V. M. Petechuk, Yu. V. Petechuk

RESIDUAL AND FIXED MODULES

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The paper presents some sufficient conditions for the commutativity of transvections with elements of linear groups over division ring in the language of residual and fixed submodules. The residual and fixed submodules of the element σ of the linear group are defined as the image and nucleus of the element $\sigma-1$ and are denoted by $R(\sigma)$ and $P(\sigma)$, respectively. It is proved that transvection σ_1 over an arbitrary body commutes with an element σ_2 for which $\dim R(\sigma_2)=\dim R(\sigma_2)\cap P(\sigma_2)+l$, $l\le 1$, if and only if the inclusion system $R(\sigma_1)\subseteq P(\sigma_2)$, $R(\sigma_2)\subseteq P(\sigma_1)$. It is shown that for $l>1$ this statement is not always true.

Introduction. The condition of switching elements of linear groups of modules over associative rings in the language of residual and fixed modules play an important role in the theory of linear groups over rings. The residual and fixed submodules of the element Σ of the linear group are defined as the image and nucleus of the element $\Sigma-1$ and are denoted by $R(\Sigma)$ and $P(\Sigma)$ respectively. From the system of simultaneous inclusions $R(\Sigma_1)\subseteq P(\Sigma_2)$ and $R(\Sigma_2)\subseteq P(\Sigma_1)$, it follows that the elements Σ_1 and Σ_2 are commutative. The reverse is not always true. This paper is devoted to finding sufficient conditions under which the commutativity of the elements Σ_1 and Σ_2 follows the system of the above-mentioned inclusions. It is proved that if transvection Σ_1 over an arbitrary body commutes with an element Σ_2 for which $\dim R(\Sigma_2)=\dim R(\Sigma_2)\cap P(\Sigma_2)+l$, $l\le 1$, then the above system of inclusions holds. It is shown that for $l>1$ this statement is not always true.

Let R be an associative ring of 1, $E(n,R)$ be a subgroup of $GL(n,R)$, generated by all elementary transvections $t_{ij}(R)=1+re_{ij}$, $r\in R$, $1\le i\ne j\le n$, e_{ij} is the standard matrix unit in which (i,j) stands at 1, and at other places there are zeros.

Let V be an arbitrary left R-module, $GL(V)$ be a group of automorphisms of module V. If V is a left free module of finite size n, then we denote $GL(V)$ by $GL(n,V)$. In the fixed base of module V, the group $GL(n,V)$ is identified with the group $GL(n,R)$.

The residual and fixed submodules of module V of the endomorphism Σ are called submodules $R(\Sigma)=(\Sigma-1)V$ and $P(\Sigma)=ker(\Sigma-1)$ in accordance. It is clear that $R(\sigma)=\{(\sigma-1)v\mid v\in V\}$ and $P(\sigma)=\{v\in V\mid \sigma v=v\}$, as well as $R(1-\sigma)=\sigma V$ and $P(1-\sigma)=\ker\sigma$.

It is easy to see that if Σ is an automorphism of module V, then $\sigma^{-1}-1=(\sigma-1)(-\sigma^{-1})$ and $R(\sigma^{-1})=R(\sigma)$, $P(\sigma^{-1})=P(\sigma)$.

If W is the R-submodule of module V, then

$$\sigma W=(\sigma-1+1)W\subseteq R(\sigma)+W.$$

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Similarly, $\sigma^{-1}W = (\sigma^{-1} - 1 + 1)\,W \subseteq R(\sigma^{-1}) + W = R(\sigma) + W$, if σ – is an automorphism of module V. In particular, if $R(\sigma) \subseteq W$, then $\sigma^{\pm 1}W \subseteq W$ and, as a consequence, $\sigma W = W$. This equality holds when $W \subseteq P(\sigma)$.

Note that if g is an arbitrary endomorphism of module V such that one of the equations $g\sigma = \sigma^{\pm 1}g$ holds, then $g(\sigma - 1) = (\sigma^{\pm 1} - 1)\,g$ and $(\sigma - 1)\,g = g(\sigma^{\pm 1} - 1)$.

