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ON PAPPIAN AND DESARGUESIAN AFFINE PLANES

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We present a complete proof of the classical Heisenberg's Theorem on the Desarguesian property of Pappian affine planes. In the proof we consider two cases, one of which was omitted in the original Hessenberg's proof, making it incomplete.

1. Introduction. The properties of Desargues and Pappus for affine or projective planes play an exceptionally important role in Incidence Geometry because they allow algebraization of Geometry and application of standard methods of Analytic Geometry for algebraic description of various geometric properties, see [4] and [3] for more information on this interesting topic.

In 1905 a German mathematician Gerhard Hessenberg ([2]) proved that every Pappian affine or projective plane is Desarguesian. However his proof contained some inaccuracies (namely one case in the proof was not considered). For projective planes those inaccuracies were filled in [1] by Arno Cronheim who published a complete proof of Hesserberg's theorem for the projective case in 1953. However, for affine planes Hessenberg's proof has not been completed (in spite of the fact that nobody doubts that the theorem is correct). In this paper we present a complete proof of Hessenberg's Theorem for affine planes, with all due details.

Theorem 1 (Hessenberg, 1905)**.** *Every Pappian affine plane is Desarguesian.*

Two cases appear in the proof of this theorem. The first case follows the lines of the original Hessenberg's proof, but the second case was omitted by Hessenberg and needed new ideas to complete the proof.

First of all we will give definitions of affine, Desarguesian and Pappian planes.

Definition 1. An *affine plane* is a pair $(X, \mathcal{L})$ consisting of a set of points X and a set of lines $\mathcal{L}$ that satisfy the following three axioms:

1. any two distinct points $x, y \in X$ belong to a unique line $L \in \mathcal{L}$;
2. there exist three points that do not belong to a single line;
3. (Playfair Axiom) for any line $L \in \mathcal{L}$ and any point $x \in X \setminus L$ there exists a unique line $\Lambda \in \mathcal{L}$ such that $x \in \Lambda \subseteq X \setminus L$.

The third axiom in Definition 1 is called *the Playfair Axiom*, it has been suggested by John Playfair in his work [5].

According to Definition 1, any distinct points x, y in an affine plane P belong to a unique line that will be denoted by $\overline{xy}$.

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Definition 2. Two lines L, Λ in an affine plane are called *parallel* (denoted by $L \parallel \Lambda$) if they are equal or disjoint.

Lemma 1. *In the affine plane P the parallelity is an equivalence relation.*

Proof. The reflexivity and symmetry axioms hold by Definition 1. We need to check the transitivity of the parallelity relation.

Assume that L_1, L_2, L_3 are three lines in an affine plane P, and L_1, L_2 are parallel to L_3. We need to show that $L_1 \parallel L_2$. So, we assume that $L_1 \nparallel L_2$ and intersect in a point x.

If L_3 is equal to L_1 or L_2, then $L_1 \parallel L_2$ by our assumption. So, assume that L_3 is disjoint with L_1 and L_2.

Since $x \in L_1 \cap L_2$ and $L_3 \cap L_1 = L_3 \cap L_2 = \varnothing$, we conclude that $x \notin L_3$. Hence, L_1 and L_2 are two lines that contain the x and are disjoint with the line L_3, which contradict the Playfair Axiom. This contradiction shows that $L_1 \parallel L_2$ and hence the transitivity of parallelity holds. □

Definition 3. Lines $L_1, \dots, L_n$ in an affine plane are called *concurrent* if they are distinct and have a common point.

The following lemma is called the *Proclus Axiom*; it has been suggested by Proclus in his comments [6] to Euclid's "Elements".

Lemma 2. *(Proclus Axiom) If a line in an affine plane is concurrent to one of two parallel lines, then it is concurrent to the other parallel line.*

Proof. Assume that L_1, L_2 are two parallel lines in an affine plane. If a line Λ has a common point x with the line L_1 but does not intersect the line L_2, then the parallel lines L_1, L_2 are distinct and hence disjoint. Then Λ and L_1 are two distinct lines that contain the point x and are disjoint with the line L_2, which contradicts the Playfair Axiom. □

Lemma 3. *Any line in an affine plane contains two or more points and all lines have the same number of points.*

Proof. First of all we will show that an affine plane does not have a line with less that two points. Assume that there exist 0-point and 1-point lines in an affine plane.

