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V. V. Kravtsiv, P. Y. Dolishniak, R. Y. Stakhiv

WARING-GIRARD FORMULAS FOR BLOCK-SYMMETRIC AND BLOCK-SUPERSYMMETRIC POLYNOMIALS

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Symmetric polynomials are objects of study in classical invariant theory. A generalization of symmetric polynomials is represented by block-symmetric polynomials, also known as MacMahon polynomials in classical theory. To study algebras of symmetric and block-symmetric polynomials, it is necessary to describe algebraic bases for these algebras.

This paper investigates the structure and properties of block-symmetric and block-supersymmetric polynomials in Banach spaces. The study extends classical results on symmetric polynomials to infinite-dimensional settings, particularly in sequence spaces such as $\ell_p(\mathbb{C}^s)$, where $1 \le p < \infty$, and in spaces of two-sided absolutely summing series of vectors in $\mathbb{C}^s$ for some positive integer $s > 1$. In this paper, we derive analogs of the Waring-Girard formulas for block-symmetric and block-supersymmetric polynomials.

Moreover, the paper discusses combinatorial implications, including new identities obtained by evaluating polynomials at specific points. The finite-dimensional case is also considered, highlighting algebraic dependencies between supersymmetric and non-supersymmetric generators. These findings contribute to the broader understanding of symmetric structures in mathematical analysis and quantum physics. The results further open avenues for applications in cryptography and functional analysis.

1. Introduction. Symmetric functions on finite-dimensional vector spaces are fundamental objects in combinatorics and classical invariant theory (see, e.g., [18, 24]). The study of symmetric polynomials in infinite-dimensional spaces was initiated by Nemirovski and Semenov in [20]. In particular, they constructed algebraic bases for algebras of symmetric real-valued polynomials on the Banach spaces ℓ_p and $L_p[0,1]$ for $1 \le p < \infty$. These results were later extended in [7] to separable sequence Banach spaces with symmetric bases, as well as to separable rearrangement-invariant Banach spaces. Algebraic basis plays a crucial role in the problem of description of spectra of algebras generated by polynomials ([3, 4, 22, 23]).

Block-symmetric polynomials, also known as MacMahon polynomials, are a natural extension of symmetric polynomials and can be interpreted as symmetric polynomials on vector-sequence linear spaces. Their combinatorial properties are discussed in [21]. An algebraic basis for the algebra of all symmetric continuous complex-valued polynomials on the Cartesian power of the complex Banach space ℓ_p for a fixed $1 \le p < \infty$ was constructed in [14]. The algebras of symmetric continuous polynomials on Cartesian products $\ell_{p_1} \times \cdots \times \ell_{p_s}$

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for different values of $p_1, \dots, p_s$ were studied in [1]. Certain generalizations of Newton's formulas for algebraic bases of block-symmetric polynomials were derived in [10, 13].

In [9], Jawad and Zagorodnyuk examined supersymmetric polynomials and analytic functions on the space $\ell_1(\mathbb{Z}_0)$ which consists of absolutely summable sequences (x_n), where $n \in \mathbb{Z}_0 = \mathbb{Z} \setminus 0$. Further generalizations of supersymmetric polynomials to more general sequence spaces were considered in [2]. Applications of algebraic bases of supersymmetric polynomials to models of ideal gases in quantum physics were proposed in [5]. Additionally, supersymmetric polynomials over finite fields and their applications in cryptography were explored in [6].

The work in [11] examined polynomial algebras that are both block- and supersymmetric in infinite-dimensional Banach spaces of absolutely summable sequences of vectors in $\mathbb{C}^s$. Algebraic bases for such algebras were developed, and Newton-type relations linking different bases were established in [11].

This paper introduces analogs of Waring-Girard formulas for block-symmetric polynomials in the spaces $\ell_1(\mathbb{C}^s)$ and $\ell_p(\mathbb{C}^s)$ for $p > 1$ and for block-supersymmetric polynomials.

1.1. Symmetric polynomials. Let X be a complex Banach space and let $S_{\mathbb{N}}$ be the semigroup of all permutations of the natural numbers $\mathbb{N}$. A function f on X is called *symmetric* if it remains unchanged under any permutation from $S_{\mathbb{N}}$, meaning that for every $\sigma \in S_{\mathbb{N}}$,

$$f(x_1, x_2, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots)$$

for each $\sigma \in S_{\mathbb{N}}$. The algebra of all continuous symmetric polynomials on X is denoted by $\mathcal{P}_s(X)$.

It is known that the polynomials

$$F_k(x) = \sum_{n=1}^{\infty} x_n^k, \quad k \in \mathbb{N}$$

form an algebraic basis in the algebra $\mathcal{P}_s(\ell_1)$ ([7]). That is, for any polynomials $P \in \mathcal{P}_s(\ell_1)$ there exists a unique polynomial $Q(t_1, \dots, t_m)$ in several complex variables such that $P(x) = Q(F_1(x), \dots, F_m(x))$. Polynomials F_k are known as *power* symmetric polynomials. The algebraic basis is not unique, and we also consider the following alternative bases in $\mathcal{P}_s(\ell_1)$: the *elementary* symmetric polynomial basis, given by

$$G_n(x) = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n},$$

and the *complete* symmetric polynomial basis, defined as

$$H_n(x) = \sum_{i_1 \le \dots \le i_n} x_{i_1} \cdots x_{i_n}.$$

These bases are connected through the well-known Newton identities:

