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ON APPROXIMATION OF SOME LAURICELLA-SARAN'S HYPERGEOMETRIC FUNCTIONS F_M AND THEIR RATIOS BY BRANCHED CONTINUED FRACTIONS

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The paper considers the problem of approximating Lauricella-Saran's hypergeometric functions $F_M(a_1, a_2, b_1, b_2; a_1, c_2; z_1, z_2, z_3)$ by rational functions, which are approximants of branched continued fraction expansions – a special family functions. Under the conditions of positive definite values of the elements of the expansions, the domain of analytic continuation of these functions and their ratios is established. Here, the domain is an open connected set. It is also proven that under the above conditions, every branched continued fraction expansion converges to the function that is analytic in a given domain of analytic continuation at least as fast as a geometric series with a ratio less then unity.

1. Introduction. Special family functions (such as Appell, Horn, Lauricella-Saran, and other) have been the subject of study by both pure and applied mathematicians from their inception to nowadays ([24, 25, 35]). The interest in these special functions is largely due to their importance in applications in various fields of science and engineering ([14, 37]).

Their representations are of considerable matter for the computation of special functions. Recently, some interesting results have appeared on the representation of Appell's functions F_2 ([1, 21]) and F_4 ([28]), Horn's functions H_3 ([3]), H_4 ([7], see also [16, 19, 20]), H_6 ([8, 15, 27]), and H_7 ([4, 29, 30]), Lauricella-Saran's functions F_K ([2, 17]) and F_M ([33]), generalization of hypergeometric functions ${}_3F_2$ ([6]) and ${}_4F_3$ ([32]), confluent hypergeometric function $\Phi_D^{(N)}$ ([5]), and other ([22, 23, 34]) as branched continued fractions (a rather intriguing special family of functions).

The Lauricella-Saran's hypergeometric functions F_M ([31, 36]) are defined by

$$F_M(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma_1, \gamma_2; \mathbf{z}) = \sum_{p,q,r=0}^{+\infty} \frac{(\alpha_1)_p(\alpha_2)_{q+r}(\beta_1)_{p+r}(\beta_2)_q}{(\gamma_1)_p(\gamma_2)_{q+r}} \frac{z_1^p z_2^q z_3^r}{p!q!r!}, \tag{1}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C}$, $\gamma_1, \gamma_2 \notin \{0, -1, -2, \ldots\}$, $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{C}^3$, $(\cdot)_k$ is the Pochhammer symbol.

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Let $\mathfrak{J}=\{1,2\}$ and for $k\geq 1$

$$\mathfrak{J}_k=\{i(k)=(i_0,i_1,i_2,\ldots,i_k)\colon i_r\in\mathfrak{J},\ 0\leq r\leq k\}.$$

The following result directly follows from Theorem 3.1 ([33]).

Theorem 1. *For each $i_0\in\mathfrak{J}$ the function*

$$\frac{F_M(\alpha_1,\alpha_2,\beta_1,\beta_2;\alpha_1,\gamma_2;\mathbf{z})}{(1-\delta_{i_0}^1z_1)F_M(\alpha_1,\alpha_2+1,\beta_1+\delta_{i_0}^1,\beta_2+\delta_{i_0}^2;\alpha_1,\gamma_2+1;\mathbf{z})},\tag{2}$$

where $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_2\in\mathbb{C}$, $\alpha_1,\gamma_2\notin\{0,-1,-2,\ldots\}$, and δ_i^j is the Kronecker symbol, has a formal branched continued fraction

$$q_{i_0}(\mathbf{z})+\sum_{i_1=1}^{2}\frac{p_{i(1)}(\mathbf{z})}{q_{i(1)}(\mathbf{z})+\sum_{i_2=1}^{2}\frac{p_{i(2)}(\mathbf{z})}{q_{i(2)}(\mathbf{z})+\ddots}},\tag{3}$$

where

$$q_{i_0}(\mathbf{z})=1-\frac{\alpha_2+\beta_{i_0}+1}{\gamma_2}\frac{(1-\delta_{i_0}^2z_1)z_{4-i_0}}{1-z_1}-\frac{\beta_{3-i_0}}{\gamma_2}\frac{(1-\delta_{i_0}^1z_1)z_{1+i_0}}{1-z_1}\tag{4}$$

