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N. M. Pyrch

FREE PRODUCTS OF TOPOLOGICAL GROUPS AND M-EQUIVALENCE

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In the paper we apply free products of topological groups for investigating the M-equivalence of Tychonoff spaces which are infinite disjoint sums of its subspaces. The main result is the following: Let $X = \bigoplus\limits_{i=1}^{\infty} X_i$ and for every $i \in \mathbb{N}$ there exists a topological group G_i such that $F(X_{i+1})$ is topologically isomorphic to the free product $F(X_i)$ and G_i. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of the natural numbers and let $\tilde{X} = \bigoplus\limits_{k=1}^{\infty} X_{n_k}$. Then $X \overset{M}{\sim} \tilde{X}$. This theorem give us many examples of topological spaces with topologically isomorphic free topological groups.

We use obtained results for constructing M-equivalent pairs and M-equivalent bundles of such spaces.

1. Notations. All the spaces in the paper are assumed to be Tychonoff. Let $\{G_i \colon i \in I\}$ be a family of topological groups. Then a topological group F is said to be a *free topological product* of $\{G_i \colon i \in I\}$, denoted by $\mathop{*}\limits_{i\in I} G_i$ if

1) for each $i \in I$ G_i is a subgroup in F;

2) F is generated by $\bigcup\limits_{i\in I} G_i$;

3) for every family $\{f_i \colon G_i \to H \colon i \in I\}$ of continuous homomorphisms to a topological group H there exists a unique continuous homomorphism $f \colon F \to H$ such that $f \restriction_{G_i} = f_i$. In particular, for finite number of topological groups $G_1, G_2, \dots, G_n$ their free product is denoted by $G_1 * G_2 * \dots * G_n$.

For a topological space X denote by $F(X)$ the free topological group on X in Markov sense. For a topological space X we denote by X^+ the space obtained from X by adding one isolated point. A subspace Y of a topological space X is called a *G-retract* of the topological space X if any continuous mapping $f \colon Y \to H$ from a topological space Y into a topological group H admits a continuous extension on X.

Topological spaces X and Y are called

M-equivalent ($X \overset{M}{\sim} Y$) if their free topological groups $F(X)$ and $F(Y)$ are topologically isomorphic;

r-equal, if the space X contains a retract homeomorphic to Y and the space Y contains a retract homeomorphic to X;

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r_G-equal, if the space X contains a G-retract homeomorphic to Y and space Y contains a G-retract homeomorphic to X.

A pair of topological spaces (X, Y) is a space X and its subspace Y. We denote by $\langle Y \rangle$ the subgroup in $F(X)$, generated by the set Y. A pair of topological spaces (X, X_1) is called *M-equivalent* to a pair (Y, Y_1), if there exists a topological isomorphism $f\colon F(X) \to F(Y)$ such that $f(\langle X_1 \rangle) = \langle Y_1 \rangle$. A bundle $(X, \{X_i\colon i \in I\})$ of topological spaces consists of the topological space X and the family of its subspaces $\{X_i\colon i \in I\}$. We say that the bundle $(X, \{X_i\colon i \in I\})$ is *M-equivalent* to a bundle $(Y, \{Y_i\colon i \in I\})$, if there exists a topological isomorphism $h\colon F(X) \to F(Y)$ such that $h(\langle X_i \rangle) = \langle Y_i \rangle$ for all $i \in I$.

A bundle $(X, \{X_s\colon s \in S\})$, where $X = \bigcup_{s\in S} X_s$, forms a *Δ-system*, if for all different $i, j \in S$ we have that $X_i \cap X_j = K$, where $K = \bigcap_{s\in S} X_s$. The bouquet $\bigvee_{s\in S}(X_s, x_s)$ of topological spaces X_s with the distinguished points $x_s \in X_s$ can be defined as quotient space $\big(\bigoplus_{s\in S} X_s\big)/\big(\bigoplus_{s\in S}\{x_s\}\big)$. It was proved in [9] that the bouquet of topological spaces does not depend up to M-equivalence on the distinguished points, so we will write shortly $\bigvee_{s\in S} X_s$. We will write $X \sim Y$ for homeomorphic topological spaces X and Y, and $G \simeq H$ for topologically isomorphic topological groups G and H.

For a topological space X with a distinguished point a we denote by $FG(X, a)$ the Graev free topological group on X with the identity a. Since $FG(X, a)$ does not depend up to topological isomorphism on a we will write shortly $FG(X)$ ([6]). For every topological space X we have that $F(X) \simeq FG(X^+)$. We refer to [2] for basic topological properties of the free topological groups.

2. Infinite unions of topological spaces and M-equivalence. From the definition of a free product of topological groups and a free topological group it follows that for every family $\{X_i\colon i \in I\}$ of disjoint topological spaces

$$F\big(\bigoplus_{i\in I} X_i\big) \simeq \mathop{*}_{i\in I} F(X_i) \text{ and } FG\big(\bigvee_{i\in I}(X_i, a_i)\big) \simeq \mathop{*}_{i\in I} FG(X_i, a_i)$$

for every family $\{(X_i, a_i)\colon i \in I\}$ of topological spaces with the distinguished points a_i.

In particular $F(X \vee Y) \simeq FG((X \vee Y)^+) \simeq FG(X) * FG(Y^+) \simeq FG(X) * F(Y)$, and similarly $F(X \vee Y) \simeq F(X) * FG(Y)$.