$$nG_n = \sum_{k=1}^{n} (-1)^{k-1} G_{n-k} F_k, \quad n \in \mathbb{N}, \tag{1}$$

$$nH_n = \sum_{k=1}^{n} H_{n-k} F_k, \quad n \in \mathbb{N}. \tag{2}$$

Let $\mathbb{Z}_+$ denote the set of all nonnegative integers, $\mathbb{Z}_+^n$ — the set of all vectors $\lambda = (\lambda_1, \dots, \lambda_n)$, where $\lambda_i \in \mathbb{Z}_+$. We will use standard notations $|\lambda|_1 = \lambda_1 + 2\lambda_2 + \ldots + n\lambda_n$, $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_n$, $\lambda! = \lambda_1! \cdot \ldots \cdot \lambda_n!$, $F^\lambda = F_1^{\lambda_1} \cdot \ldots \cdot F_n^{\lambda_n}$. Let we denote by $1^{\lambda_1} \cdot 2^{\lambda_2} \cdot \ldots \cdot n^{\lambda_n} = z_n^\lambda$ for $z_n = (1, 2, \dots, n)$. It is well known that elementary and complete symmetric polynomials can be expressed in terms of power symmetric polynomials using the Waring-Girard formulas (see, e.g., [19], pp. 6–7):

$$G_n = \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda \lambda!} F^\lambda \tag{3}$$

and

$$H_n = \sum_{|\lambda|_1=n} \frac{1}{z_n^\lambda \lambda!} F^\lambda$$

where $\lambda \in \mathbb{Z}_+^n$.

Let we denote by $z_{p,n} = (p, \ldots, n)$, $\lambda_{p,n} = (\lambda_p, \ldots, \lambda_n)$, $|\lambda_{p,n}|_1 = p\lambda_p + \ldots + n\lambda_n$, $|\lambda_{p,n}| = \lambda_p + \ldots + \lambda_n$, $z_{p,n}^{\lambda_{p,n}} = p^{\lambda_p} \cdot \ldots \cdot n^{\lambda_n}$, $\lambda_{p,n}! = \lambda_p! \cdot \ldots \cdot \lambda_n!$, $F_{p,n} = (F_p, \ldots, F_n)$ and $F_{p,n}^{\lambda_{p,n}} = (F_p)^{\lambda_p} \ldots (F_n)^{\lambda_n}$. In [8], the Waring-Girard formulas were extended to the Banach space $\ell_p, p > 1$:

$$G_n^{(p)} = \sum_{|\lambda_{p,n}|_1=n} \frac{(-1)^{n+|\lambda_{p,n}|}}{z_{p,n}^{\lambda_{p,n}} \lambda_{p,n}!} F_{p,n}^{\lambda_{p,n}}$$

and

$$H_n^{(p)} = \sum_{|\lambda_{p,n}|_1=n} \frac{-1}{z_{p,n}^{\lambda_{p,n}} \lambda_{p,n}!} F_{p,n}^{\lambda_{p,n}},$$

where $G_n^{(p)}$ and $H_n^{(p)}$ are elementary and complete symmetric polynomials on ℓ_p.

1.2. Block-symmetric polynomials. We denote by $\ell_p(\mathbb{C}^s) = \ell_p(\mathbb{N}, \mathbb{C}^s)$, $1 \le p < \infty$ the linear space of all sequences

$$x = (x_1, x_2, \ldots, x_j, \ldots), \tag{4}$$

such that $x_j = (x_j^{(1)}, \ldots, x_j^{(s)}) \in \mathbb{C}^s$ for $j \in \mathbb{N}$, and the series $\sum_{j=1}^{\infty} \sum_{r=1}^{s} \left| x_j^{(r)} \right|^p$ converges.

Vectors x_j in (4) are referred to as the vector coordinates of x. The linear space $\ell_p(\mathbb{C}^s)$ endowed with the norm

$$\|x\| = \left(\sum_{j=1}^{\infty} \sum_{r=1}^{s} \left| x_j^{(r)} \right|^p \right)^{1/p}$$

is a Banach space. A polynomial P on the space $\ell_p(\mathbb{C}^s)$ is called *block-symmetric (or vector-symmetric)* if:

$$P(x_1, x_2, \ldots, x_m, \ldots) = P(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)}, \ldots)$$

for every permutation $\sigma \in S_{\mathbb{N}}$ and $x_m \in \mathbb{C}^s$. We denote by $\mathcal{P}_{\mathrm{vs}}(\ell_p(\mathbb{C}^s))$ the algebra of all block-symmetric polynomials on $\ell_p(\mathbb{C}^s)$.

More information about algebra $\mathcal{P}_{\mathrm{vs}}(\ell_p(\mathbb{C}^s))$ can be found in [1, 14, 16] and in references therein. Note that in combinatorics, block-symmetric polynomials on finite-dimension spaces are known as *MacMahon symmetric polynomials* (see [21]).

Throughout this paper, we consider multi-indexes $\mathbf{k} = (k_1, k_2, \ldots, k_s)$ with non-negative integer components $k_1, k_2, \ldots, k_s$ and we will use the standard notations $|\mathbf{k}| = k_1 + k_2 + \cdots + k_s$, and $\mathbf{k}! = k_1! k_2! \cdots k_s!$.