and for $i(k)\in\mathfrak{J}_k$ and $k\geq 1$

$$p_{i(k)}(\mathbf{z})=\frac{(\alpha_2+k)\left(\beta_{i_k}+\sum_{r=0}^{k-1}\delta_{i_r}^{i_k}\right)}{(\gamma_2+k-1)(\gamma_2+k)}\frac{(1-\delta_{i_k}^2z_1)^2z_{4-i_k}(1-\delta_{i_k}^1z_1-z_{4-i_k})}{(1-z_1)^2},\tag{5}$$

$$q_{i(k)}(\mathbf{z})=1-\frac{\alpha_2+\beta_{i_k}+k+1+\sum_{r=0}^{k-1}\delta_{i_r}^{i_k}}{\gamma_2+k}\frac{(1-\delta_{i_k}^2z_1)z_{4-i_k}}{1-z_1}-$$
$$-\frac{\beta_{3-i_k}+\sum_{r=0}^{k-1}\delta_{i_r}^{3-i_k}}{\gamma_2+k}\frac{(1-\delta_{i_k}^1z_1)z_{1+i_k}}{1-z_1}.\tag{6}$$

This study delves the convergence problem of branched continued fractions (3), which, in turn, provides, in particular, the domains of extension of functions (2). In [24] one can find analytical extensions of the Lauricella-Saran's hypergeometric functions F_M through their integral representations. Some results on the convergence of various branched continued fractions can be found in [9, 12] (see also [10, 11, 26]).

2. Convergence and analytic continuation. The following result is true.

Theorem 2. *Let α_1 be complex number herewith $\alpha_1\notin\{0,-1,-2,\ldots\}$ and α_2, β_1, β_2, and γ_2 real numbers such that*

$$0<\alpha_2+1\leq\gamma_2,\ \alpha_2+\beta_1+\beta_2+1\leq 2\gamma_2,\ 0\leq\beta_{i_0},\ \beta_1^2+\beta_2^2\neq 0,\ \beta_{i_0}\leq\gamma_2,\quad i_0\in\mathfrak{J}.\tag{7}$$

Then, for each $i_0\in\mathfrak{J}$, the following statements are true:

(a) *the branched continued fraction (3) converges to a finite value* $g^{(i_0)}(\mathbf{z})$ *for each* $\mathbf{z}\in\mathfrak{D}_\eta$, *where*

$$\mathfrak{D}_\eta=\left\{\mathbf{z}\in\mathbb{R}^3\colon z_1<1,\ 0\le\frac{z_3}{1-z_1}\le\frac{1-\eta}{2},\ 0\le z_2\le\frac{1-\eta}{2}\right\},\quad 0<\eta<1; \tag{8}$$

(b) *the convergence is uniformly on every compact subset of the domain* $\operatorname{Int}(\mathfrak{D}_\eta)$, *and* $g^{(i_0)}(\mathbf{z})$ *is analytic on* $\operatorname{Int}(\mathfrak{D}_\eta)$;

(c) *if* $g_n^{(i_0)}(\mathbf{z})$ *denotes the* n*th approximant of (3), then for each* $\mathbf{z}\in\mathfrak{D}_\eta$

$$|g^{(i_0)}(\mathbf{z})-g_n^{(i_0)}(\mathbf{z})|\le\frac{\mu^{n+1}(\mathbf{z})}{(1-2\nu(\mathbf{z}))((1-2\nu(\mathbf{z}))^2+\mu(\mathbf{z}))^n},\quad n\ge1,$$

where

$$\nu(\mathbf{z})=\max\left\{\frac{z_3}{1-z_1},\ z_2\right\}, \tag{9}$$

$$\mu(\mathbf{z})=\frac{z_3}{1-z_1}\left(1-\frac{z_3}{1-z_1}\right)+z_2(1-z_2); \tag{10}$$

(d) *the function* $g^{(i_0)}(\mathbf{z})$ *is an analytic continuation of the function (2) in the domain* $\operatorname{Int}(\mathfrak{D}_\eta)$.