Lemma 1. *The free product of topological groups is associative, i.e. if topological group H_j is isomorphic to the free product $\mathop{*}_{i\in I_j} G_i$ of the family of topological groups $\{G_i\colon i \in I_j\}$, where $I_s \cap I_k = \varnothing$ for $s \neq k$, then $\mathop{*}_{j\in J} H_j$ is topologically isomorphic to $\mathop{*}_{i\in I} G_i$, where $I = \bigcup_{j\in J} I_j$.*

Proof. Consider the topological group $G = \mathop{*}_{j\in J} H_j$. For each $i \in I$ G_i is a subgroup in some H_j and hence is a subgroup in G. The set $\bigcup_{i\in I_j} G_i$ generates H_j, the set $\bigcup_{j\in J} H_j$ generates G, hence the set $\bigcup_{i\in I} G_i$ generates G. Every family of continuous homomorphisms $\{f_i\colon G_i \to H\colon i \in I\}$ to a topological group H admits extension to the family of continuous homomorphisms $\{h_j\colon H_j \to H\colon j \in J\}$, which admits extension to continuous homomorphism $h\colon G \to H$. From the uniqueness of the free product of topological groups the group $\mathop{*}_{j\in J} H_j$ is isomorphic to the free product $\mathop{*}_{i\in I} G_i$. □

Lemma 2. *A free product of topological groups is commutative, i.e. if I is an ordered set, $\{G_i\colon i \in I\}$ is a family of topological groups, $\sigma\colon I \to I$ is permutation defined on I, then $\mathop{*}_{i\in I} G_i$ is topologically isomorphic to $\mathop{*}_{i\in I} G_{\sigma(i)}$.*

Proof. Consider the topological group $G = \underset{i\in I}{*} G_{\sigma(i)}$. For each $i \in I$ G_i is a subgroup in G. The set $\bigcup_{i\in I} G_i \simeq \bigcup_{i\in I} G_{\sigma(i)}$ generates G. Every family of continuous homomorphisms $\{f_i \colon G_i \to H \colon i \in I\}$ to a topological group H generates a family $\{h_i \colon G_{\sigma(i)} \to H \colon i \in I\}$ of continuous homomorphisms and hence admits extension to continuous homomorphism $h \colon G \to H$. From the uniqueness of the free product of topological groups we conclude that the group G is topologically isomorphic to the free product $\underset{i\in I}{*} G_i$. □

Proposition 1. *Let $X = \bigoplus_{i=1}^{\infty} X_i$ and K be an arbitrary topological space. If for every $i \in \mathbb{N}$ there exists a topological group G_i such that $F(X_i) \simeq F(K) * G_i$, then $X \overset{M}{\sim} X \oplus K$.*

Proof.

$$F\Big(\bigoplus_{i=1}^{\infty} X_i\Big) \simeq \overset{\infty}{\underset{i=1}{*}} F(X_i) \simeq \overset{\infty}{\underset{i=1}{*}} (G_i * F(K)) \simeq \Big(\overset{\infty}{\underset{i=1}{*}} G_i\Big) * \Big(\overset{\infty}{\underset{i=1}{*}} F(K)\Big) \simeq$$
$$\simeq \Big(\overset{\infty}{\underset{i=1}{*}} G_i\Big) * \Big(\overset{\infty}{\underset{i=1}{*}} F(K)\Big) * F(K) \simeq \Big(\overset{\infty}{\underset{i=1}{*}} (G_i * F(K))\Big) * F(K) \simeq \Big(\overset{\infty}{\underset{i=1}{*}} F(X_i)\Big) * F(K) \simeq$$
$$\simeq F(X) * F(K) \simeq F(X \oplus K).$$

□

Theorem 1. *Let $X = \bigoplus_{i=1}^{\infty} X_i$ and for every $i \in \mathbb{N}$ there exists a topological group G_i such that $F(X_{i+1}) \simeq F(X_i) * G_i$. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of the natural numbers and let $\tilde{X} = \bigoplus_{k=1}^{\infty} X_{n_k}$. Then $X \overset{M}{\sim} \tilde{X}$.*

Proof. Obviously if $j > i$ then there exists a topological group G such that $F(X_j) \simeq F(X_i) * G$. Denote $P = \{n_k \colon k \in \mathbb{N}\}$, $S = \mathbb{N} \setminus P$. The set S as a subset of the ordered set $\mathbb{N}$ is ordered, so we denote its elements $s_1, s_2, \ldots$, with the order inherited from $\mathbb{N}$. Put $Y = \bigoplus_{i\in S} X_i$ and $Y_m = X_{s_m}$. Then $Y = X \setminus \tilde{X}$. For every natural number $i \in S$ define the natural number $k(i)$ and the set K_i by the induction

$$K_0 = \varnothing, \quad k(i) = \min_{\substack{n_l > i, \\ l \in \mathbb{N}\setminus K_{i-1}}} l \ , \quad K_i = \{k_1, k_2, \ldots, k_i\}.$$

So for every $i \in \mathbb{N}$ there exists $k(i) \in \mathbb{N}$ such that $n_{k_i} > i$ and from the fact $i > j$ it follows that $k(i) > k(j)$. Hence, for every element Y_i, for every $j \geq i$ there exists a topological group G_j such that $F(X_{n_{k(j)}}) \simeq F(Y_i) * G_j$. Put $K = \{k_i \colon i \in \mathbb{N}\}$. Denote by $W = \bigoplus_{i\in K} X_i$. If we denote $W_i = X_{n_{k(i)}}$, then $W = \bigoplus_{i\in\mathbb{N}} W_i$. Denote by $V = \tilde{X} \setminus W$. We match to each element from Y the sequence from W in the following way:

$Y_1 \colon Z_1 = W_1 \oplus W_3 \oplus W_5 \oplus W_7 \oplus \ldots$
$Y_2 \colon Z_2 = W_2 \oplus W_6 \oplus W_{10} \oplus W_{14} \oplus \ldots$
$Y_3 \colon Z_3 = W_4 \oplus W_{14} \oplus W_{20} \oplus W_{28} \oplus \ldots$
$\ldots$
$Y_n \colon Z_n = W_{2^{n-1}} \oplus W_{2^{n-1}+2^{n+1}} \oplus W_{2^{n-1}+2\times 2^{n+1}} \oplus W_{2^{n-1}+3\times 2^{n+1}} \oplus \ldots$
$\ldots$

In these sums every W_i meets exactly one time. If $i = 2^m \cdot l$, where l is odd then W_i is in $(m+1)$-th row.