According to polynomials ([14])

$$H^{\mathbf{k}}(x) = H^{k_1, k_2, \ldots, k_s}(x) = \sum_{j=1}^{\infty} \prod_{\substack{r=1 \\ |\mathbf{k}| \ge \lceil p \rceil}}^{s} (x_j^{(r)})^{k_r}$$

form an algebraic basis in $\mathcal{P}_{\mathrm{vs}}(\ell_p(\mathbb{C}^s))$, $1 \le p < \infty$, where $x = (x_1, \ldots, x_m, \ldots)$ are in $\ell_p(\mathbb{C}^s)$, $x_j = (x_j^{(1)}, \ldots, x_j^{(s)}) \in \mathbb{C}^s$ and $\lceil p \rceil$ is a ceiling of p. According to [21], these polynomials are known as The Power Sum MacMahon Symmetric Functions.

In the case of the space $\ell_1(\mathbb{C}^s)$ there exist another important algebraic basis (see [17, 21, 24]):

$$R^{\mathbf{k}}(x) = R^{k_1,k_2,\ldots,k_s}(x) = \sum_{\substack{i_1^j<\cdots<i_{k_j}^j,\\ 1\le j\le s}}^{\infty} \prod_{j=1}^{s} x_{i_1^j}^{(j)} \ldots x_{i_{k_j}^j}^{(j)}. \tag{5}$$

In [21] these polynomials are referred to as the Elementary MacMahon Symmetric Functions.

Let $\mathcal{H}(x)(t)$ and $\mathcal{R}(x)(t)$ be the generating functions for $H^{\mathbf{k}}(x)$ and $R^{\mathbf{k}}(x)$ according (see [21])

$$\mathcal{H}(x)(t) = \sum_{|\mathbf{k}|=1}^{\infty} \prod_{i=1}^{s} t_i^{k_i} H^{\mathbf{k}}(x),$$

$$\mathcal{R}(x)(t) = \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(x), \quad R^{\mathbf{0}} = 1.$$

As shown in [17] and [21],

$$\mathcal{R}(x)(t) = \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(x) = \prod_{i=1}^{\infty} (1 + x_i^{(1)} t_1 + \cdots + x_i^{(s)} t_s).$$

Furthermore, from [21], another algebraic basis of homogeneous polynomials $E^{\mathbf{k}}(x)$ is derived from the generating function

$$\mathcal{E}(x)(t) = \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} E^{\mathbf{k}}(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i^{(1)} t_1 - \cdots - x_i^{(s)} t_s}, \quad E^{\mathbf{0}} = 1.$$

These polynomials are known as the Complete Homogeneous MacMahon Symmetric Functions.

Conversely, each polynomial $F_m(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)})$ and $G_m(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)})$ can be expressed as a linear combination of block-symmetric polynomials $H^{\mathbf{k}}(x)$ and $R^{\mathbf{k}}(x)$, respectively. This follows directly from explicit calculations,

$$G_n(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)}) = \sum_{|\mathbf{k}|=n} t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s} R^{\mathbf{k}}(x) \tag{6}$$

and

$$F_n(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)}) = \sum_{|\mathbf{k}|=n} \frac{|\mathbf{k}|!}{\mathbf{k}!} t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s} H^{\mathbf{k}}(x), \tag{7}$$

where $x = (x^{(1)}, \ldots, x^{(s)})$.

Let ω be the isomorphism of $\mathcal{P}_{\mathrm{vs}}(\ell_1(\mathbb{C}^s))$ to itself defined so that $\omega(H^{\mathbf{k}}) = (-1)^{|k|+1} H^{\mathbf{k}}$ for every multi-index $\mathbf{k}$. In other words, if $P \in \mathcal{P}_{\mathrm{vs}}(\ell_1(\mathbb{C}^s))$ is of the form

$$P(x) = Q(H^{\mathbf{k}}, H^{\mathbf{m}}, \ldots, H^{\mathbf{r}}),$$

where Q is a polynomial in several variables, then

$$\omega(P)(x) = Q\big(\omega(H^{\mathbf{k}}), \omega(H^{\mathbf{m}}), \ldots, \omega(H^{\mathbf{r}})\big).$$

The following proposition, proved in [11], establishes an important property of ω.

Proposition 1. *For every multi-index* $\mathbf{k}$,

$$\omega(R^{\mathbf{k}}) = E^{\mathbf{k}} \quad \textit{and} \quad \omega(E^{\mathbf{k}}) = R^{\mathbf{k}}.$$

For given multi-indexes $\mathbf{k}$ and $\mathbf{q}$ we denote by $\mathbf{k} - \mathbf{q} = (k_1 - q_1, k_2 - q_2, \ldots, k_s - q_s)$. In addition, we write $\mathbf{k} \ge \mathbf{q}$ whenever $k_1 \ge q_1$, $k_2 \ge q_2$, ..., $k_s \ge q_s$. From [10, 13] follows the following generalization of Newton's formula (1)

$$nR^{\mathbf{k}} = \sum_{j=1}^{|\mathbf{k}|}(-1)^{j-1}\sum_{|\mathbf{q}|=j,\ \mathbf{k}\geq\mathbf{q}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}R^{\mathbf{k}-\mathbf{q}}.$$

In [11], the following generalization of Newton's formula (2) was proved

$$nE^{\mathbf{k}} = \sum_{j=1}^{|\mathbf{k}|}\sum_{|\mathbf{q}|=j,\ \mathbf{k}\geq\mathbf{q}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}E^{\mathbf{k}-\mathbf{q}}.$$