This means that $gR(\sigma) \subseteq R(\sigma^{\pm 1}) = R(\sigma)$ and $gP(\sigma) \subseteq P(\sigma^{\pm 1}) = P(\sigma)$.

In particular, if g is an automorphism of module V and one of the equations $g\sigma g^{-1} = \sigma^{\pm 1}$ holds, then $gR(\sigma) = R(\sigma)$ and $gP(\sigma) = P(\sigma)$. This statement also follows from the general formulas

$$R\left(g\sigma g^{-1}\right) = gR(\sigma)\,, P\left(g\sigma g^{-1}\right) = gP(\sigma)\,,$$

where g is the automorphism of module V and σ is the endomorphism of module V.

If g is an automorphism of the module V, then $gV = V$ and $g\sigma g^{-1} - 1 = g(\sigma - 1)\,g^{-1}$.

Since $\sigma_1\sigma_2 - 1 = (\sigma_1 - 1)\,\sigma_2 + \sigma_2 - 1 = \sigma_1(\sigma_2 - 1) + \sigma_1 - 1$, then

$$R(\sigma_1\sigma_2) \subseteq R(\sigma_1) + R(\sigma_2)\,, P(\sigma_1\sigma_2) \supseteq P(\sigma_1) \cap P(\sigma_2)\,.$$

In particular, if $[\sigma_1, \sigma_2] = \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}$, then

$$R([\sigma_1, \sigma_2]) \subseteq R(\sigma_1) + R\left(\sigma_2\sigma_1^{-1}\sigma_2^{-1}\right) = R(\sigma_1) + \sigma_2 R(\sigma_1) \subseteq R(\sigma_1) + R(\sigma_2)\,,$$
$$P([\sigma_1, \sigma_2]) \supseteq P(\sigma_1) \cap P(\sigma_2)\,.$$

Lemma 1. *The inclusion of $R(\sigma_1) \subseteq P(\sigma_2)$ holds if and only if $(\sigma_2 - 1)(\sigma_1 - 1) = 0$, that is, when $\sigma_2\sigma_1 = \sigma_1 + \sigma_2 - 1$.*

Proof. The proof is obvious. □

It follows from Lemma 1 that the system

$$\begin{cases} R(\sigma_1) \subseteq P(\sigma_2) \\ R(\sigma_2) \subseteq P(\sigma_1) \end{cases} \tag{1}$$

occurs if and only if

$$(\sigma_2 - 1)(\sigma_1 - 1) = (\sigma_1 - 1)(\sigma_2 - 1) = 0 \tag{2}$$

This means that system (1) holds if and only if

$$\sigma_2\sigma_1 = \sigma_1 + \sigma_2 - 1 = \sigma_1\sigma_2 \tag{3}$$

Thus, conditions (1)–(3) are equivalent and from them follows the commutativity of the endomorphisms Σ_1 and Σ_2. On the contrary, it is not always true. Finding the conditions under which the commutativity of endomorphisms Σ_1 and Σ_2 implies system (1) is the main purpose of this article.

Obviously, if endomorphisms σ_1 and σ_2 are commuting and at least one inclusion of system (1) takes place, then according to (3) there is also a second inclusion of system (1).

Lemma 2. *Let σ_1 and σ_2 commute, $R(\sigma_1) \cap R(\sigma_2) = 0$ or $(\sigma_1) + P(\sigma_2) = V$. Then system (1) takes place.*

Proof. Since inclusion

$$(\sigma_1 - 1)(\sigma_2 - 1) V \subseteq R(\sigma_1) \cap R(\sigma_2)$$

follows the first assertion of Lemma 2, and from equality

$$(\sigma_1 - 1)(\sigma_2 - 1)(P(\sigma_1) + P(\sigma_2)) = (\sigma_2 - 1)(\sigma_1 - 1)(P(\sigma_1) + P(\sigma_2)) = 0$$

the second assertion of Lemma 2 follows. □

It is easy to see that Σ is an involution, that is, $\Sigma^2 = 1$, if and only if $\Sigma\big|_{R(\Sigma)} = -1$.