Obviously, $W = \overset{\infty}{\underset{i=1}{\oplus}} Z_i$. From Proposition 1 it follows that $Z_n \overset{M}{\sim} Z_n \oplus Y_n$. So,

$$X = \tilde{X} \oplus Y = V \oplus W \oplus Y = V \oplus \Big(\overset{\infty}{\underset{n=1}{\oplus}} Z_n \Big) \oplus \Big(\overset{\infty}{\underset{n=1}{\oplus}} Y_n \Big) = V \oplus \Big(\overset{\infty}{\underset{n=1}{\oplus}} (Z_n \oplus Y_n) \Big) \overset{M}{\sim}$$
$$\overset{M}{\sim} V \oplus \Big(\overset{\infty}{\underset{n=1}{\oplus}} Z_n \Big) = V \oplus W = \bar{X}.$$

□

Subsets K_1 and K_2 of topological spaces X are called *parallel G-retracts* of X if there exist continuous homomorphisms $R_1 \colon F(X) \to \langle K_1 \rangle$ and $R_2 \colon F(X) \to \langle K_2 \rangle$ such that

1) $R_1(x) = x$ for all $x \in K_1$;
2) $R_2(x) = x$ for all $x \in K_2$;
3) $R_1 \circ R_2 = R_1$, $R_2 \circ R_1 = R_2$.

Recall that a continuous surjective mapping $p \colon X \to Y$ is called R-quotient if a real-valued function ϕ on Y is continuous iff the composition $\phi \circ p \colon X \to \mathbb{R}$ is continuous. If $p \colon X \to Y$ is mapping of a space X onto a set Y, then there exists and unique completely regular topology on Y, called the R-quotient topology, which makes the mapping p R-quotient. This topology can be described as the finest completely regular topology on Y making the mapping p continuous. If K is a subspace of Tychonoff space X, then R-quotient space X/K is Tychonoff ([9]). Obviously, if quotient space X/K is Tychonoff then quotient and R-quotient topologies coincide. So, in this article if Y is a closed subspace of Tychonoff space X under X/Y we will understand R-quotient space. If K_1 and K_2 are parallel G-retracts of X then R-quotient spaces X/K_1 and X/K_2 are M-equivalent ([11]). This proposition is a generalization of the Okunev's theorem from [9] and similarly was proved for R-quotient topologies on X/K_1 and X/K_2.

Proposition 2. *If K is a G-retract of a topological space X, then X is M-equivalent to $(X/K) \vee K$.*

Proof. If K is a G-retract in X, then there exists a continuous homomorphism $R \colon F(X) \to F(K)$ such that $R(a) = a$ for all $a \in K$ [10]. Let $b \in K$, $h \colon K \to K_1$ be some homeomorphism, $H \colon F(K) \to F(K_1)$ continuous homomorphic extension of the mapping h, $b_1 = h(b)$. Consider the bouquet $Y = (X, b) \vee (K_1, b_1)$. Define the mapping $r_1 \colon Y \to F(K)$ in the following way: $r_1(x) = R(x)$ if $x \in X$ and $r_1(x) = h^{-1}(x)$ if $x \in K_1$. Denote by $R_1 \colon F(Y) \to F(K)$ the homomorphic extension of the mapping r_1. Put $R_2 = H \circ R_1 \colon F(Y) \to F(K_1)$. Then $R_1 \circ R_2 = H^{-1} \circ R_2 \circ R_2 = H^{-1} \circ R_2 = R_1$, $R_2 \circ R_1 = H \circ R_1 \circ R_1 = H \circ R_1 = R_2$. So, $R_1(Y)$ and $R_2(Y)$ are parallel G-retracts and hence $Y/K \overset{M}{\sim} Y/K_1$, i.e., $X \overset{M}{\sim} (X/K) \vee K$. □

Proposition 3. *If a topological space X contains a G-retract K, M-equivalent to Y, then there exists a topological group G such that $F(X) \simeq F(Y) * G$.*

Proof. Let K be an G-retract of a topological space X. From the fact that $X \overset{M}{\sim} (X/K) \vee K$, so $F(X) \simeq F(K) * FG(X/K)$. Since $F(K)$ and $F(Y)$ are topologically isomorphic then $F(X) \simeq F(Y) * FG(X/K)$. □

Proposition 4. *If a topological space X contains a G-retract K such that R-quotient space X/K is M-equivalent to Y, then there exists a topological group G such that*

$$F(X) \simeq F(Y) * G.$$

Proof. Let K be an G-retract of topological space X. From the fact that $X \stackrel{M}{\sim} (X/K) \vee K$ it follows that $F(X) \simeq F(X/K) * FG(K)$. Since $F(X/K)$ and $F(Y)$ are topologically isomorphic then $F(X) \simeq F(Y) * FG(K)$. □

Retractions r_1 and r_2 of a space X are called *orthogonal* if the mappings $r_1 \circ r_2$ and $r_2 \circ r_1$ are constant. The images of the space X under orthogonal retractions K_1 and K_2 are called *orthogonal* retracts of the space X.

Proposition 5. *If a topological space X contains orthogonal G-retracts K_1 and K_2 ([10]) such that $X/\{K_1, K_2\}$ is M-equivalent to Y, then there exists a topological group G such that $F(X) \simeq F(Y) * G$.*

Proof. By Proposition 2 $F(X) \simeq FG(K_1) * F(X/K_1)$. Let $p\colon X \to X/K_1$ be a R-quotient mapping. By Proposition 2 from [10] $p(K_2)$ is homeomorphic to K_2 and is G-retract in X/K_1. Hence, $F(X/K_1) \simeq F(X/\{K_1, K_2\}) * FG(p(K_2))$. Thus, $F(X) \simeq FG(K_1) * F(X/\{K_1, K_2\}) * FG(p(K_2)) \simeq F(Y) * FG(K_1) * FG(p(K_2))$. □

Topological space X is called an *absolute G-retract* if X is G-retract of every topological space Y that contains X as closed subspace. Similarly to proposition 5 we can prove

Proposition 6. *If a topological space X contains disjoint absolute G-retracts $K_1, K_2, \ldots, K_n$ such that $X/\{K_1, K_2, \ldots, K_n\}$ is M-equivalent to Y, then there exists a topological group H such that $F(X) \simeq F(Y) * H$.*

From Theorem 1 and Proposition 3 it follows

Theorem 2. *Let $X = \bigoplus_{i=1}^{\infty} X_i$ and every X_{i+1} contains G-retract (retract) homeomorphic to X_i. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of the natural numbers and let $\tilde{X} = \bigoplus_{k=1}^{\infty} X_{n_k}$. Then $X \stackrel{M}{\sim} \tilde{X}$.*