For the first case with 0-point line Λ we can take three distinct points x, y, z such that $\overline{x\,y} \cap \overline{x\,z} = \{x\}$ by Axioms 1, 2 of Definition 1. Since $x \in \overline{x\,y} \cap \overline{x\,z}$ and $\Lambda \cap \overline{x\,y} = \Lambda \cap \overline{x\,z} = \varnothing$, we conclude that $\overline{x\,y}$ and $\overline{x\,z}$ are two lines that contain the point x and are disjoint with the line Λ, which contradict the Playfair Axiom.

For the second case with 1-point line Λ let x be a unique point of a line Λ, then we can take two more distinct points y, z such that $x \notin \overline{y\,z}$ by Axioms 1, 2 of Definition 1. Since $x \notin \overline{y\,z}$ and Λ containt only one point x we conclude that $\Lambda \cap \overline{y\,z} = \varnothing$, and hence $\Lambda \parallel \overline{y\,z}$ by Definition 2. By the Playfair Axiom there exist line L such that $z \in L \parallel \overline{x\,y}$. Since $\Lambda \cap L = \varnothing$, we conclude that Λ and L are two parallel lines. Hence the line Λ is parallel to the two concurrent lines L and $\overline{y\,z}$, which contradict the Playfair Axiom.

These two contradictions show that any line in an affine plane contains at least two points.

Next, we will show that all lines in an affine plane have the same number of points.

Take any lines L and Λ that have a unique common point o. Choose any points $a \in L \setminus \{o\} = L \setminus \Lambda$ and $b \in \Lambda \setminus \{o\} = \Lambda \setminus L$. Consider the sets of points $L^\circ := L \setminus \{o, a\}$ and $\Lambda^\circ := \Lambda \setminus \{o, b\}$ and the relation

$$F := \{(x, y) \in L^\circ \times \Lambda^\circ : \overline{x\,y} \cap \overline{a\,b} = \varnothing\} \cup \{(o, o), (a, b)\}.$$

The Playfair Axiom and Lemma 2 imply that the relation F is a bijective function between the lines L and Λ. So, these lines have the same number of points.

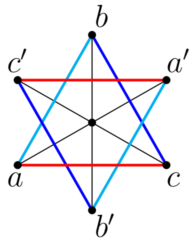
If L and Λ are two disjoint lines in X, then fix any points $a \in L$, $b \in \Lambda$, and observe that the line $\overline{a\,b}$ intersects both lines L and Λ. By the preceding case, $|L| = |\overline{a\,b}| = |\Lambda|$.

Therefore all lines in an affine plane have the same number of points. $\square$

Result of the first part of Lemma 3 was first obtained by Sala Weinlös in her doctoral thesis defended in Lviv in 1927 (see [7]).

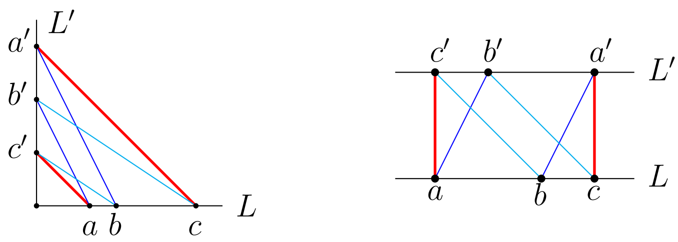
Definition 4. An affine plane P is called *Desarguesian*, if it satisfies the *Affine Desargues Axiom*:

(ADA) for any concurrent lines A, B, C in P and points $a, a' \in A \setminus (B \cup C)$, $b, b' \in B \setminus (A \cup C)$, $c, c' \in C \setminus (A \cup B)$, if $\overline{a\,b} \parallel \overline{a'\,b'}$ and $\overline{b\,c} \parallel \overline{b'\,c'}$, then $\overline{a\,c} \parallel \overline{a'\,c'}$.



