w_j \in E(u_0, \alpha^*)$ and assertion (36) holds. We also need the estimate

$$F(w_j) = F(w_j) - F(u_j) - F'(u_j)(v_j - u_j) =$$
$$= F(w_j) - F(u_j) - F'(u_j)(w_j - u_j) + F'(u_j)(w_j - v_j),$$

which gives

$$\|M^{-1}F(w_j)\| \leq \int_0^1 \tilde{f}(\theta\|w_j - u_j\|)\,d\theta\|w_j - u_j\| \quad + (1 + \tilde{f}_0(\|u_j - u_0\|))\|w_j - v_j\| \leq$$
$$\leq \int_0^1 \tilde{f}(\theta(\gamma_j - \alpha_j))\,d\theta(\gamma_j - \alpha_j) + (1 + \tilde{f}_0(\alpha_j))(\gamma_j - \beta_j) = \lambda_j$$

and

$$\|M^{-1}F'(u_j)\| = \|M^{-1}(F'(u_j) - M + M)\| \leq 1 + \|M^{-1}(F'(u_j) - M)\| \leq 1 + \tilde{f}_0(\|u_j - u_0\|).$$

Moreover, as in the local case,

$$\|M^{-1}(C_j - M)\| \leq \tilde{q}_0(\|u_j - u_0\|, \|w_j - u_0\|) + \tilde{q}_2(\|u_j - u_0\|, \|v_j - u_0\|, \|w_j - u_0\|) \leq$$
$$\leq \tilde{q}_0(\alpha_j, \gamma_j) + \tilde{q}_2(\alpha_j, \beta_j, \gamma_j).$$

So, $\|C_j^{-1}M\| \leq \frac{1}{1-\tilde{h}_j^{(2)}}$, which implies

$$\|u_{j+1} - w_j\| \leq \|C_j^{-1}M\|\|M^{-1}F(w_j)\| \leq \frac{\lambda_j}{1-\tilde{h}_j^{(2)}} = \alpha_{j+1} - \gamma_j$$

and $\|u_{j+1} - u_0\| \leq \|u_{j+1} - w_j\| + \|w_j - u_0\| \leq \alpha_{j+1} - \gamma_j + \gamma_j - \alpha_0 = \alpha_{j+1} < \alpha^*$. Thus, the iterate $\{u_{j+1}\} \in E(u_0, \alpha^*)$ and the assertion (37) holds. Next, iterate v_{j+1} exits by the first substep of the method (7). We need the Ostrowski type representation

$$F(u_{j+1}) = F(u_{j+1}) - F(u_j) - F'(u_j)(v_j - u_j) =$$
$$= F(u_{j+1}) - F(u_j) - F'(u_j)(u_{j+1} - u_j) + F'(u_j)(u_{j+1} - v_j),$$

which implies

$$\|M^{-1}F(u_{j+1})\| \leq \int_0^1 \tilde{f}(\theta\|u_{j+1} - u_j\|)\,d\theta\|u_{j+1} - u_j\| + (1 + \tilde{f}_0(\|u_j - u_0\|))\|u_{j+1} - v_j\| \leq$$

$$\leq \int_0^1 \tilde{f}(\theta(\alpha_{j+1}-\alpha_j))\,d\theta(\alpha_{j+1}-\alpha_j)+(1+\tilde{f}_0(\alpha_j))(\alpha_{j+1}-\beta_j)=\mu_{j+1}. \tag{38}$$

In view of first substep of the (2) for $j+1$ replacing j, we get in turn

$$\|v_{j+1}-u_{j+1}\| \leq \|F'(u_{j+1})^{-1}M\|\,\|M^{-1}F(u_{j+1})\| \leq \frac{\mu_{j+1}}{1-\tilde{f}_0(\|u_{j+1}-u_0\|)}$$
$$\leq \frac{\mu_{j+1}}{1-\tilde{f}_0(\alpha_{j+1})}=\beta_{j+1}-\alpha_{j+1}$$

and

$$\|v_{j+1}-u_0\| \leq \|v_{j+1}-u_{j+1}\|+\|u_{j+1}-u_0\| \leq \beta_{j+1}-\alpha_{j+1}+\alpha_{j+1}-\alpha_0=\beta_{j+1}<\alpha^*.$$