From the fact that every closed subspace of a zero-dimensionally metrizable space is a retract in this space we obtain

Corollary 1. *Let $X = \bigoplus_{i=1}^{\infty} X_i$, where every space X_{i+1} is zero-dimensionally metrizable and contains a closed subspace homeomorphic to X_i. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of the natural numbers and let $\tilde{X} = \bigoplus_{k=1}^{\infty} X_{n_k}$. Then $X \stackrel{M}{\sim} \tilde{X}$.*

Denote by $I = [0, 1]$ the closed unit interval with the topology generated by the standard euclidian metric. Since I is an absolute retract we obtain

Corollary 2. *Let $X = \bigoplus_{i=1}^{\infty} X_i$ and for every $i \in \mathbb{N}$ the space X_i is nontrivial path connected. Then $X \stackrel{M}{\sim} X \oplus I$.*

From Theorem 1 and Proposition 4 it follows

Theorem 3. *Let $X = \bigoplus_{i=1}^{\infty} X_i$ and every X_{i+1} contains G-retract (retract) K_i such that quotient space X_{i+1}/K_i is homeomorphic to X_i. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of the natural numbers and let $\tilde{X} = \bigoplus_{k=1}^{\infty} X_{n_k}$. Then $X \stackrel{M}{\sim} \tilde{X}$.*

Lemma 3. *If for all $k = 1, \dots, n$ there exists a topological group G_k, such that $F(X_k) \simeq F(Y_k * G_k)$. The there exists a topological group G such that $F(X_1 \oplus X_2 \oplus \dots \oplus X_n) \simeq F(Y_1 \oplus Y_2 \oplus \dots \oplus Y_n) * G$.*

Proof. $F(X_1 \oplus X_2 \oplus \dots \oplus X_n) \simeq F(X_1) * F(X_2) * \dots * F(X_n) \simeq$
$\simeq F(Y_1) * G_1 * F(Y_2) * G_2 * \dots * F(Y_n) * G_n \simeq$
$\simeq (F(Y_1) * F(Y_2) * \dots * F(Y_n)) * (G_1 * G_2 * \dots * G_n) \simeq F(Y_1 \oplus Y_2 \oplus \dots \oplus Y_n) * G,$
where $G = G_1 * G_2 * \dots * G_n$. □

Denote by $\mathbb{N}$ the set of natural numbers with the discrete topology.

Proposition 7. *Let $X = \overset{\infty}{\underset{i=1}{\oplus}} X_i$ and for every $i \in \mathbb{N}$ there exists a topological group G_i such that $F(X_{i+1}) \simeq F(X_i) * G_i$. Then $X \overset{M}{\sim} \mathbb{N} \times X$.*

Proof. Apply Theorem 1 for the disjoint sum of the topological spaces $Z = A_1 \oplus A_2 \oplus A_3 \oplus \dots$ defined as

$$Z = (X_1) \oplus (X_1) \oplus (X_2) \oplus (X_1 \oplus X_2) \oplus (X_3 \oplus X_3) \oplus (X_1 \oplus X_2 \oplus X_3 \oplus X_3) \oplus$$
$$\oplus (X_4 \oplus X_4 \oplus X_4 \oplus X_4) \oplus (X_1 \oplus X_2 \oplus X_3 \oplus X_4 \oplus X_4 \oplus X_4 \oplus X_4) \oplus \dots$$

Denote by D_i a discrete space containing i elements. Here $A_{2k} = D_{a_k} \times X_k$, $A_{2k-1} = X_1 \oplus X_2 \oplus \dots \oplus X_k \oplus D_{a_{k-1}} \times X_k$, where a_k is a sequence satisfying recurrence relation $a_k = a_{k-1} + k - 2$, $a_1 = 1$. Solving this relation we obtain that $a_k = \frac{1}{2}k^2 - \frac{3}{2}k + 1$ ([1, chapter 11.2]).

Note that $Z \sim \mathbb{N} \times X$. By Theorem 1 $Z \overset{M}{\sim} \overset{\infty}{\underset{i=1}{\oplus}} A_{2i} = \overset{\infty}{\underset{i=1}{\oplus}} D_{a_i} \times X_i$.

Denote $Z_1 = \overset{\infty}{\underset{i=1}{\oplus}} A_{2i}$.

Apply Theorem 1 for the disjoint sum of the topological spaces

$$Z_1 = X_1 \oplus X_2 \oplus X_3 \oplus X_3 \oplus X_4 \oplus X_4 \oplus X_4 \oplus X_4 \oplus \dots$$

(here X_i meet a_i times).

This sequence contains a subsequence $\{X_i\}$, so $Z_1 \overset{M}{\sim} \overset{\infty}{\underset{i=1}{\oplus}} X_i$. Hence, $\mathbb{N} \times X \sim Z \overset{M}{\sim} Z_1 \overset{M}{\sim} X$. □

Corollary 3. *Let $X = \overset{\infty}{\underset{i=1}{\oplus}} X_i$ and for every $i \in \mathbb{N}$ there exists a topological group G_i such that $F(X_{i+1}) \simeq F(X_i) * G_i$. Let D be a finite discrete topological space. Then $X \overset{M}{\sim} D \times X$.*

Proposition 8. *Let Y be a G-retract of the space X. Then $\mathbb{N} \times (X \oplus Y) \overset{M}{\sim} \mathbb{N} \times X$.*

Proof. By Proposition 1 we have that $Y \oplus \mathbb{N} \times X \overset{M}{\sim} \mathbb{N} \times X$. Since the relation of M-equivalence is additive, then from the fact $X_1 \overset{M}{\sim} X_2$ it follows $\mathbb{N} \times X_1 \overset{M}{\sim} \mathbb{N} \times X_2$.