If $e^2 = e \epsilon GL(V)$, then $V = eV \oplus (1-e)V$, $R(e) = (1-e)V$, $P(e) = eV$.

In particular, if $m \epsilon K^*$ and $\sigma^m = 1$, then $e = \dfrac{1 + \cdots + \sigma^{m-1}}{m}$ is an idempotent and

$$R(\sigma) = R(e), P(\sigma) = P(e), V = R(\sigma) \oplus P(\sigma).$$

Properties of excess and fixed submodules are widely used. In particular, they are applied to describe the homomorphisms of matrix groups over associative rings with units [1–3].

In [2, 3] the authors proved

Theorem 1. *Let R, K be associative rings of 1, V be a left (not necessarily free) K-module,*

$$E(n,R) \subseteq G \subseteq GL(n,R), E(n,R) = \langle t_{ij}(r) |\, 1 \leq i \neq j \leq n,\ r \epsilon R \rangle,$$

$\Lambda\colon G \to GL(V)$ is a homomorphism with condition (*), that is, for an arbitrary nonzero nilpotent element $m \epsilon EndV$, $m^2 = 0$ there are natural numbers s_1 and s_2 that are reversible in K and $A \epsilon G$ such that $\Lambda A = 1 + s_1 m$ and from the equality $\Lambda A \bullet \Lambda B = \Lambda B \bullet \Lambda A$, $B \epsilon G$ implies that $A^{s_2} B = B A^{s_2}$.

Then, at $n \geq 4$, there is an isomorphism $g\colon V \to V_g = \underbrace{L \oplus \cdots \oplus L}\oplus P$ such that

$$\Lambda(x) = g^{-1}\left[\overline{\delta}(x)e + \overline{v}\left(x^{-1}\right)(1-e) + e_1\right] g, x \epsilon E(n,R)$$

where L occurs n times, e is the central idempotent of the $EndL$, ring, e_1 is the unit of the $EndP$, $\delta\colon R \to EndL$ is the ring homomorphism, $v\colon R \to EndL$ is the ring antihomomorphism, $\overline{\delta}$ and $\overline{v}$ are the annular matrix homomorphism and antihomomorphism induced δ and V, respectively.

Theorem 1 implies a description of the isomorphisms of the groups $GL(n,R)$ and $GL(m,K)$ at $n, m \geq 4$ over the associative rings R and K of 1, which, by developing the technique associated with idempotents, was carried out by I.Z. Golubchik ([4]). It turned out that they allow a standard description on the group $E(n,R)$.

The method of excess and fixed submodules was initiated by O'Meara ([5]).

O'Meara first proposed the use of residual and fixed subspaces to describe isomorphisms of matrix groups rich in projective transvections in dimensions ≥ 5, and Yu. V. Sosnowski ([6]) extended this description to dimensions ≥ 3.

A shorter version of the proof O'Meara-Sosnovsky's theorem has proposed by V. M. Petechuk ([7]).

It turned out that all the isomorphisms are standard.

The theory of residual and fixed modules is most systematically presented in [8]. The statement in [8] on p. 122 is not true.

An endomorphism $\sigma \in GL(V)$ is unipotent if $(\sigma-1)^k = 0$ for some $k \geq 0$. It is clear that the unipotent element is reversible. If $(\sigma-1)^{k-1} \neq 0$, and $(\sigma-1)^k = 0$, then K is called the degree (height) of nilpotency of the element $\sigma - 1$ and, accordingly, the level (height) of unipotency of the element σ. The level of 1 is 1. Any matrix in the group of upper triangular matrices of the group $UT(n, R)$ with units on the diagonal satisfies the equality $(\sigma-1)^n = 0$. Therefore, the level of unipotency of any matrix of the group $UT(n, R)$ does not exceed n.

Obviously, $\sigma = 1$ if and only if $R(\sigma) = 0$ or $P(\sigma) = V$.

Lemma 3. *The element $\sigma \in GL(V)$ is unipotent of level 2 if and only if $0 \subset R(\sigma) \subseteq P(\sigma)$.*

Proof. If σ is a unipotent element of level 2, then $\sigma - 1 \neq 0$, but $(\sigma-1)^2 = 0$ and $0 \subset R(\sigma) = (\sigma-1)V \subseteq \ker(\sigma-1) = P(\sigma)$.