Definition 5. An affine plane P is called *Pappian*, if it satisfies the *Affine Pappus Axiom*:

(APA) for every distinct lines L, L' in P and points $a, b, c \in L \setminus L'$ and $a', b', c' \in L' \setminus L$, if $\overline{a\,b'} \parallel \overline{a'\,b}$ and $\overline{b\,c'} \parallel \overline{b'\,c}$, then $\overline{a\,c'} \parallel \overline{a'\,c}$.



2. Proof of Hessenberg's Theorem. In order to prove Hessenberg's Theorem 1, take any concurrent lines A, B, C in a Pappian affine plane P and points $a, a' \in A \setminus (B \cup C)$, $b, b' \in B \setminus (A \cup C)$, $c, c' \in C \setminus (A \cup B)$ such that $\overline{a\,b} \parallel \overline{a'\,b'}$ and $\overline{b\,c} \parallel \overline{b'\,c'}$. We have to prove that $\overline{a\,c} \parallel \overline{a'\,c'}$. Let o be a unique common point of the concurrent lines A, B, C.

According to Lemma 3, any line in an affine plane contains at least two points. First of all we should check cases with 2-points and 3-points lines.

Since $a \neq o \neq a'$, $b \neq o \neq b'$ and $c \neq o \neq c'$, if we have 2-points lines in an affine plane P, then $a = a'$, $b = b'$ and $c = c'$ (otherwise we have more than 2 points in the lines), and $\overline{a\,b} = \overline{a'\,b'}$, $\overline{b\,c} = \overline{b'\,c'}$ by Definition 2, and hence $\overline{a\,c} = \overline{a'\,c'}$ and $\overline{a\,c} \parallel \overline{a'\,c'}$.

If we have 3-points lines in an affine plane P, then for the line $\overline{a\,c}$ we can choose a parallel line $\ell_{a'}$ that contains the point a', by the Playfair Axiom. If $a = a'$, then $\overline{a\,c} = \ell_{a'}$, $b = b'$ and $c = c'$, which implies $\overline{a\,c} \parallel \overline{a'\,c'}$. So, assume that $a \neq a'$, then by the Proclus Axiom there exist a unique common point x, such that $\ell_{a'} \cap C \setminus \{o, c\} = \{x\}$. Since lines in this case have only three points, then either $c = c'$, which implies that $a = a'$ and contradicts our assumption, or $x = c'$, which mean that $\overline{a\,c} \parallel \overline{a'\,x} = \overline{a'\,c'}$.

So, assume that every line in the affine plane P contains at least four points.

If $\overline{a\,b} \parallel \overline{b\,c}$, then $\overline{a\,c} = \overline{a\,b} = \overline{b\,c}$. By transitivity of parallelity, the parallelity relations $\overline{a'\,b'} \parallel \overline{a\,b}$ and $\overline{b'\,c'} \parallel \overline{b\,c}$ imply $\overline{a'\,b'} \parallel \overline{b'\,c'}$ and hence the line $\overline{a'\,c'} = \overline{a'\,b'}$ is parallel to the line $\overline{a\,b} = \overline{a\,c}$. So, we assume that $\overline{a\,b} \nparallel \overline{b\,c}$, which implies $\overline{a'\,b'} \nparallel \overline{b'\,c'}$.

If $b = b'$, then $\overline{a\,b} \parallel \overline{a'\,b'}$ implies $\overline{a\,b} = \overline{a'\,b'}$ and hence $\{a\} = \overline{o\,a} \cap \overline{a\,b} = \overline{o\,a'} \cap \overline{a'\,b'} = \{a'\}$. By analogy we can prove that $c = c'$ and hence $\overline{a\,c} = \overline{a'\,c'}$ and $\overline{a\,c} \parallel \overline{a'\,c'}$. So, we assume that $b \neq b'$. In this case $a \neq a'$ and $c \neq c'$.