So, $\mathbb{N} \times (Y \oplus \mathbb{N} \times X) \overset{M}{\sim} \mathbb{N} \times \mathbb{N} \times X$, i.e., $\mathbb{N} \times (X \oplus Y) \overset{M}{\sim} \mathbb{N} \times X$. □

Corollary 4. *For every nonempty space X we have that $\mathbb{N} \times X \overset{M}{\sim} \mathbb{N} \times X \oplus \mathbb{N}$.*

Proposition 9. *If Y is a G-retract of the space X. Then $\mathbb{N} \times (X \oplus X/Y) \overset{M}{\sim} \mathbb{N} \times X$.*

Proof. By Corollary 1 from [11] we have that $Y \oplus X/Y \overset{M}{\sim} X^+$. From Corollary 4 it follows that $\mathbb{N} \times X \sim \mathbb{N} \times X \oplus \mathbb{N} \times X \overset{M}{\sim} \mathbb{N} \times X \oplus \mathbb{N} \times X \oplus \mathbb{N} = \mathbb{N} \times X \oplus \mathbb{N} \times X^+$. So $\mathbb{N} \times X \overset{M}{\sim} \mathbb{N} \times X \oplus \mathbb{N} \times X^+ \overset{M}{\sim} \mathbb{N} \times X \oplus \mathbb{N} \times (X/Y \oplus Y) = \mathbb{N} \times (X \oplus Y) \oplus \mathbb{N} \times (X/Y)$. From Proposition 8 it follows that $\mathbb{N} \times (X \oplus Y) \overset{M}{\sim} \mathbb{N} \times X$. Hence, $\mathbb{N} \times X \overset{M}{\sim} \mathbb{N} \times (X \oplus Y) \oplus \mathbb{N} \times (X/Y) \overset{M}{\sim} \mathbb{N} \times X \oplus \mathbb{N} \times (X/Y) = \mathbb{N} \times (X \oplus X/Y)$. □

From Proposition 8 it follows

Proposition 10. *Let X and Y be r_G-equal spaces. Then $\mathbb{N} \times X \overset{M}{\sim} \mathbb{N} \times Y$.*

In [3] Borsuk study r-equal spaces, in particular, r-invariants. Since many topological properties, in particular, cardinal invariants, for a topological spaces X and $\mathbb{N} \times X$ coincide, we can conclude that the long list of M-invariants ([2, chapter 7.10]), are also r-invariants.

Corollary 5. *Let X and Y be a zero-dimensional metrizable spaces such that*
1) X contains closed subspace homeomorphic to Y,
2) Y contains closed subspace homeomorphic to X.
Then $\mathbb{N} \times X \overset{M}{\sim} \mathbb{N} \times Y$.

Proposition 11. *Let X and Y be a spaces such that X contains G-retract K_X such that X/K_X is homeomorphic to Y and Y contains G-retract K_Y such that Y/K_Y is homeomorphic to X. Then $\mathbb{N} \times X \overset{M}{\sim} \mathbb{N} \times Y$.*

Proof. From Proposition 9 it follows that $\mathbb{N} \times X \overset{M}{\sim} \mathbb{N} \times (X \oplus X/K_X) \overset{M}{\sim} \mathbb{N} \times (X \oplus Y)$ and $\mathbb{N} \times Y \overset{M}{\sim} \mathbb{N} \times (Y \oplus Y/K_Y) \overset{M}{\sim} \mathbb{N} \times (Y \oplus X)$. Hence, $\mathbb{N} \times X \overset{M}{\sim} \mathbb{N} \times Y$. □

Denote by $\mathbb{Z}$ the group of integers with the discrete topology. We say that the isomorphism $i\colon F(X) \to F(Y)$ is *special* if the mapping $e_Y^* \circ i$ is constant mapping, where $e_Y^*\colon F(Y) \to \mathbb{Z}$ is homomorphism extending function $e_Y\colon Y \to \mathbb{Z}$, equal 1 on Y.

Proposition 12. *For every Tychonof X and its G-retract Y we have that $F(X) \overset{M}{\sim} F(X) \oplus Y$. In particular, $F(X) \overset{M}{\sim} F(X) \oplus X$ for every space.*

Proof. Let $e\colon X \to \mathbb{Z}$ be a continuous mapping defined as $e\restriction_X = 1$, $E\colon F(X) \to \mathbb{Z}$ is homomorphism, extending the mapping e. Put $G_i(X) = E^{-1}(i)$. For every $n, m \in \mathbb{N}$ subspaces $G_m(X)$ and $G_n(X)$ are homeomorphic (homeomorphisms we can define by the formula $h(x) = x \cdot a^{n-m}$, where $a \in X$). Since $X \subset G_1(X)$, and X is a G-retract of the space $F(X)$, then X is a G-retract in the space $G_1(X)$. Subspace Y is G-retract in X, and space X is a G-retract of the space $G_1(X)$, so Y is G-retract in the space $G_1(X)$. Since $F(X) = \bigoplus_{i \in \mathbb{Z}} G_i(X)$, and every space $G_i(X)$ contains G-retract homeomorphic to Y, so by Proposition 1 we have that $F(X) \overset{M}{\sim} F(X) \oplus Y$. □

Proposition 13. *For every space X and its G-retract Y we have that $F(X) \overset{M}{\sim} F(X) \oplus F(Y)$.*

Proof. By Proposition 3 from [10] we obtain that there exists a special topological homomorphism $h\colon F(X) \to F(Y)$ such that $h(x) = x$ for all $x \in F(Y)$. Since h is special, then $h(G_1(X)) = G_1(Y)$, and subspace $G_1(Y)$ is retract of the space $G_1(X)$. From Proposition 7 it follows that $F(X) \sim \mathbb{N} \times G_1(X) \overset{M}{\sim} \mathbb{N} \times (G_1(X) \oplus G_1(Y)) \sim F(X) \oplus F(Y)$. □

Proposition 14. *Let X and Y be r_G-equal spaces. Then $F(X) \overset{M}{\sim} F(Y)$.*

Proof. From Proposition 13 it follows that $F(X) \overset{M}{\sim} F(X) \oplus F(Y)$ and $F(Y) \overset{M}{\sim} F(X) \oplus F(Y)$. Hence, $F(X) \overset{M}{\sim} F(Y)$. □

For a topological space X denote by $F_p(X)$ the free paratopological group on X ([5]), by $F_q(X)$ the free quasitopological group on X ([4]).

Proposition 15. *For every topological space X and its G-retract Y we have that*

$$F_p(X) \overset{M}{\sim} F_p(X) \oplus Y, \quad F_q(X) \overset{M}{\sim} F_q(X) \oplus Y.$$

In particular, $F_p(X) \overset{M}{\sim} F_p(X) \oplus X$ and $F_q(X) \overset{M}{\sim} F_q(X) \oplus X$ for every space X.