If $0 \subset R(\sigma) \subseteq P(\sigma)$, then $(\sigma-1)V \subseteq P(\sigma)$. Therefore $(\sigma-1)^2 = 0$, but $\sigma - 1 \neq 0$. □

Definition 1. A submodule W of module V is called a *hyperplane* if there is an element $v \in V$, $v \neq 0$ such that module V is a direct sum of $V = W \oplus \langle v \rangle$, where $\langle v \rangle$ is a submodule of module V generated by the element v.

Definition 2. The endomorphism σ of module V is called *transvection* if $P(\sigma)$ is a hyperplane of module V and $R(\sigma) \subseteq P(\sigma)$.

Lemma 4. *Transvection Σ is a unipotent element of level 2.*

Proof. By definition, $V = P(\sigma) \oplus \langle v \rangle$. Therefore, $v \neq 0$, $\sigma \neq 1$, $R(\sigma) \neq 0$, $0 \subset R(\sigma) \subseteq P(\sigma)$. According to Lemma 3, Σ is a unipotent element of level 2.

It is clear that not every unipotent element of level 2 is transvection.

Let R be a division ring, V be a finite-dimensional vector space over R. As is known from linear algebra $\dim V = \dim R(\sigma) + \dim P(\sigma)$. In particular, if σ is transvection, then $R(\sigma) = \langle v \rangle$, $\dim R(\sigma) = 1$, $\dim P(\sigma) = n - 1$.

Based on $e_1, \ldots, e_n$ of module V, where $R(\sigma) = \langle e_1 \rangle$, $P(\sigma) = \langle e_1, \ldots, e_{n-1} \rangle$ transvection σ has the form $t_{1n}(r)$, where $r \in R$. □

Lemma 5. *Let R be a division ring, V be a finite-dimensional vector space over R,* $\dim V = n \geq 2$, *σ_1 be a transvection, and σ_2 be an element of the group $GL(n, V)$ such that $\sigma_1\sigma_2 = \sigma_2\sigma_1$. Then system* (1) *or $R(\sigma_1) \subseteq R(\sigma_2)$ takes place.*

Proof. Let $e_1, \ldots, e_n$ be the base of a vector space V. We assume that $R(\sigma_1) = \langle e_1 \rangle$, $P(\sigma_1) = \langle e_1, \ldots, e_{n-1} \rangle$.

Therefore, without limitation of generality, with the accuracy of conjugation, we can assume that $\sigma_1 = t_{1n}(r)$. Then

$$\sigma_2 = \begin{pmatrix} r & * & * \\ 0 & * & * \\ & & r \end{pmatrix}, \quad \text{where } r \in R^{*} \ *.$$

If $r = 1$, then $\langle e_1 \rangle \in P(\sigma_2)$, $R(\sigma_1) \subseteq P(\sigma_2)$ and system (1) holds.

If $r \neq 1$, then from inclusion $(r-1)e_1 = (\sigma_2 - 1)e_1 \in R(\sigma_2)$ implies that $e_1 \in R(\sigma_2)$. Therefore $R(\sigma_1) \subseteq R(\sigma_2)$.

Note that if in Lemma 5 the element Σ_2 is unipotent or $R = Z_2$, then $r = 1$.

This means that transvection Σ_1 commutes with the unipotent element Σ_2 if and only if system (1) holds. In particular, two transvections commute if and only if system (1) takes place. □

Theorem 2. *Let R be a division ring, V be a finite-dimensional vector space over R, Σ_1 be a transvection, $\Sigma_1\Sigma_2{=}\Sigma_2\Sigma_1$, $\dim R\,(\Sigma_2) = \dim R\,(\Sigma_2) \cap P\,(\Sigma_2) + l$, where $l{\leq}1$. Then system (1) holds.*

Proof. By Lemma 5, system (1) or $R\,(\sigma_1) \subseteq R\,(\sigma_2)$ holds. Therefore, it is sufficient to consider the case when $R\,(\sigma_1) \subseteq R\,(\sigma_2)$.