Let us set (see [13])

$$Q_{i(n)}^{(n)}(\mathbf{z})=q_{i(n)}(\mathbf{z}),\quad i(n)\in\mathfrak{I}_n,\ n\ge1, \tag{11}$$

and

$$Q_{i(k)}^{(n)}(\mathbf{z})=q_{i(k)}(\mathbf{z})+\sum_{i_{k+1}=1}^{2}\cfrac{p_{i(k+1)}(\mathbf{z})}{q_{i(k+1)}(\mathbf{z})+\sum\limits_{i_{k+2}=1}^{2}\cfrac{p_{i(k+2)}(\mathbf{z})}{q_{i(k+2)}(\mathbf{z})+\ddots+\sum\limits_{i_n=1}^{2}\cfrac{p_{i(n)}(\mathbf{z})}{q_{i(n)}(\mathbf{z})}}},$$

where $i(k)\in\mathfrak{I}_k$, $1\le k\le n-1$, $n\ge2$. Then

$$Q_{i(k)}^{(n)}(\mathbf{z})=q_{i(k)}(\mathbf{z})+\sum_{i_{k+1}=1}^{2}\frac{p_{i(k+1)}(\mathbf{z})}{Q_{i(k+1)}^{(n)}(\mathbf{z})}, \tag{12}$$

where $i(k)\in\mathfrak{I}_k$, $1\le k\le n-1$, $n\ge2$, and the approximants of (3) are written as

$$g_n^{(i_0)}(\mathbf{z})=q_{i_0}(\mathbf{z})+\sum_{i_1=1}^{2}\frac{p_{i(1)}(\mathbf{z})}{Q_{i(1)}^{(n)}(\mathbf{z})},\quad n\ge1.$$

Proof of Theorem 2. Let i_0 be an arbitrary index in $\mathfrak{I}$.

(a) To prove this statement, we will use the method of establishing the truncation error bounds from Theorem 1 ([6]). Let $\mathbf{z}$ be an arbitrary fixed point in (8). It is obvious that under conditions (7), the elements (5) are nonnegative.

Let us evaluate (6). Using (7) and (8), for any $i(k) \in \mathfrak{J}_k$, $k \geq 1$, we obtain

$$q_{i(k)}(\mathbf{z}) \geq 1 - \frac{\alpha_2 + \beta_{i_k} + k + \sum\limits_{r=0}^{k-1} \delta_{i_r}^{i_k} + 1}{\gamma_2 + k} \nu(\mathbf{z}) - \frac{\beta_{3-i_k} + \sum\limits_{r=0}^{k-1} \delta_{i_r}^{3-i_k}}{\gamma_2 + k} \nu(\mathbf{z}) =$$
$$= 1 - \frac{\alpha_2 + \beta_1 + \beta_2 + 2k + 1}{\gamma_2 + k} \nu(\mathbf{z}) \geq 1 - 2\nu(\mathbf{z}) \geq \eta.$$

Here we used that

$$\sum_{r=0}^{k-1} \delta_{i_r}^{i_k} + \sum_{r=0}^{k-1} \delta_{i_r}^{3-i_k} = k$$

for any $i(k) \in \mathfrak{J}_k$, $k \geq 1$.

Thus,

$$Q_{i(k)}^{(n)}(\mathbf{z}) \geq q_{i(k)}(\mathbf{z}) \geq 1 - 2\nu(\mathbf{z}) \geq \eta > 0, \quad i(k) \in \mathfrak{J}_k,\ 1 \leq k \leq n,\ n \geq 1. \tag{13}$$