Proof. Since every continuous mapping from Tychonoff space X into topological group H admits continuous extension into $F_p(X)$, we make the conclusion that subspace X is G-retract of the space $F_p(X)$. The rest proof is similar to the proof of Proposition 12. □

Denote by $S(X) = \bigoplus_{n=1}^{\infty} X^n$ the free topological semigroup on X. Taking in this sum the powers of X divisible by some natural number k, we obtain subspace in $S(X)$ homeomorphic to $S(X^k)$.

Proposition 16. *Let X be a Tychonof space, $n, m \in \mathbb{N}$. Then $S(X) \overset{M}{\sim} X^m \times S(X^n)$.*

Proof. Apply Theorem 1 for the sequence $Y_i = X^i$ and its subsequence $Z_j = X^m \cdot X^{nj}$. □

From Proposition 7 it follows

Proposition 17. *Let X be a space. Then $S(X) \overset{M}{\sim} \mathbb{N} \times S(X) \sim S(\mathbb{N} \times X) \sim S(S(X))$.*

Proposition 18. *For every space X and its retract Y we have that*

$$S(X) \overset{M}{\sim} S(X) \oplus S(Y) \overset{M}{\sim} S(X) \oplus Y.$$

Proof. Let $k \in \mathbb{N}$. Then $S(X) = \bigoplus_{n=1}^{\infty} X^n = \left(\bigoplus_{n=1}^{k-1} X^n\right) \oplus \left(\bigoplus_{n=k}^{\infty} X^n\right)$. Since for $n \geq k$ X^n contains retract homeomorphic Y^k then by Proposition 1 we have $\bigoplus_{n=k}^{\infty} X^n \overset{M}{\sim} \left(\bigoplus_{n=k}^{\infty} X^n\right) \oplus Y^k$. So $S(X) \overset{M}{\sim} \left(\bigoplus_{n=1}^{k-1} X^n\right) \oplus \left(\bigoplus_{n=k}^{\infty} X^n\right) \oplus Y^k = S(X) \oplus Y^k$. By the additivity of the relation of M-equivalence we have $\mathbb{N} \times S(X) \overset{M}{\sim} \bigoplus_{k=1}^{\infty} (S(X) \oplus Y^k) = \mathbb{N} \times S(X) \oplus \left(\bigoplus_{k=1}^{\infty} Y^k\right) = S(X) \oplus S(Y)$. □

Corollary 6. *Let X and Y be an r-equal spaces. Then $S(X) \overset{M}{\sim} S(Y)$.*

Proposition 19. *For every space X and its retract Y we have that $S(X) \overset{M}{\sim} S(X \times Y)$.*

Proof. Apply Theorem 2 for disjoint sum of the sequence

$$S(X) = X \oplus X^2 \oplus X^3 \oplus \dots$$

By this theorem $S(X)$ is M-equivalent to the space $Z_1 = \overset{\infty}{\underset{n=1}{\oplus}} X^{2^n}$.

Similarly the space $S(X \times Y)$ is M-equivalent to the space $Z_2 = \overset{\infty}{\underset{n=1}{\oplus}} (X \times Y)^{2^n}$.

Apply Theorem 2 for sequence

$$Z = X \oplus X \times Y \oplus X^2 \oplus X^2 \times Y^2 \oplus X^4 \oplus X^4 \times Y^4 \oplus \dots$$

By this theorem the space Z is M-equivalent to the space $Z_1 = \overset{\infty}{\underset{n=1}{\oplus}} X^{2^n}$ and to the space $Z_2 = \overset{\infty}{\underset{n=1}{\oplus}} (X \times Y)^{2^n}$. The space Z_1 is M-equivalent to $S(X)$, space Z_2 is M-equivalent to $S(X \times Y)$, so

$$S(X) \overset{M}{\sim} Z_1 \overset{M}{\sim} Z \overset{M}{\sim} Z_2 \overset{M}{\sim} S(X \times Y).$$

□

Proposition 20. *Let $X_1, X_2, \dots, X_n$ be such topological spaces, that there exist a topological groups $G_1, G_2, \dots, G_n$ with*

$$F(X_2) \simeq F(X_1) * G_1, \quad F(X_3) \simeq F(X_2) * G_2, \dots ., F(X_1) \simeq F(X_n) * G_n.$$

Then $\mathbb{N} \times X_1 \overset{M}{\sim} \mathbb{N} \times X_2 \overset{M}{\sim} \dots, \overset{M}{\sim} \mathbb{N} \times X_n$.

Proof. Consider the space

$$Z = X_1 \oplus X_2 \oplus \dots \oplus X_n \oplus X_1 \oplus X_2 \oplus \dots \oplus X_n \oplus X_1 \oplus X_2 \oplus \dots \oplus X_n \oplus \dots$$

By Theorem 1 $Z \overset{M}{\sim} \mathbb{N} \times X_1$, $Z \overset{M}{\sim} \mathbb{N} \times X_2$, $\dots$, $Z \overset{M}{\sim} \mathbb{N} \times X_n$. □

3. Equivalence of the bundles and infinite unions.

Proposition 21. *If a topological space X is the disjoint infinite sum $X = \underset{i \in I}{\oplus} X_i$ of its nonempty subspaces X_i, then $FG(X) \simeq F(X)$.*

Proof. Let $a_i \in X_i$ be an arbitrary points. Then the space $K = \{a_i \colon i \in I\}$ is infinite discrete and is retract in X. Applying Proposition 2 for Graev free topological groups we obtain

$$FG(X) \simeq FG((X/K) \vee K) \simeq FG((X/K) \vee K \oplus \{a_0\}) \simeq FG(X \oplus \{a_0\}) \simeq F(X).$$

□

From Proposition 21 it follows that for spaces being disjoint infinite sum of its nonempty subspaces the relations M-equivalence and M^*-equivalence ([12]) coincide. So, we make a conclusion that the spaces X and $\tilde{X}$ are M^*-equivalent.