Let we denote by $\mathbf{q}^m = (q_1^m, q_2^m, \ldots, q_s^m)$ – multi-index with non-negative integer entries $q_1^m, q_2^m, \ldots, q_s^m$, $|\mathbf{q}^m| = q_1^m + q_2^m + \cdots + q_s^m$, $\mathbf{q}^m! = q_1^m!q_2^m!\cdots q_s^m!$.

$$\|\lambda^{\mathbf{q}^i}\|_1 = \sum_{|\mathbf{q}^i|=i}\lambda_i^{\mathbf{q}^i},\ \|\lambda^{\mathbf{q}_r}\|_2 = \sum_{j=1}^{n}\sum_{|\mathbf{q}^j|=j} q_r^j\lambda_j^{\mathbf{q}^j},\ \|\lambda_{p,n}^{\mathbf{q}_r}\|_2 = \sum_{j=p}^{n}\sum_{|\mathbf{q}^j|=j} q_r^j\lambda_j^{\mathbf{q}^j}.$$

In [12] was proved next analogues of Waring-Girard formulas for the block-symmetric polynomials on the space $\ell_1(\mathbb{C}^2)$.

Theorem 1. *For every $\lambda \in \mathbb{Z}_+^n, \lambda_i^{\mathbf{q}^i} \in \mathbb{Z}_+$, $i \in \{1, \ldots, n\}$, $\mathbf{k} = (k_1, k_2), \mathbf{q}^i = (q_1^i, q_2^i) \in \mathbb{Z}_+^2$ we have*

$$R^{\mathbf{k}} = \sum_{|\lambda|_1=n}\frac{(-1)^{n+|\lambda|}}{z_n^{\lambda}}\sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\leq i\leq n\\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ r=1,2}}\prod_{i=1}^{n}\prod_{|\mathbf{q}^i|=i}\left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}}\frac{1}{\lambda_i^{\mathbf{q}^i}}\left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}$$

and

$$E^{\mathbf{k}} = \sum_{|\lambda|_1=n}\frac{1}{z_n^{\lambda}}\sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\leq i\leq n\\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ r=1,2}}\prod_{i=1}^{n}\prod_{|\mathbf{q}^i|=i}\left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}}\frac{1}{\lambda_i^{\mathbf{q}^i}}\left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}.$$

In the case of the spaces $\ell_p(\mathbb{C}^2)$, where p is positive integer, analogs of the Waring-Girard formulas were proven in [12].

$$R_{(p)}^{\mathbf{k}} = \sum_{|\lambda_{p,n}|_1=n}\frac{(-1)^{n+|\lambda_{p,n}|}}{z_{p,n}^{\lambda_{p,n}}}\sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ p\leq i\leq n\\ \|\lambda_{p,n}^{\mathbf{q}_r}\|_2=k_r,\ r=1,2}}\prod_{i=1}^{n}\prod_{|\mathbf{q}^i|=i}\left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}}\frac{1}{\lambda_i^{\mathbf{q}^i}}\left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}.$$

and

$$E_{(p)}^{\mathbf{k}} = \sum_{|\lambda_{p,n}|_1=n}\frac{1}{z_{p,n}^{\lambda_{p,n}}}\sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ p\leq i\leq n\\ \|\lambda_{p,n}^{\mathbf{q}_r}\|_2=k_r,\ r=1,2}}\prod_{i=1}^{n}\prod_{|\mathbf{q}^i|=i}\left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}}\frac{1}{\lambda_i^{\mathbf{q}^i}}\left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}$$

1.3. Block-supersymmetric polynomials. In [11] was studied algebra of block-supersymmetric polynomials. Let us denote by $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$ the space of sequences

$$z = (\ldots, z_{-n}, \ldots, z_{-1}, z_1, \ldots, z_n, \ldots) = (y|x) = (\ldots, y_n, \ldots, y_1|x_1, \ldots, x_n, \ldots)$$

with

$$||z|| = \sum_{i=-\infty}^{\infty}||z_i|| = \sum_{i=-\infty}^{\infty}\sum_{j=1}^{s}|z_i^{(j)}|,$$

where both $x = (x_1, \ldots, x_n, \ldots)$ and $y = (y_1, \ldots, y_n, \ldots)$ are in $\ell_1(\mathbb{C}^s)$, $x_i = (x_i^{(1)}, \ldots, x_i^{(s)})$ and $y_i = (y_i^{(1)}, \ldots, y_i^{(s)})$ are in $\mathbb{C}^s$ with $z_n = x_n$, $z_{-n} = y_n$ for $n \in \mathbb{N}$.

Let us consider the next polynomials on $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$:

$$T^{\mathbf{k}}(z) = H^{\mathbf{k}}(x) - H^{\mathbf{k}}(y) = \sum_{j=1}^{\infty} \prod_{\substack{r=1 \\ |k|\geq 1}}^{s} (x_j^{(r)})^{k_r} - \sum_{j=1}^{\infty} \prod_{\substack{r=1 \\ |k|\geq 1}}^{s} (y_j^{(r)})^{k_r}, \mathbf{k} = (k_1, \ldots, k_s).$$

A polynomial P on $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$ is called *block-supersymmetric* if it is an algebraic combination of polynomials $\left\{T^{\mathbf{k}}\right\}_{|\mathbf{k}|=1}^{\infty}$. We denote by $\mathcal{P}_{\text{vsup}}$ the algebra of all block-supersymmetric polynomials on $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$.