If $\overline{a\,c} \parallel B \parallel \overline{a'\,c'}$, then $\overline{a\,c} \parallel \overline{a'\,c'}$, by transitivity of parallelity. So, we assume that either $\overline{a\,c} \nparallel B$ or $B \nparallel \overline{a'\,c'}$. We lose no generality assuming that

$$B \nparallel \overline{a'\,c'}. \tag{1}$$

Since P is an affine plane, by the Playfair Axiom, there exists a unique line $\ell_a \subset P$ such that $a \in \ell_a$ and $\ell_a \parallel B$. It follows from $\ell_a \parallel B \nparallel \overline{a'\,c'}$ that the line ℓ_a is not parallel to the line $\overline{a'\,c'}$. Hence, the lines ℓ_a and $\overline{a'\,c'}$ have a unique common point x.

Claim 1. *$a \neq x$ and $x \notin \overline{a'\,b'}$.*

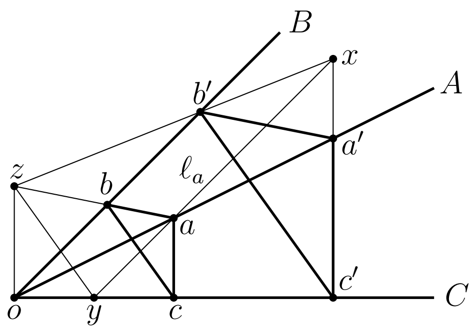
Proof. Assuming that $x = a$, we conclude that $a = x \in \overline{a'\,c'}$ and hence $a \in A \cap \overline{a'\,c'} = \{a'\}$, which contradicts our assumption. This contradiction shows that $x \neq a$.

Assuming that $x \in \overline{a'\,b'}$, we conclude that $x \in \overline{a'\,b'} \cap \overline{a'\,c'} = \{a'\}$ and hence $\ell_a = \overline{a\,a'} = \overline{o\,a}$. Then $\overline{o\,a} = \ell_a \parallel \overline{o\,b}$ and hence $A = \overline{o\,a} = \overline{o\,b} = B$, which contradicts the choice of the point $a \in A \setminus B$. This contradiction shows that $x \notin \overline{a'\,b'}$. $\square$

Since, $\ell_a \parallel B$ and $B \cap C = \{o\}$, the lines ℓ_a and C have a unique common point y, by the Proclus Axiom. Assuming that $y = o$ and taking into account that $\ell_a \parallel B$, we conclude that $y = o \in \ell_a \cap B$ and hence $a \in \ell_a = B$, which contradicts the choice of the point a. This contradiction shows that

$$y \neq o. \tag{2}$$

Claim 1 implies that $\overline{x\,b'} \cap \overline{a'\,b'} = \{b'\}$. Since $\overline{a\,b} \parallel \overline{a'\,b'}$, the Proclus Axiom ensures that $\overline{x\,b'} \cap \overline{a\,b} = \{z\}$ for some point z.



Claim 2. *The points x, b', z do not belong to the line A.*

Proof. The point b' does not belong to A by the choice of b'. Assuming that $x \in A$, we obtain $x \in \ell_a \cap A = \{a\}$, which contradicts Claim 1. Assuming that $z \in A$, we conclude that $z \in \overline{a\,b} \cap A = \{a\}$ and hence $a = z \in \overline{x\,b'}$ and $b' \in \overline{a\,x} = \ell_a$. Taking into account that $\ell_a \parallel B$, we conclude that $a \in \ell_a = B$, which contradicts the choice of the point a. This contradiction shows that $z \notin A$. $\square$

Claim 3. *The points o, a, a' do not belong to the line $\overline{\{z, b', x\}}$.*

Proof. If $o \in \overline{\{z, b', x\}}$, then $x \in \ell_a \cap \overline{o\,b'} = \ell_a \cap B$. Taking into account that $\ell_a \parallel B$, we conclude that $a \in \ell_a = B$, which contradicts the choice of the point a.

If $a \in \overline{\{z, b', x\}}$, then $b' \in \overline{a\,x} = \ell_a$. Taking into account that $\ell_a \parallel B$, we conclude that $a \in \ell_a = B$, which contradicts the choice of the point a.