Next, let us evaluate $|g_{n+k}^{(i_0)}(\mathbf{z}) - g_n^{(i_0)}(\mathbf{z})|$ for $n \geq 1$ and $k \geq 1$. To do this, we will use the well-known formula for the difference of two approximants of a branched continued fraction ([13]), and write it in the form

$$g_{n+k}^{(i_0)}(\mathbf{z}) - g_n^{(i_0)}(\mathbf{z}) =$$
$$= (-1)^n \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_{n+1}=1}^{2} \frac{p_{i(1)}(\mathbf{z})}{Q_{i(1)}^{(q)}(\mathbf{z})} \prod_{r=1}^{[(n+1)/2]} \frac{p_{i(2r)}(\mathbf{z})}{Q_{i(2r-1)}^{(p)}(\mathbf{z}) Q_{i(2r)}^{(p)}(\mathbf{z})} \prod_{r=1}^{[n/2]} \frac{p_{i(2r+1)}(\mathbf{z})}{Q_{i(2r)}^{(q)}(\mathbf{z}) Q_{i(2r+1)}^{(q)}(\mathbf{z})},$$

where $q = n + k$, $p = n$, if $n = 2s$, and $q = n$, $p = n + k$, if $n = 2s - 1$, $s \geq 1$.

Using (11)–(13), for any $m \geq 1$ we have

$$\sum_{i_1=1}^{2} \frac{p_{i(1)}(\mathbf{z})}{Q_{i(1)}^{(m)}(\mathbf{z})} \leq \sum_{i_1=1}^{2} \frac{p_{i(1)}(\mathbf{z})}{q_{i(1)}(\mathbf{z})} \leq \sum_{i_1=1}^{2} \frac{p_{i(1)}(\mathbf{z})}{1 - 2\nu(\mathbf{z})}$$

and for any $i(k) \in \mathfrak{J}_k$, $1 \leq k \leq m$,

$$\sum_{i_{k+1}=1}^{2} \frac{p_{i(k+1)}(\mathbf{z})}{Q_{i(k)}^{(m+1)}(\mathbf{z}) Q_{i(k+1)}^{(m+1)}(\mathbf{z})} = \frac{\sum\limits_{i_{k+1}=1}^{2} \frac{p_{i(k+1)}(\mathbf{z})}{Q_{i(k+1)}^{(m+1)}(\mathbf{z})}}{q_{i(k)}(\mathbf{z}) + \sum\limits_{i_{k+1}=1}^{2} \frac{p_{i(k+1)}(\mathbf{z})}{Q_{i(k+1)}^{(m+1)}(\mathbf{z})}} \leq \frac{\sum\limits_{i_{k+1}=1}^{2} \frac{p_{i(k+1)}(\mathbf{z})}{q_{i(k)}(\mathbf{z}) q_{i(k+1)}(\mathbf{z})}}{1 + \sum\limits_{i_{k+1}=1}^{2} \frac{p_{i(k+1)}(\mathbf{z})}{q_{i(k)}(\mathbf{z}) q_{i(k+1)}(\mathbf{z})}} \leq$$
$$\leq \frac{\sum\limits_{i_{k+1}=1}^{2} p_{i(k+1)}(\mathbf{z})}{(1 - 2\nu(\mathbf{z}))^2 + \sum\limits_{i_{k+1}=1}^{2} p_{i(k+1)}(\mathbf{z})}.$$

Next again, for any $i(k)\in\mathfrak{J}_k$, $k\ge 1$, by (5), (7) and (8) we obtain

$$\sum_{i_k=1}^{2} p_{i(k)}(\mathbf{z}) = \sum_{i_k=1}^{2} \frac{(\alpha_2+k)\left(\beta_{i_k}+\sum\limits_{r=0}^{k-1}\delta_{i_r}^{i_k}\right)}{(\gamma_2+k-1)(\gamma_2+k)} \frac{(1-\delta_{i_k}^2 z_1)^2 z_{4-i_k}(1-\delta_{i_k}^1 z_1 - z_{4-i_k})}{(1-z_1)^2} \le$$

$$\le \sum_{i_k=1}^{2} \frac{(1-\delta_{i_k}^2 z_1)^2 z_{4-i_k}(1-\delta_{i_k}^1 z_1 - z_{4-i_k})}{(1-z_1)^2} = \frac{z_3}{1-z_1}\left(1-\frac{z_3}{1-z_1}\right) + z_2(1-z_2) = \mu(\mathbf{z}).$$

Thus,

$$|g_{n+k}^{(i_0)}(\mathbf{z}) - g_n^{(i_0)}(\mathbf{z})| \le \frac{\mu^{n+1}(\mathbf{z})}{(1-2\nu(\mathbf{z}))((1-2\nu(\mathbf{z}))^2+\mu(\mathbf{z}))^n}, \quad n\ge 1,\ k\ge 1. \tag{14}$$