In [11] was proved that polynomials $T^{\mathbf{k}}$ form an algebraic basis in $\mathcal{P}_{\text{vsup}}$ and that

$$W^{\mathbf{n}}(y|x) = \sum_{\mathbf{k}\leq\mathbf{n}} R^{\mathbf{k}}(x) E^{\mathbf{n}-\mathbf{k}}(-y), \quad \mathbf{n} = (n_1, \ldots, n_s).$$

form other algebraic basis on $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$.

Let us denote by Λ an algebraic isomorphism from $\mathcal{P}_{\text{vs}} = \mathcal{P}_{\text{vs}}(\ell_1(\mathbb{C}^s))$ to $\mathcal{P}_{\text{vsup}}$ such that $\Lambda\colon H^{\mathbf{k}} \longmapsto T^{\mathbf{k}}, \quad \mathbf{k} = (k_1, \ldots, k_s)$. Note that $\Lambda(R^{\mathbf{k}}) = W^{\mathbf{k}}$.

From [11] follows Newton-type formulas for block-supersymmetric polynomials

$$nW^{\mathbf{k}} = \sum_{j=1}^{|\mathbf{k}|} (-1)^{j-1} \sum_{|\mathbf{q}|=j,\ \mathbf{k}\geq\mathbf{q}} \frac{|\mathbf{q}|!}{\mathbf{q}!} T^{\mathbf{q}} W^{\mathbf{k}-\mathbf{q}}.$$

and

$$n\widetilde{W}^{\mathbf{k}} = \sum_{j=1}^{|\mathbf{k}|} \sum_{|\mathbf{q}|=j,\ \mathbf{k}\geq\mathbf{q}} \frac{|\mathbf{q}|!}{\mathbf{q}!} T^{\mathbf{q}} \widetilde{W}^{\mathbf{k}-\mathbf{q}},$$

where $\widetilde{W}^{\mathbf{k}} = \Lambda\big(E^{\mathbf{k}}\big)$.

2. Main results.

2.1. Analogues of Waring-Girard formulas for the block-symmetric polynomials on the space $\ell_1(\mathbb{C}^s)$. In a similar way as in [8] we obtain next theorem

Theorem 2. *For every $\lambda_i, \lambda_i^{\mathbf{q}^i}, k_j, q_j^i \in \mathbb{Z}_+$, $i \in \{1, \ldots, n\}, j \in \{1, \ldots, s\}$ we have*

$$R^{\mathbf{k}} = \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^{\lambda}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\leq i\leq n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\leq r\leq s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} \tag{8}$$

and

$$E^{\mathbf{k}} = \sum_{|\lambda|_1=n} \frac{1}{z_n^{\lambda}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\leq i\leq n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\leq r\leq s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}. \tag{9}$$

Proof. From formulas (6) and (7) we have that

$$F_n(t_1x^{(1)} + t_2x^{(2)} + \ldots + t_sx^{(s)}) = \sum_{|\mathbf{q}^n|=n} \frac{|\mathbf{q}^n|!}{\mathbf{q}^n!} t_1^{q_1^n} t_2^{q_2^n} \ldots t_s^{q_s^n} H^{\mathbf{q}^n}(x_1, x_2, \ldots, x_s)$$

and

$$G_n(t_1x^{(1)} + t_2x^{(2)} + \ldots + t_sx^{(s)}) = \sum_{|\mathbf{k}|=n} t_1^{k_1} t_2^{k_2} \ldots t_s^{k_s} R^{\mathbf{k}}(x_1, x_2, \ldots, x_s). \tag{10}$$

From direct calculations we obtain

$$\big(F_n(t_1x^{(1)} + t_2x^{(2)} + \ldots + t_sx^{(s)})\big)^{\lambda_n} = \left(\sum_{|\mathbf{q}^n|=n} \frac{|\mathbf{q}^n|!}{\mathbf{q}^n!} t_1^{q_1^n} t_2^{q_2^n} \ldots t_s^{q_s^n} H^{\mathbf{q}^n}(x_1, x_2, \ldots, x_s)\right)^{\lambda_n} =$$

$$= \sum_{\|\lambda^{\mathbf{q}_n}\|_1=\lambda_n} t_1^{\sum\limits_{|\mathbf{q}^n|=n} q_1^n \lambda_n^{\mathbf{q}^n}} \ldots t_s^{\sum\limits_{|\mathbf{q}^n|=n} q_s^n \lambda_n^{\mathbf{q}^n}} \frac{\lambda_n!}{\prod\limits_{|\mathbf{q}^n|=n} \lambda_n^{\mathbf{q}^n}} \prod_{|\mathbf{q}^n|=n} \left(\frac{|\mathbf{q}^n|!}{\mathbf{q}^n!}\right)^{\lambda_n^{\mathbf{q}^n}} \left(H^{\mathbf{q}^n}\right)^{\lambda_n^{\mathbf{q}^n}}. \tag{11}$$