Assuming that $a' \in \overline{\{z, b', x\}}$, we conclude that $z \in \overline{a'\,b'} \cap \overline{a\,b}$ which is not possible because the lines $\overline{a\,b}$ and $\overline{a'\,b'}$ are parallel and disjoint.

Therefore, $o, a, a' \notin \overline{\{z, b', x\}}$. $\square$

Claims 2 and 3 ensure that $\{o, a, a'\} \cap \overline{\{z, b', x\}} = \varnothing = \{z, b', x\} \cap \overline{\{o, a, a'\}}$. Since $\overline{a\,x} \parallel \overline{o\,b'}$ and $\overline{a\,z} \parallel \overline{a'\,b'}$, the Affine Pappus Axiom applied to the triples o, a, a' and x, b', z ensures that

$$\overline{o\,z} \parallel \overline{a'\,x} = \overline{a'\,c'}. \tag{3}$$

Claim 4. $z \notin A \cup B \cup C$.

Proof. By Claim 3, $o \neq z$. Assuming that $z \in A$, we conclude that $\overline{o\,z} = A$. Then $A = \overline{o\,z} \parallel \overline{a'\,c'}$ and hence $c' \in \overline{a'\,c'} = A$, which contradicts the choice of c'.

Assuming that $z \in C$, we conclude that $\overline{o\,z} = C$. Then $C = \overline{o\,z} \parallel \overline{a'\,c'}$ and hence $a' \in \overline{a'\,c'} = C$, which contradicts the choice of the point a'.

Assuming that $z \in B$, we conclude that $\overline{o\,z} = B$ and hence $B = \overline{o\,z} \parallel \overline{a'\,c'}$, which contradicts the assumption (1). $\square$

Claim 5. *The points o, y, c do not belong to the line $\overline{a\,b} = \overline{\{a, b, z\}}$.*

Proof. Assuming that $o \in \overline{a\,b}$, we conclude that $a \in \overline{o\,b} = B$, which contradicts the choice of the point a. Assuming that $c \in \overline{a\,b}$, we conclude that $\overline{a\,b} = \overline{b\,c}$, which contradicts our assumption $\overline{a\,b} \nparallel \overline{b\,c}$. Assuming that $y \in \overline{a\,b}$, we conclude that $y \in \ell_a \cap \overline{a\,b} = \{a\}$ and hence $a = y \in C$, which contradicts the choice of the point a. Those contradictions show that $o, y, c \notin \overline{a\,b}$. $\square$

Two cases are possible.

1. First we assume that $x \notin C$. This case follows the lines of the original Hessenberg's proof ([2]).

Claim 6. *The points x, b', z do not belong to the line $\overline{\{o, y, c'\}} = C$.*

Proof. By the assumption, $x \notin C$. The choice of the point b' ensures that $b' \notin C$. By Claim 4, $z \notin C$. $\square$

Claim 7. *The points o, y, c' do not belong to the line $\overline{\{x, b', z\}}$.*

Proof. By Claim 3, $o \notin \overline{\{x, b', z\}}$. It follows from $x \notin C = \overline{o\,y}$ that $x \neq y$. Assuming that $y \in \overline{\{x, b', z\}}$ and taking into account that $\ell_a \parallel B$, we conclude that $b' \in \overline{y\,x} = \ell_a$ and hence $a \in \ell_a = B$, which contradicts the choice of the point a.

Assuming that $c' \in \overline{\{x, b', z\}}$ and taking into account that $x \in \overline{a'\,c'} \setminus C$, we conclude that $b' \in \overline{x\,c'} = \overline{a'\,c'}$ and hence $\overline{a'\,b'} \parallel \overline{b'\,c'}$, which contradicts our assumption.