Therefore, due to the arbitrariness of k and given that for arbitrary fixed $\mathbf{z}\in\mathfrak{D}_\eta$,

$$\frac{\mu^{n+1}(\mathbf{z})}{(1-2\nu(\mathbf{z}))((1-2\nu(\mathbf{z}))^2+\mu(\mathbf{z}))^n} \to 0 \quad \text{as} \quad n\to+\infty,$$

it follows (a).

(b) Let $\mathfrak{L}$ be an arbitrary compact subset of $\mathrm{Int}(\mathfrak{D}_\eta)$. Then, using (8)–(10) and (13), from (14) for $n\ge 1$ and $k\ge 1$ we have

$$|g_{n+k}^{(i_0)}(\mathbf{z}) - g_n^{(i_0)}(\mathbf{z})| < \frac{(1-\eta^2)^{n+1}}{\eta(\eta^2+1-\eta^2)^n} = \frac{(1-\eta^2)^{n+1}}{\eta}$$

for all $\mathbf{z}\in\mathfrak{L}$. Next, if q and p are arbitrary integer numbers such that $q\ge 1$ and $p\ge n\ge 1$, then, for all $\mathbf{z}\in\mathfrak{L}$,

$$|g_{p+q}^{(i_0)}(\mathbf{z}) - g_p^{(i_0)}(\mathbf{z})| \le |g_{p+q}^{(i_0)}(\mathbf{z}) - g_n^{(i_0)}(\mathbf{z})| + |g_p^{(i_0)}(\mathbf{z}) - g_n^{(i_0)}(\mathbf{z})|.$$

Therefore, due to

$$\frac{(1-\eta^2)^{n+1}}{\eta} \to 0 \quad \text{as} \quad n\to+\infty,$$

it follows (b).

(c) follows from (14) if we pass to the limit as $k\to+\infty$.

(d) Here we will use the PF method ([2, 5]). It known that the series (1) converges in the domain (see, [36])

$$\mathfrak{D}_{\kappa,\tau} = \{\mathbf{z}\in\mathbb{C}^3\colon |z_1|<\kappa,\ |z_2|<1,\ |z_3|<\tau\},$$

where κ and τ are positive numbers such that $\kappa+\tau=1$. In addition, it is clear that

$$F_M(\alpha_1,\alpha_2,\alpha_1,\beta_2;\gamma_1,\gamma_2;\mathbf{0}) = F_M(\alpha_1,\alpha_2+1,\beta_1+\delta_{i_0}^1,\beta_2+\delta_{i_0}^2;\alpha_1,\gamma_2+1;\mathbf{0}) = 1.$$

Therefore, there exists $0<\varepsilon<1$ such that function (2) is analytic in domain

$$\mathfrak{D}_{\kappa,\tau,\eta,\varepsilon} =$$

$$= \left\{\mathbf{z}\in\mathbb{R}^3\colon 0<z_1<\varepsilon\min\{\kappa,\eta\},\ 0<z_2<\frac{\varepsilon(1-\eta)}{4},\ 0<z_3<\varepsilon\min\left\{\tau,\frac{(1-\eta)^2}{4}\right\}\right\},$$

and $\mathfrak{D}_{\kappa,\tau,\eta,\varepsilon} \subset (\mathfrak{D}_{\kappa,\tau} \cap \mathrm{Int}(\mathfrak{D}_\eta))$, in particular, $\mathfrak{D}_{\kappa,\tau,\eta,1/2} \subset (\mathfrak{D}_{\kappa,\tau} \cap \mathrm{Int}(\mathfrak{D}_\eta))$.