If we put (10) and (11) to the formula (3) we obtain

$$\sum_{|\mathbf{k}|=n} t_1^{k_1} t_2^{k_2} \dots t_s^{k_s} R^{\mathbf{k}}(x_1, x_2, \dots, x_s) = \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda \lambda!} \times$$

$$\times \left(F_1(t_1 x^{(1)} + \dots + t_s x^{(s)})\right)^{\lambda_1} \times \dots \times \left(F_n(t_1 x^{(1)} + \dots + t_s x^{(s)})\right)^{\lambda_n} = \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda \lambda!} \times$$

$$\times \Bigg(\sum_{\|\lambda^{\mathbf{q}^1}\|_1=\lambda_1} t_1^{\sum\limits_{|\mathbf{q}^1|=1} q_1^1 \lambda_1^{\mathbf{q}^1}} \dots t_s^{\sum\limits_{|\mathbf{q}^1|=1} q_s^1 \lambda_1^{\mathbf{q}^1}} \frac{\lambda_1!}{\prod\limits_{|\mathbf{q}^1|=1} \lambda_1^{\mathbf{q}^1}} \prod_{|\mathbf{q}^1|=1} \left(\frac{|\mathbf{q}^1|!}{\mathbf{q}^1!}\right)^{\lambda_1^{\mathbf{q}^1}} \left(H^{\mathbf{q}^1}\right)^{\lambda_1^{\mathbf{q}^1}} \Bigg) \times \dots$$

$$\times \Bigg(\sum_{\|\lambda^{\mathbf{q}^n}\|_1=\lambda_n} t_1^{\sum\limits_{|\mathbf{q}^n|=n} q_1^n \lambda_n^{\mathbf{q}^n}} \dots t_s^{\sum\limits_{|\mathbf{q}^n|=n} q_s^n \lambda_n^{\mathbf{q}^n}} \frac{\lambda_n!}{\prod\limits_{|\mathbf{q}^n|=n} \lambda_n^{\mathbf{q}^n}} \prod_{|\mathbf{q}^n|=n} \left(\frac{|\mathbf{q}^n|!}{\mathbf{q}^n!}\right)^{\lambda_n^{\mathbf{q}^n}} \left(H^{\mathbf{q}^n}\right)^{\lambda_n^{\mathbf{q}^n}} \Bigg) = \tag{12}$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda \lambda!} \times$$

$$\times \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i \\ 1\le i\le n}} t_1^{\sum\limits_{r=1}^{n} \sum\limits_{|\mathbf{q}^r|=r} q_1^r \lambda_r^{\mathbf{q}^r}} \dots t_s^{\sum\limits_{r=1}^{n} \sum\limits_{|\mathbf{q}^r|=r} q_s^r \lambda_r^{\mathbf{q}^r}} \prod_{i=1}^{n} \lambda_i! \prod_{|\mathbf{q}^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i \\ 1\le i\le n}} t_1^{\|\lambda^{\mathbf{q}_1}\|_2} \dots t_s^{\|\lambda^{\mathbf{q}_s}\|_2} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}.$$

If equate multipliers at the all powers of $t_i, 1 \le i \le s$ we obtain the required formula (8). By applying the isomorphism ω to Equation (8), we have

$$E^{\mathbf{k}} = \omega(R^{\mathbf{k}}) = \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda} \times \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(\omega(H^{\mathbf{q}^i})\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} (-1)^{|\mathbf{q}^i|\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} (-1)^{2\lambda_1+3\lambda_2+\dots+(n+1)\lambda_n} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|_1+2|\lambda|}}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{1}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}.$$

□

Formula (3) is useful in combinatorics. It is well-known combinatorial identity

$$\sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^{\lambda}\lambda!} = 0.$$

It can be obtained if we compute $G_n(e_1)$ using (3), where $e_1 = (1,0,0,\ldots)$. In [12] we obtained some new combinatorial identity

$$\sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^{\lambda}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n\\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ r=1,2}} \prod_{i=1}^{n}\prod_{|q^i|=i} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \frac{1}{\lambda_i^{\mathbf{q}^i}} = 0.$$

In the case $\ell_1(\mathbb{C}^s)$ let $e_1 = (1,0,\ldots,0,\ldots)$, where $1 = (\underbrace{1,\ldots,1}_{s})$ and $0 = (\underbrace{0,\ldots,0}_{s})$. Then $H^{\mathbf{k}}(e_1) = 1$ and $R^{\mathbf{k}}(e_1) = 0$. By the similar way we obtain next combinatorial identity

$$\sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^{\lambda}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n\\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n}\prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} = 0.$$

2.2. Analogues of Waring-Girard formulas for the block-symmetric polynomials on the space $\ell_p(\mathbb{C}^s)$. In the case of spaces $\ell_p(\mathbb{C}^s)$, where p is positive integer numbers we can obtain analog of the Waring-Girard formulas. If p is not integer, then we can take $\lceil p \rceil$ instead of p.

Proposition 2. *For every $\lambda_i, \lambda_i^{\mathbf{q}^i}, k_j, q_j^i \in \mathbb{Z}_+$, $i \in \{p,\ldots,n\}, j \in \{1,\ldots,s\}$ we have*

$$R_{(p)}^{\mathbf{k}} = \sum_{|\lambda_{p,n}|_1=n} \frac{(-1)^{n+\lambda_{p,n}}}{z_{p,n}^{\lambda_{p,n}}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ p\le i\le n\\ \|\lambda_{p,n}^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=p}^{n}\prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} \tag{13}$$

and

$$E_{(p)}^{\mathbf{k}} = \sum_{|\lambda_{p,n}|_1=n} \frac{1}{z_{p,n}^{\lambda_{p,n}}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ p\le i\le n\\ \|\lambda_{p,n}^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=p}^{n}\prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}. \tag{14}$$