Those contradictions show that the points o, y, c' do not belong to the line $\overline{\{x, b', z\}}$. □

Claims 6 and 7 ensure that $\{c', o, y\} \cap \overline{\{z, x, b'\}} = \varnothing = \{z, x, b'\} \cap \overline{\{c', o, y\}}$. Since $\overline{y\,x} \parallel \overline{o\,b'}$ and $\overline{z\,o} \parallel \overline{a'\,c'} = \overline{x\,c'}$, the Affine Pappus Axiom implies $\overline{z\,y} \parallel \overline{b'\,c'}$. Since $\overline{b\,c} \parallel \overline{b'\,c'} \parallel \overline{z\,y}$, transitivity of parallelity implies that

$$\overline{z\,y} \parallel \overline{b\,c}.$$

Claim 8. *The points a, b, z do not belong to the line C.*

Proof. The choice of the points $a \in A \setminus (B \cup C)$, $b \in B \setminus (A \cup C)$ ensures that $a, b \notin C$. Assuming that $z \in C$ and taking into account that $y \in C$, we conclude that $C = \overline{z\,y}$ and hence $C \parallel \overline{b\,c}$ and $b \in \overline{b\,c} = C$, which contradicts the choice of the point b. Therefore, $a, b, z \notin C$. □

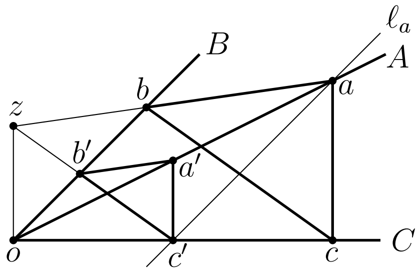
Claims 5 and 8 ensure that $\{o, y, c\} \cap \overline{\{a, b, z\}} = \varnothing = \{a, b, z\} \cap \overline{\{o, y, c\}}$. Since $\overline{o\,b} \parallel \overline{y\,a}$ and $\overline{y\,z} \parallel \overline{b\,c}$, Definition 5 implies $\overline{o\,z} \parallel \overline{a\,c}$. Taking into account that $\overline{o\,z} \parallel \overline{a'\,c'}$, we conclude that $\overline{a\,c} \parallel \overline{a'\,c'}$.

2. Next, consider the second case: $x \in C$. This brings us to a new configuration of points and lines, which has been omitted in Hessenberg's proof ([2]).

It follows that $x \in C \cap \overline{a'\,c'} = \{c'\}$ and $y \in \ell_a \cap C = \{c'\} = \{x\}$. Then

$$\overline{z\,c'} = \overline{z\,x} = \overline{b'\,x} = \overline{b'\,c'} \parallel \overline{b\,c}$$

and $\overline{a\,c'} = \ell_a \parallel \overline{b\,o}$. The choice of the points $a, b \notin C$ and Claim 4 ensure that the points a, b, z do not belong to the line $C = \overline{\{o, c', c\}}$.



Claim 9. *The points o, c', c do not belong to the line $\overline{\{a, b, z\}} = \overline{a\,b}$.*

Proof. Claim 5 implies that $o, c \notin \overline{\{a, b, z\}}$. Assuming that $c' \in \overline{a\,b}$ and taking into account that $z \in \overline{a\,b} \setminus C$, we conclude that $\overline{a\,b} = \overline{z\,c'} \parallel \overline{b\,c}$, which contradicts our assumption. □

Claims 4 and 9 ensure that $\{a, b, z\} \cap \overline{\{o, c', c\}} = \varnothing = \{o, c', c\} \cap \overline{\{a, b, z\}}$. Also $\overline{a\,c'} = \ell_a \parallel \overline{o\,b}$ and $\overline{z\,c'} \parallel \overline{b\,c}$. Applying the Affine Pappus Axiom to the triples a, b, z and o, c', c, we conclude that $\overline{z\,o} \parallel \overline{a\,c}$ and hence $\overline{a\,c} \parallel \overline{z\,o} \parallel \overline{a'\,c'}$. This completes the proof of Hessenberg's Theorem 1.

3. An example of a non-Pappian Desarguesian plane. The following well-known example shows that the reverse implication in the Hessenberg Theorem 1 is false.

Example 1. The quaternion plane $\mathbb{H} \times \mathbb{H}$ is Desarguesian but not Pappian.

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