If $l = 0$, then $R\,(\sigma_1) \subseteq R\,(\sigma_2) \subseteq P\,(\sigma_2)$ and system (1) holds.

Consider the case $l = 1$. Then $R\,(\sigma_2) \cap P\,(\sigma_2)$ is a hyperplane in $R\,(\sigma_2)$.

Of course, it is sufficient to consider the case when $R\,(\sigma_1) \nsubseteq P\,(\sigma_2)$.

Since $R\,(\sigma_1) \nsubseteq R\,(\sigma_2)$, we have $R\,(\sigma_2) = R\,(\sigma_1) \oplus R\,(\sigma_2) \cap P\,(\sigma_2)$.

We choose the base $R\,(\sigma) = \langle e_1 \rangle$, $R\,(\sigma_2) \cap P\,(\sigma_2) = \langle e_1, \ldots, e_k \rangle$ and supplement the base of the subspace $R\,(\sigma_2) = \langle e_1, \ldots, e_k \rangle$ to the base of the whole space $V = \langle e_1, e_2, \ldots, e_k, \ldots, e_n \rangle$.

In the base V, the matrices σ_1 and σ_2 have the form

$$\sigma_1 = \begin{pmatrix} 1 & x & y \\ 0 & E & 0 \\ 0 & 0 & E \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} r & 0 & * \\ 0 & E & * \\ 0 & 0 & E \end{pmatrix}, \quad \text{where } r \neq 1.$$

Since $\sigma_1 - 1$ and $\sigma_2 - 1$ switch, so are the matrices

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} r-1 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & \end{pmatrix}$$

also commute.

Hence $(r-1)\,x = 0$ and, as a consequence, $x = 0$. In this case, $(r-1)\,y = 0$ and, as a consequence, $y = 0$, $\sigma_1 = 1$. But, this contradicts the fact that σ_1 is transvection.

This proves that the case $R\,(\sigma_1) \nsubseteq P\,(\sigma_2)$ is not possible. This means that at $l = 1$, system (1) also holds. □

For $l > 1$, Theorem 2 does not always true. This follows from the examples.

Example 1. Let R be a division ring, $R \neq \mathbb{Z}_2$,

$$\sigma_1 = \begin{pmatrix} E & & & \\ & t_{1l} & & \\ & & E & \\ & & & E \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} E & & E & \\ & \alpha E & & \\ & & E & \\ & & & E \end{pmatrix}$$

where $\alpha \neq 0, 1$ is taken l times, $l > 1$. After all, $\sigma_1\sigma_2 = \sigma_2\sigma_1$, but $R\,(\sigma_1) \nsubseteq P\,(\sigma_2)$. In this case, $P\,(\sigma_1) \cap R\,(\sigma_2) \neq 0$.

Example 2. Let R be a division ring, $R \neq \mathbb{Z}_2$,

$$\sigma_1 = \mathrm{diag}(E, t_{1l}\,(1)\,, E, E), \quad \sigma_1 = \mathrm{diag}(E, \alpha E, E, E)$$

where $\alpha \neq 0, 1$ is taken l times, $l > 1$. After all, $\sigma_1\sigma_2 = \sigma_2\sigma_1$, $R\,(\sigma_1) \nsubseteq P\,(\sigma_2)$ and $P\,(\sigma_1) \cap R\,(\sigma_2) = 0$.

We note that the case when σ_1 is a transvection commuting with σ_2, where $\dim R\,(\sigma_2) = 2$ and $R\,(\sigma_2) \cap P\,(\sigma_2) \neq 0$, which follows from [8], follows from Theorem 2. After all, in this case

$$1 \leq \dim R\,(\sigma_2) \cap P\,(\sigma_2) \leq 2, 0 \leq \dim R\,(\sigma_2) - \dim R\,(\sigma_2) \cap P\,(\sigma_2) = l \leq 1.$$

Therefore, by Theorem 2, system (1) holds.

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Transcarpathian Regional Institute of Postgraduate Pedagogical Education, Uzhhorod, Ukraine
Uzhhorod National University, Uzhhorod, Ukraine
vasil.petechuk@gmail.com

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