Proof. If we put $F_i = 0$, for all $1 \le i \le p$ to the formulas (12) we obtain

$$\sum_{|\mathbf{k}|=n} t_1^{k_1} t_2^{k_2} \ldots t_s^{k_s} R^{\mathbf{k}}(x_1,x_2,\ldots,x_s) = \sum_{|\lambda_{p,n}|_1=n} \frac{(-1)^{n+|\lambda_{p,n}|}}{z_{p,n}^{\lambda_{p,n}}\lambda_{p,n}!} \times$$

$$\times \left(F_p(t_1x^{(1)}+\ldots+t_sx^{(s)})\right)^{\lambda_p} \times \ldots \times \left(F_n(t_1x^{(1)}+\ldots+t_sx^{(s)})\right)^{\lambda_n} = \sum_{|\lambda_{p,n}|_1=n} \frac{(-1)^{n+|\lambda_{p,n}|}}{z_{p,n}^{\lambda_{p,n}}\lambda_{p,n}!} \times$$

$$\times \Bigg(\sum_{\|\lambda^{\mathbf{q}^p}\|_1=\lambda^{\mathbf{q}^p}} t_1^{\sum\limits_{|\mathbf{q}^p|=p} q_1^p \lambda_p^{\mathbf{q}^p}} \ldots t_s^{\sum\limits_{|\mathbf{q}^p|=p} q_s^p \lambda_p^{\mathbf{q}^p}} \frac{\lambda_p!}{\prod\limits_{|\mathbf{q}^p|=p} \lambda_p^{\mathbf{q}^p}} \prod_{|\mathbf{q}^p|=p} \left(\frac{|\mathbf{q}^p|!}{\mathbf{q}^p!}\right)^{\lambda_p^{\mathbf{q}^p}} \left(H^{\mathbf{q}^p}\right)^{\lambda_p^{\mathbf{q}^p}} \Bigg) \times \ldots \times$$

$$\times \Bigg(\sum_{\|\lambda^{\mathbf{q}^n}\|_1=\lambda^{\mathbf{q}^n}} t_1^{\sum\limits_{|\mathbf{q}^n|=n} q_1^n \lambda_n^{\mathbf{q}^n}} \ldots t_s^{\sum\limits_{|\mathbf{q}^n|=n} q_s^n \lambda_n^{\mathbf{q}^n}} \frac{\lambda_n!}{\prod\limits_{|\mathbf{q}^n|=n} \lambda_n^{\mathbf{q}^n}} \prod_{|\mathbf{q}^n|=n} \left(\frac{|\mathbf{q}^n|!}{\mathbf{q}^n!}\right)^{\lambda_n^{\mathbf{q}^n}} \left(H^{\mathbf{q}^n}\right)^{\lambda_n^{\mathbf{q}^n}} \Bigg) =$$

$$\sum_{|\lambda_{p,n}|_1=n} \frac{(-1)^{n+|\lambda_{p,n}|}}{z_{p,n}^{\lambda_{p,n}}} \sum_{\|\lambda^{\mathbf{q}^n}\|_1=\lambda_i,\ p\le i\le n} t_1^{\|\lambda_{p,n}^{\mathbf{q}_1}\|_2} \ldots t_s^{\|\lambda_{p,n}^{\mathbf{q}_s}\|_2} \prod_{i=p}^{n}\prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}.$$

If equate multipliers at the all powers of $t_i, 1 \le i \le s$ we obtain the required formula (13).

By applying the isomorphism ω to equation (13) as in Theorem 2 we obtain (14). □

2.3. Analogues of Waring-Girard formulas for the block-supersymmetric polynomials on the space $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$. To obtain some Waring-Girard formulas for block-supersymmetric polynomials, we have to apply the isomorphism Λ to corresponding Waring-Girard formulas for block-symmetric polynomials. Applying Λ to (8) and (9), we obtain

$$W^{\mathbf{k}} = \Lambda(R^{\mathbf{k}}) = \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^{\lambda}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(\Lambda(H^{\mathbf{q}^i})\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^{\lambda}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(T^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}$$

and

$$\widetilde{W}^{\mathbf{k}} = \Lambda(E^{\mathbf{k}}) = \sum_{|\lambda|_1=n} \frac{1}{z_n^{\lambda}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(\Lambda(H^{\mathbf{q}^i})\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{1}{z_n^{\lambda}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(T^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}.$$

2.4. The finite dimensional case for block-supersymmetric polynomials. Let us denote by $\big(\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})\big)_m$ the finite-dimensional Banach space of all sequences

$$z = (z_{-m}, \ldots, z_{-1}|z_1, \ldots, z_m) = (y|x) = (y_m, \ldots, y_1|x_1, \ldots, x_m)$$

and $\mathcal{P}^m_{\mathrm{vsup}}$ — the algebra of block-supersymmetric polynomials on the $\big(\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})\big)_m$. Clearly, $\big(\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})\big)_m$ is a subspace of $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$. The number of homogeneous polynomials $T^{\mathbf{k}}$ for $|\mathbf{k}| \le 2m$ and $2ms$ independent variables is given by

$$\sum_{l=1}^{2m} \frac{(l+1)(l+2)\ldots(l+s-1)}{(s-1)!}.$$

The set of generators consists of supersymmetric polynomials and block-supersymmetric polynomials that are not supersymmetric. The number of supersymmetric polynomials $T^{\mathbf{k}}$ in $\big(\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})\big)_m$ is equal to $2ms$. These supersymmetric polynomials are algebraically independent. The other elements of the system generators are block-supersymmetric but not supersymmetric polynomials. We say that a system τ^m_{vsup} of generating elements in $\mathcal{P}^m_{\mathrm{vsup}}$ is *reasonable* if it contains $2ms$ algebraically independent supersymmetric polynomials.