From Theorem 1 and Corollary 5 from [12] for one-element system S we obtain

Proposition 22. *Let $X = \overset{\infty}{\underset{i=1}{\oplus}} X_i$ and each X_{i+1} contains G-retract homeomorphic X_i, I_1, I_2 be infinite subsets of the set of natural numbers with infinite complements. $K_1 = \underset{i \in I_1}{\oplus} X_i$, $K_2 = \underset{i \in I_2}{\oplus} X_i$. Then the pairs (X, K_1) and (X, K_2) are M-equivalent.*

Proof. Denote $J_1 = \mathbb{N}\backslash I_1$, $J_2 = \mathbb{N}\backslash I_2$, $T_1 = \underset{i\in J_1}{\oplus} X_i$, $T_2 = \underset{i\in J_2}{\oplus} X_i$. Then $X \sim K_1 \oplus T_1 \sim K_2 \oplus T_2$. By Theorem 1 $T_1 \overset{M}{\sim} T_2 \overset{M}{\sim} K_1 \overset{M}{\sim} K_2 \overset{M}{\sim} X$. Let $s_1\colon F(K_1) \to F(K_2)$, $s_2\colon F(T_1) \to F(T_2)$ be a topological isomorphisms. Define the mapping $s\colon K_1 \oplus T_1 \to F(K_2 \oplus T_2)$ by putting $s(x) = s_1(x)$ if $x \in K_1$ and $s(x) = s_2(x)$ if $x \in T_1$. The extension $s^*\colon F(K_1\oplus T_1) \to F(K_2\oplus T_2)$ of the mapping s to the homomorphism of the free topological groups is an isomorphism ([7, Exercise 8.8]). By construction $s(\langle K_1\rangle) = \langle K_2\rangle$. □

Corollary 7. *Let X be a space, $n_1, n_2, m_1, m_2 \in \mathbb{N}$, $n_1, n_2 \geq 2$. Then*

$$(S(X), X^{m_1} \times S(X^{n_1})) \overset{M}{\sim} (S(X), X^{m_2} \times S(X^{n_2})).$$

Theorem 4. *Let $X = \overset{\infty}{\underset{i=1}{\oplus}} X_i$ and for every $i \in \mathbb{N}$ there exists a topological group G_i such that $F(X_{i+1}) \simeq F(X_i) * G_i$ and*

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \subseteq \mathbb{N},\ J_1 \subseteq J_2 \subseteq \cdots \subseteq J_m \subseteq \mathbb{N}$$

are subsets of the set of natural numbers such that the sets I_1, J_1, $I_k \setminus I_{k-1}$, $J_k \setminus J_{k-1}$ $(k = 2, \dots, n)$, $\mathbb{N} \setminus I_m$, $\mathbb{N} \setminus J_m$ are infinite. Put $A_s = \underset{k\in I_s}{\oplus} X_k$. $B_s = \underset{k\in j_s}{\oplus} X_k$. Then

$$(X, A_1, A_2, \dots A_m) \overset{M}{\sim} (Y, B_1, B_2, \dots B_m).$$

Proof. The R-quotient space A_k/A_{k-1} is homeomorphic to $\Big(\underset{i\in I_k\setminus I_{k-1}}{\oplus} X_i\Big)^+$, similarly B_k/B_{k-1} is homeomorphic to $\Big(\underset{i\in J_k\setminus J_{k-1}}{\oplus} X_i\Big)^+$. By Proposition 1,

$$\Big(\underset{i\in I_k\setminus I_{k-1}}{\oplus} X_i\Big)^+ \text{ is } M\text{-equivalent to } \underset{i\in I_k\setminus I_{k-1}}{\oplus} X_i$$

and

$$\Big(\underset{i\in J_k\setminus J_{k-1}}{\oplus} X_i\Big)^+ \text{ is } M\text{-equivalent to } \underset{i\in J_k\setminus J_{k-1}}{\oplus} X_i.$$

Both the spaces are $\underset{i\in I_k\setminus I_{k-1}}{\oplus} X_i$ and $\underset{i\in J_k\setminus J_{k-1}}{\oplus} X_i$ are M-equivalent to X, so the spaces

$$\Big(\underset{i\in I_k\setminus I_{k-1}}{\oplus} X_i\Big)^+ \text{ and } \Big(\underset{i\in J_k\setminus J_{k-1}}{\oplus} X_i\Big)^+ \text{ are } M\text{-equivalent}$$

and hence M^*-equivalent. Similarly $A_1 \overset{M^*}{\sim} B_1$ and $X/A_m \overset{M^*}{\sim} Y/B_m$.

Hence, by Theorem 4 from [12] we obtain $(X, A_1, A_2, \dots A_m) \overset{M}{\sim} (Y, B_1, B_2, \dots B_m)$. □

Corollary 8. *Let X be a space, $1 < n_1 < n_2 < \cdots < n_k$, $1 < m_1 < m_2 < \cdots < m_k$ be natural numbers such that $n_i | n_{i+1}$, $m_i | m_{i+1}$ for $i \in \{1, \dots, k-1\}$. Then*

$$(S(X), S(X^{n_1}), S(X^{n_2}), \dots S(X^{n_k})) \overset{M}{\sim} (S(X), S(X^{m_1}), S(X^{m_2}), \dots S(X^{m_k})).$$

Proposition 23. *Let $X = \overset{\infty}{\underset{i=1}{\oplus}} X_i$ and for every $i \in \mathbb{N}$ there exists a topological group G_i such that $F(X_{i+1}) \simeq F(X_i) * G_i$. Let $\{I_s\colon s \in S\}$, $\{J_s\colon s \in S\}$ be subsets of the set of natural numbers, which form Δ-system and satisfying following conditions:*

1) the sets $I_s \setminus I_0$ and $J_s \setminus J_0$ are both infinite or equal;

2) the sets I_0 and J_0 are both infinite or equal, where $I_0 = \underset{s\in S}{\cap} I_s$, $J_0 = \underset{s\in S}{\cap} J_s$.

Put $A_s = \underset{i\in I_s}{\oplus} X_i$, $B_s = \underset{i\in J_s}{\oplus} X_i$, $A_0 = \underset{i\in I_0}{\oplus} X_i$, $B_0 = \underset{i\in J_0}{\oplus} X_i$.

Then the bundles $(X, \{A_s\colon s \in S\})$ and $(X, \{B_s\colon s \in S\})$ are M-equivalent.