Thus, the iterate $v_{j+1}\in E(u_0,\alpha^*)$ and the assertion (35) holds. It follows that the induction for the assertions (34)–(37) is completed and all iterates $w_j, v_j, u_j \in E(u_0,\alpha^*)$. Notice that by the triangle inequality and (34)–(37) we also have $\|u_{j+1}-u_j\| \leq \alpha_{j+1}-\alpha_j$ and for $k=1,2,\ldots$.

$$\|u_{j+k}-u_j\| \leq \alpha_{j+k}-\alpha_j. \tag{39}$$

The sequence $\{u_j\}$ is fundamental, since $\{\alpha_j\}$ is convergent by the condition $(\mathcal{C}_3)$ therefore, there exists $u^*\in E[u_0,\alpha^*]$ such that $\lim_{n\to\infty} u_n=u^*$. By letting $n\to\infty$ in (38) and using the continuity of F, we get $F(u^*)=0$. Finally, by $k\to\infty$ in (39) we show the assertion (33). □

A domain is established next, inside which there is only one solution of the equation $F(x)=0$.

Proposition 2. *There exists a solution $u^*\in E(u_0,\delta_1)$ of the equation $F(x)=0$, the condition $\mathcal{C}_4$ holds in the ball $E(u_0,\delta_1)$ and there exists $\delta_2\geq\delta_1$ such that*

$$\int_0^1 \tilde{f}_0((1-\theta)\delta_1+\theta\delta_2)\,d\theta<1. \tag{40}$$

Define the domain $D_3=E[u_0,\delta_2]\cap D$. Then, u^* is the only solution of the equation $F(x)=0$ in the domain D_3.

Proof. Suppose that there exists a solution $v^*\in D_3$ such that $v^*\neq u^*$ and $F(v^*)=0$. Define linear function $L_1=\int_0^1 F'(u^*+\theta(v^*-u^*))\,d\theta$, $\|M^{-1}(L_1-M)\|\leq\int_0^1\tilde{f}_0((1-\theta)\delta_1+\theta\delta_2)\,d\theta<1$. Hence, L_1^{-1} exists. Finally, from the identity

$$v^*-u^*=L_1^{-1}(F(v^*)-F(u^*))=L_1^{-1}(0)=0,$$

we conclude that $v^*=u^*$. □

Remark 4. 1) The limit point α^* can be replaced by $\tilde{\delta}$ in the condition $(\mathcal{C}_3)$.
2) Under all the conditions $(\mathcal{C}_1)$–$(\mathcal{C}_6)$ one can set $u^*=v^*$ and $\delta_1=\alpha^*$ in the Proposition 2.

We have compared our method by numerical experiments with some other known existing seventh order methods ([6, 9, 24]), and we have used $b_1=0, b_2=-2$ and $b_3=0$ in our computations. For our comparison we have chosen following test functions $f_1(u)=\cos(u)-u$, $f_2(u)=10ue^{-u^2}-1$, $f_3(u)=\sin(u)-\frac{1}{3}u$, $f_4(u)=e^{-u}+\cos(u)$, $f_5(u)=e^u-1.5-\tan^{-1}(u)$, $f_6(u)=\ln(u^2+u+2)-u-1$. The calculated results are in excellent agreement with the theoretical predictions, thereby confirming the robustness and efficiency of the proposed algorithm, if we compare the number of iteration and approximated computational order of convergence.

5. Conclusion. In this paper, we have presented a seventh order iterative method for solving systems of nonlinear equations. The proposed scheme comprises three steps: an initial Newton iteration followed by two weighted Newton steps incorporating power series expansions. Some of the challenges encountered during the convergence analysis are discussed, along with potential solutions, by presenting a detailed local and semilocal convergence analysis. The technique of Section 3 can be used to extend similarly the application of other methods ([12, 16–24]).

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Received 11.05.2025
Revised 02.11.2025