Let $\mathbf{z}$ be an arbitrary fixed point in $\mathfrak{D}_{\kappa,\tau,\eta,\varepsilon}$. Then it is obvious that the elements (4)–(6) are positive. This means that (see, [2, 13])

$$g_{2n}^{(i_0)}(\mathbf{z}) < g_{2n+2}^{(i_0)}(\mathbf{z}) < g_{2n+1}^{(i_0)}(\mathbf{z}) < g_{2n-1}^{(i_0)}(\mathbf{z}), \quad n \geq 1,$$

which in turn means that the sequences of even and odd approximants of (3) converge to a finite value $g^{(i_0)}(\mathbf{z})$. It should be note that this system of inequalities expresses the so-called property of fork for branched continued fractions.

Next, we will consider

$$\frac{F_M(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \gamma_2; \mathbf{z})}{(1-\delta_{i_0}^1 z_1) F_M(\alpha_1, \alpha_2+1, \beta_1+\delta_{i_0}^1, \beta_2+\delta_{i_0}^2; \alpha_1, \gamma_2+1; \mathbf{z})} - g_n^{(i_0)}(\mathbf{z}), \quad n \geq 1,$$

where (see [33])

$$\frac{F_M(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \gamma_2; \mathbf{z})}{(1-\delta_{i_0}^1 z_1) F_M(\alpha_1, \alpha_2+1, \beta_1+\delta_{i_0}^1, \beta_2+\delta_{i_0}^2; \alpha_1, \gamma_2+1; \mathbf{z})} =$$
$$= q_{i_0}(\mathbf{z}) + \sum_{i_1=1}^{2} \cfrac{p_{i(1)}(\mathbf{z})}{q_{i(1)}(\mathbf{z}) + \sum_{i_2=1}^{2} \cfrac{p_{i(2)}(\mathbf{z})}{q_{i(2)}(\mathbf{z}) + \ddots + \sum_{i_{n+1}=1}^{2} \cfrac{p_{i(n+1)}(\mathbf{z})}{G_{i(n+1)}^{(n+1)}(\mathbf{z})}}},$$

and, for $i(n+1) \in \mathfrak{I}_{n+1}$, $n \geq 1$,

$$G_{i(n+1)}^{(n+1)}(\mathbf{z}) = \frac{F_M\left(\alpha_1, \alpha_2+n+1, \beta_1+\sum_{r=0}^{n}\delta_{i_r}^1, \beta_2+\sum_{r=0}^{n}\delta_{i_r}^2; \alpha_1, \gamma_2+n+1; \mathbf{z}\right)}{(1-\delta_{i_{n+1}}^1 z_1) F_M\left(\alpha_1, \alpha_2+n+2, \beta_1+\sum_{r=0}^{n+1}\delta_{i_r}^1, \beta_2+\sum_{r=0}^{n+1}\delta_{i_r}^2; \alpha_1, \gamma_2+n+2; \mathbf{z}\right)}.$$

Let

$$G_{i(k)}^{(n+1)}(\mathbf{z}) = q_{i(k)}(\mathbf{z}) + \sum_{i_{k+1}=1}^{2} \cfrac{p_{i(k+1)}(\mathbf{z})}{q_{i(k+1)}(\mathbf{z}) + \sum_{i_{k+2}=1}^{2} \cfrac{p_{i(k+2)}(\mathbf{z})}{q_{i(k+2)}(\mathbf{z}) + \ddots + \sum_{i_{n+1}=1}^{2} \cfrac{p_{i(n+1)}(\mathbf{z})}{G_{i(n+1)}^{(n+1)}(\mathbf{z})}}},$$

where $i(k) \in \mathfrak{I}_k$, $1 \leq k \leq n$, $n \geq 1$. Then

$$G_{i(k)}^{(n+1)}(\mathbf{z}) = q_{i(k)}(\mathbf{z}) + \sum_{i_{k+1}=1}^{2} \frac{p_{i(k+1)}(\mathbf{z})}{G_{i(k+1)}^{(n+1)}(\mathbf{z})}, \quad i(k) \in \mathfrak{I}_k,\ 1 \leq k \leq n,\ n \geq 1. \tag{15}$$