Proof. The R-quotient space A_s/A_0 is homeomorphic to $\Big(\underset{i\in I_s\setminus I_0}{\oplus} X_i\Big)^+$, similarly B_s/B_0 is homeomorphic to $\Big(\underset{i\in J_s\setminus J_0}{\oplus} X_i\Big)^+$. If the sets $I_s\setminus I_0$ and $J_s\setminus J_0$ are equal then A_s/A_0 and B_s/B_0 are homeomorphic. If the sets $I_s\setminus I_0$ and $J_s\setminus J_0$ are both infinite, then by Proposition 1 $\Big(\underset{i\in I_s\setminus I_0}{\oplus} X_i\Big)^+$ is M-equivalent to $\underset{i\in I_s\setminus I_0}{\oplus} X_i$ and $\Big(\underset{i\in J_s\setminus J_0}{\oplus} X_i\Big)^+$ is M-equivalent to $\underset{i\in J_s\setminus J_0}{\oplus} X_i$. Both the spaces are $\underset{i\in I_s\setminus I_0}{\oplus} X_i$ and $\underset{i\in J_s\setminus J_0}{\oplus} X_i$ are M-equivalent to X, so the spaces $\Big(\underset{i\in I_s\setminus I_0}{\oplus} X_i\Big)^+$ and $\Big(\underset{i\in J_s\setminus J_0}{\oplus} X_i\Big)^+$ are M-equivalent and hence M^*-equivalent. Similarly, the spaces A_0 and B_0 are M-equivalent. So, by Theorem 3 from [12] the bundles $(X,\{A_s\colon s\in S\})$ and $(X,\{B_s\colon s\in S\})$ are M-equivalent. □

Proposition 24. *Let $X=\overset{\infty}{\underset{i=1}{\oplus}} X_i$ and for every $i\in\mathbb{N}$ there exists topological group G_i such that $F(X_{i+1})\simeq F(X_i)*G_i$. Let I_1, I_2, J_1, J_2 be a subsets of the set of natural numbers such that*

1) sets $I_1\cap I_2$ and $J_1\cap J_2$ are both infinite or equal;

2) sets $I_1\setminus I_2$ and $J_1\setminus J_2$ are both infinite or equal;

3) sets $I_2\setminus I_1$ and $J_2\setminus J_1$ are both infinite or equal;

4) sets $\mathbb{N}\setminus(I_1\cup I_2)$ and $\mathbb{N}\setminus(J_1\cup J_2)$ are both infinite or equal.

Put $A_1=\underset{i\in I_1}{\oplus} X_i$, $A_2=\underset{i\in I_2}{\oplus} X_i$, $B_1=\underset{i\in J_1}{\oplus} X_i$, $B_2=\underset{i\in J_2}{\oplus} X_i$. Then $(X,A_1,A_2)\overset{M}{\sim}(X,B_1,B_2)$.

Proof. Similarly to Proposition 23 we check that from the condition 1 it follows that

$$A_1/A_2 \overset{M^*}{\sim} B_1/B_2,$$

from the condition 2 follows that $A_2/A_1 \overset{M^*}{\sim} B_2/B_1$, from the condition 3 follows that $A_1\cap A_2 \overset{M^*}{\sim} B_1\cap B_2$, from the condition 4 follows $X/(A_1\cup A_2) \overset{M^*}{\sim} X/(B_1\cup B_2)$. Applying Theorem 6 from [12] we obtain that $(X,A_1,A_2)\overset{M}{\sim}(X,B_1,B_2)$. □

Corollary 9. *Let X be a space. Then the following are equivalent:*

1) $(S(X),S(X^{n_1}),S(X^{n_2}))\overset{M}{\sim}(S(X),S(X^{m_1}),S(X^{m_2}))$;

2) $(n_1=1\iff m_1=1)\wedge(n_2=1\iff m_2=1)\wedge(n_1|n_2\iff m_1|m_2)\wedge(n_2|n_1\iff m_2|m_1)$.

Proof. $(1\implies 2)$

$$n_1=1\iff S(X)=S(X^{n_1})\iff S(X)=S(X^{m_1})\iff m_1=1;$$

$$n_2=1\iff S(X)=S(X^{n_2})\iff S(X)=S(X^{m_2})\iff m_2=1;$$

$$n_1|n_2\iff S(X^{n_1})\subseteq S(X^{n_2})\iff S(X^{m_1})\subseteq S(X^{m_2})\iff m_1|m_2;$$

$$n_2|n_1\iff S(X^{n_2})\subseteq S(X^{n_1})\iff S(X^{m_2})\subseteq S(X^{m_1})\iff m_2|m_1.$$

$(2\implies 1)$ Let us check that conditions 1–4 from proposition 24 hold. Consider the sequence $X_n=X^n$ and the sets $I_1=\{n_1k\colon k\in\mathbb{N}\}$, $I_2=\{n_2k\colon k\in\mathbb{N}\}$, $J_1=\{m_1k\colon k\in\mathbb{N}\}$, $J_2=\{m_2k\colon k\in\mathbb{N}\}$.

Condition 1. The set $I_1\cap I_2=\{l_1k\colon k\in\mathbb{N}\}$, where l_1 is least common multiple of n_1 and n_2, and the set $J_1\cap J_2=\{l_2k\colon k\in\mathbb{N}\}$, where l_2 is least common multiple of m_1 and m_2 are both nonempty. So, both the sets $I_1\cap I_2$ and $J_1\cap J_2$ are nonempty.

Condition 2. The set $I_1 \setminus I_2$ is empty if $n_2|n_1$ and infinite otherwise. The set $J_1 \setminus J_2$ is empty if $m_2|m_1$ and infinite otherwise. Condition 3 is checked similarly.

Condition 4. The set $\mathbb{N} \setminus (I_1 \cup I_2)$ is empty if $(n_1 = 1) \vee (n_2 = 1)$ and infinite otherwise, the set $\mathbb{N} \setminus (J_1 \cup J_2)$ is empty if $(m_1 = 1) \vee (m_2 = 1)$ and infinite otherwise. □

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Lviv Politechnic National University
Lviv, Ukraine
nazar.m.pyrch@lpnu.ua

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