Thus, the system of generators consisting of the restrictions of $T^{\mathbf{k}}$ to $\big(\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})\big)_m$ must have at least

$$N = \sum_{l=1}^{2m} \frac{(l+1)(l+2)\ldots(l+s-1)}{(s-1)!} - 2sm$$

algebraic dependencies. The same is true if we take another algebraic basis instead of $T^{\mathbf{k}}$.

It is known [11] in the case $\big(\ell_1(\mathbb{C}^2_{\mathbb{Z}_0})\big)_1$ with vectors

$$(y|x) = (y_1|x_1) = \left(\begin{pmatrix} y_1^{(1)} \\ y_1^{(2)} \end{pmatrix} \middle| \begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix} \right)$$

the following identities holds

$$\xi_5\xi_1\xi_2 - \frac{1}{2}\xi_1^2\xi_4 - \frac{1}{2}\xi_2^2\xi_3 \equiv 0,$$

where $T^{(1,0)} = x_1^{(1)} - y_1^{(1)} = \xi_1$, $T^{(0,1)} = x_1^{(2)} - y_1^{(2)} = \xi_2$, $T^{(2,0)} = (x_1^{(1)})^2 - (y_1^{(1)})^2 = \xi_3$, $T^{(0,2)} = (x_1^{(2)})^2 - (y_1^{(2)})^2 = \xi_4$, $T^{(1,1)} = x_1^{(1)}x_1^{(2)} - y_1^{(1)}y_1^{(2)} = \xi_5$,

and

$$\omega_5\omega_1\omega_2 - \omega_1^2\omega_4 - \omega_2^2\omega_3 \equiv 0,$$

where $W^{(1,0)} = x_1^{(1)} - y_1^{(1)} = \omega_1$, $W^{(0,1)} = x_1^{(2)} - y_1^{(2)} = \omega_2$, $W^{(2,0)} = -x_1^{(1)}y_1^{(1)} + (y_1^{(1)})^2 = \omega_3$, $W^{(0,2)} = -x_1^{(2)}y_1^{(2)} + (y_1^{(2)})^2 = \omega_4$, $W^{(1,1)} = 2y_1^{(1)}y_1^{(2)} - x_1^{(1)}y_1^{(2)} - x_1^{(2)}y_1^{(1)} = \omega_5$.

The following theorem demonstrates algebraic dependencies between generating elements of $\mathcal{P}^m_{\text{vsup}}$.

Theorem 3. *Let τ^m_{vsup} be a reasonable system of the generating elements of $\mathcal{P}^m_{\text{vsup}}$. Then for every nonsupersymmetric polynomial ψ in τ^m_{vsup} there exist supersymmetric polynomials b_k such that*

$$\psi^{(m!)^{r-1}} - a_1\psi^{(m!)^{r-1}-1} + \ldots + (-1)^{(m!)^{r-1}-1}a_{(m!)^{r-1}-1}\psi + (-1)^{(m!)^{r-1}}a_{(m!)^{r-1}} \equiv 0,$$

where r is the number of nonzero elements $k_i, i = 1, \ldots, s$ in the multi-index $\mathbf{k}$.

Proof. In [15], a method for determining algebraic dependencies in a reasonable system of polynomial generators is presented. It was proven that for every block-symmetric but nonsymmetric polynomial ξ in a reasonable system of the form (5), there exist symmetric polynomials a_k such that

$$\xi^{(m!)^{r-1}} - a_1\xi^{(m!)^{r-1}-1} + \ldots + (-1)^{(m!)^{r-1}-1}a_{(m!)^{r-1}-1}\xi + (-1)^{(m!)^{r-1}}a_{(m!)^{r-1}} \equiv 0, \tag{15}$$

where r is the number of nonzero elements $k_i, i = 1, \ldots, s$ in the multi-index $\mathbf{k}$. Applying the isomorphism Λ to identity (15), we obtain

$$\Lambda(\xi)^{(m!)^{r-1}} - \Lambda(a_1)\Lambda(\xi)^{(m!)^{r-1}-1} + \ldots + (-1)^{(m!)^{r-1}-1}\Lambda(a_{(m!)^{r-1}-1})\Lambda(\xi) + (-1)^{(m!)^{r-1}}\Lambda(a_{(m!)^{r-1}})$$

$$= \psi^{(m!)^{r-1}} - b_1\psi^{(m!)^{r-1}-1} + \ldots + (-1)^{(m!)^{r-1}-1}b_{(m!)^{r-1}-1}\psi + (-1)^{(m!)^{r-1}}b_{(m!)^{r-1}} \equiv 0,$$

where $\psi = \Lambda(\xi)$ is block-supersymmetric but nonsupersymmetric polynomials and $b_k = \Lambda(a_k)$ are the supersymmetric polynomials. □

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Vasyl Stefanyk Carpathian National University
Ivano-Frankivsk, Ukraine

viktoriia.kravtsiv@pnu.edu.ua
pavlo.dolishniak.22@pnu.edu.ua
rostyslav.stakhiv.22@pnu.edu.ua

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