It is obvious that $Q_{i(k)}^{(n)}(\mathbf{z}) \neq 0$ and $G_{i(k)}^{(n)}(\mathbf{z}) \neq 0$ for all indices and for all $\mathbf{z} \in \mathfrak{D}_{\kappa,\tau,\eta,\varepsilon}$. Then, using (11), (12), (15), and the above-mentioned formula for the difference of two approximants of the branched continued fraction ([13]), for $n \geq 1$ we obtain

$$\frac{F_M(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \gamma_2; \mathbf{z})}{(1-\delta_{i_0}^1 z_1) F_M(\alpha_1, \alpha_2+1, \beta_1+\delta_{i_0}^1, \beta_2+\delta_{i_0}^2; \alpha_1, \gamma_2+1; \mathbf{z})} - g_n^{(i_0)}(\mathbf{z}) =$$

$$= (-1)^n \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_{n+1}=1}^{2} \frac{\prod_{r=1}^{n+1} p_{i(r)}(\mathbf{z})}{\prod_{r=1}^{n+1} G_{i(r)}^{(n+1)}(\mathbf{z}) \prod_{r=1}^{n} Q_{i(r)}^{(n)}(\mathbf{z})}.$$

Therefore, for all $\mathbf{z} \in \mathfrak{D}_{\kappa,\tau,\eta,\varepsilon}$ we have

$$g_{2n}^{(i_0)}(\mathbf{z}) < \frac{F_M(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \gamma_2; \mathbf{z})}{(1 - \delta_{i_0}^1 z_1) F_M(\alpha_1, \alpha_2 + 1, \beta_1 + \delta_{i_0}^1, \beta_2 + \delta_{i_0}^2; \alpha_1, \gamma_2 + 1; \mathbf{z})} < g_{2n-1}^{(i_0)}(\mathbf{z}), \quad n \geq 1.$$

Since by the property of fork for all $\mathbf{z} \in \mathfrak{D}_{\kappa,\tau,\eta,\varepsilon}$

$$\lim_{n\to+\infty} g_{2n}^{(i_0)}(\mathbf{z}) = \lim_{n\to+\infty} g_{2n-1}^{(i_0)}(\mathbf{z}) = g^{(i_0)}(\mathbf{z}),$$

then also for all $\mathbf{z} \in \mathfrak{D}_{\kappa,\tau,\eta,\varepsilon}$,

$$g^{(i_0)}(\mathbf{z}) = \frac{F_M(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \gamma_2; \mathbf{z})}{(1 - \delta_{i_0}^1 z_1) F_M(\alpha_1, \alpha_2 + 1, \beta_1 + \delta_{i_0}^1, \beta_2 + \delta_{i_0}^2; \alpha_1, \gamma_2 + 1; \mathbf{z})}.$$

Finally, applying Theorem 2 ([2]) gives (d). □

When $\alpha_2 = \beta_1 = 0$ and γ_2 is replaced by $\gamma_2 - 1$, from Theorem 2 we have the following:

Corollary 1. *If $i_0 = 1$, α_1 be complex number herewith $\alpha_1 \notin \{0, -1, -2, \ldots\}$, β_2 and γ_2 be real numbers such that $0 < \beta_2 \leq \gamma_2 - 1$ and $\gamma_2 \geq 2$, then the following statements are true:*

(a) the branched continued fraction

$$\cfrac{1}{q_{i_0}(\mathbf{z}) + \sum_{i_1=1}^{2} \cfrac{p_{i(1)}(\mathbf{z})}{q_{i(1)}(\mathbf{z}) + \sum_{i_2=1}^{2} \cfrac{p_{i(2)}(\mathbf{z})}{q_{i(2)}(\mathbf{z}) + \ddots}}} \tag{16}$$

converges to a finite value $g^{(i_0)}(\mathbf{z})$ for each $\mathfrak{D}_\eta$, where

$$q_{i_0}(\mathbf{z}) = 1 - \frac{\beta_{i_0} + 1}{\gamma_2 - 1} \frac{(1 - \delta_{i_0}^2 z_1) z_{4-i_0}}{1 - z_1} - \frac{\beta_{3-i_0}}{\gamma_2 - 1} \frac{(1 - \delta_{i_0}^1 z_1) z_{1+i_0}}{1 - z_1}$$

and for $i(k) \in \mathfrak{I}_k$ and $k \geq 1$

$$p_{i(k)}(\mathbf{z}) = \frac{k \left(\beta_{i_k} + \sum_{r=0}^{k-1} \delta_{i_r}^{i_k} \right)}{(\gamma_2 + k - 2)(\gamma_2 + k - 1)} \frac{(1 - \delta_{i_k}^2 z_1)^2 z_{4-i_k} (1 - \delta_{i_k}^1 z_1 - z_{4-i_k})}{1 - z_1},$$

$$q_{i(k)}(\mathbf{z}) = 1 - \frac{\beta_{i_k} + k + 1 + \sum_{r=0}^{k-1} \delta_{i_r}^{i_k}}{\gamma_2 + k - 1} \frac{(1 - \delta_{i_k}^2 z_1) z_{4-i_k}}{1 - z_1} - \frac{\beta_{3-i_k} + \sum_{r=0}^{k-1} \delta_{i_r}^{3-i_k}}{\gamma_2 + k - 1} \frac{(1 - \delta_{i_k}^1 z_1) z_{1+i_k}}{1 - z_1},$$

herewith $\beta_1 = 0$, and $\mathfrak{D}_\eta$ is defined by (8);

(b) *the convergence is uniformly on every compact subset of (8) to the function* $g^{(i_0)}(\mathbf{z})$ *analytic in* $\mathrm{Int}(\mathfrak{D}_\eta)$*;*

(c) *if* $g_n^{(i_0)}(\mathbf{z})$ *denotes the nth approximant of (16), then for each* $\mathbf{z} \in \mathfrak{D}_\eta$

$$|g^{(i_0)}(\mathbf{z}) - g_n^{(i_0)}(\mathbf{z})| \le \frac{\mu^n(\mathbf{z})}{(1-\nu(\mathbf{z}))^3((1-2\nu(\mathbf{z}))^2+\mu(\mathbf{z}))^{n-1}}, \quad n \ge 1,$$

where $\nu(\mathbf{z})$ *and* $\mu(\mathbf{z})$ *are defined by (9) and (10), respectively;*

(d) *the function* $g^{(i_0)}(\mathbf{z})$ *is an analytic continuation of the function*

$$(1-z_1)F_M(\alpha_1, 1, 1, \beta_2; \alpha_1, \gamma_2; \mathbf{z})$$

in the domain $\mathrm{Int}(\mathfrak{D}_\eta)$.

It should be note that the similar consequence is also true if $i_0 = 2$, $\alpha_2 = \beta_2 = 0$, and γ_2 is replaced by $\gamma_2 - 1$. In Theorem 2 and Corollary 1, statements (a) and (c) are also true without the condition $z_1 < 1$ in (8). The validity of (b) and (d) for complex $\mathbf{z}$ will be considered in the next paper.

Example 1. From Example 3.1 in [33] we have

$$\ln \frac{1-z_1-z_3}{(1-z_1)(1-z_2)} = \left(z_2 - \frac{z_3}{1-z_1}\right)(1-z_1)F_M(1,1,1,1;1,2;\mathbf{z}) =$$

$$= \cfrac{z_2 - \cfrac{z_3}{1-z_1}}{1 - z_2 - \cfrac{z_3}{1-z_1} + \cfrac{\frac{1}{2}\frac{z_3}{1-z_1}\left(1-\frac{z_3}{1-z_1}\right)}{1-\frac{1}{2}z_2 - \frac{3}{2}\frac{z_3}{1-z_1} + \ddots} + \cfrac{\frac{1}{2}z_2(1-z_2)}{1-\frac{3}{2}z_2 - \frac{1}{2}\frac{z_3}{1-z_1} + \ddots}}. \tag{17}$$

By Corollary 1, the branched continued fraction (17) converges and represents a single-valued branch of the function

$$\ln \frac{1-z_1-z_3}{(1-z_1)(1-z_2)}$$

in the domain $\mathfrak{T}_\eta = \mathfrak{D}_\eta \cap \mathfrak{R}$, where $\mathfrak{D}_\eta$ is defined by (8) and

$$\mathfrak{R} = \{\mathbf{z} \in \mathbb{R}^3 \colon (1-z_1-z_3)(1-z_1)(1-z_2) > 0\}.$$

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