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JOST SOLUTIONS OF SCHRÖDINGER OPERATORS WITH REFLECTIONLESS OPERATOR-VALUED POTENTIALS

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Let H be a separable Hilbert space, and let $\mathcal{H}$ be the Hilbert space of square integrable functions $f\colon \mathbb{R}\to H$. In this paper, we consider the reflectionless Schrödinger operator $T_q f = -f''+qf$ acting in $\mathcal{H}$ and study the corresponding Jost solutions, i.e., solutions of the equation

$$-y''+qy=\lambda^2 y$$

with a reflectionless operator-valued potential q. In particular, we provide an explicit formula for the Jost solutions in terms of solutions of the Riccati equation

$$S'(x)=KS(x)+S(x)K-2S(x)KS(x), \qquad x\in\mathbb{R},$$

where $K\in\mathcal{B}_+(H)\backslash\{0\}$, $S\colon\mathbb{R}\to\mathcal{B}(H)$. Here $\mathcal{B}(H)$ is the Banach algebra of all linear continuous operators acting in H, and $\mathcal{B}_+(H)=\{A\in\mathcal{B}(H)\mid A\ge 0\}$.

1. Introduction. This paper continues the research presented in [1] and is devoted to the investigation of Jost solutions of the equation

$$-y''+qy=\lambda^2 y \qquad (\lambda\in\mathbb{C}) \tag{1}$$

with a reflectionless operator-valued potential q. The obtained results are intended to be used for solving inverse spectral problems for the operators $T_q=-d^2/dx^2+q$.

Reflectionless potentials of the Schrödinger operator. Let H be a separable Hilbert space with the inner product $(\cdot\mid\cdot)$ that is linear in the first argument. Denote by $\mathcal{B}(H)$ the Banach algebra of all everywhere-defined linear continuous operators $A\colon H\to H$, and by $\mathcal{B}_{\rm inv}(H)$ the group of all invertible operators in $\mathcal{B}(H)$. Furthermore, let $\mathcal{B}_+(H)$ be the cone of nonnegative operators, and let I be the identity operator in $\mathcal{B}(H)$. The domain, range, kernel, and the spectrum of a linear operator will be denoted by $\operatorname{dom}(\cdot)$, $\operatorname{ran}(\cdot)$, $\ker(\cdot)$, and $\sigma(\cdot)$, respectively. For arbitrary operators $A,B\in\mathcal{B}(H)$, we write $A<B$ if $A\le B$ and $\ker(B-A)=\{0\}$. If a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(H)$ converges to an operator A in the strong operator topology, we write $A=\operatorname*{s-lim}_{n\to\infty}A_n$.

Let $\mathcal{H}:=L_2(\mathbb{R},H)$ be the Hilbert space of square integrable functions $f\colon\mathbb{R}\to H$ with the inner product

$$(f\mid g)_{\mathcal{H}}:=\int_{\mathbb{R}}(f(x)\mid g(x))\,dx, \qquad f,g\in\mathcal{H}.$$

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Denote by $C_b(\mathbb{R},\mathcal{B}(H))$ the linear space of all continuous bounded functions $f\colon \mathbb{R}\to\mathcal{B}(H)$.

To simplify notation, we will use the following abbreviations for a function $z\mapsto F(z)\in\mathcal{B}(H)$: $(F(z))^*:=F^*(z)$, $(F(z))^{-1}:=F^{-1}(z)$.

We associate every potential $q\in C_b(\mathbb{R},\mathcal{B}(H))$ with the Schrödinger operator $T_q\colon \mathcal{H}\to\mathcal{H}$ defined by

$$T_q f = -f'' + qf$$

on the domain $\operatorname{dom} T_q := W_2^2(\mathbb{R},H)$, where $W_2^2(\mathbb{R},H)$ is the Sobolev space of H-valued functions. If the potential q belongs to the set

$$C_{b,s} := \{q\in C_b(\mathbb{R},\mathcal{B}(H)) \mid \forall x\in\mathbb{R}\quad q^*(x)=q(x)\},$$

then the operator T_q is self-adjoint.

Let $q\in C_{b,s}$ and $z\in\mathbb{C}$. Consider the equation

$$-y''+qy=zy. \tag{2}$$

As shown in [2], for every $z\in\mathbb{C}\setminus\mathbb{R}$, there exist the Weyl–Titchmarsh $\mathcal{B}(H)$-valued right $f_+(z,\cdot)$ and left $f_-(z,\cdot)$ normalized solutions of the equation (2), i.e., the solutions that satisfy the condition

$$f_+(z,0)=f_-(z,0)=I,$$

and for every $h\in H$

$$\int_{\mathbb{R}_\pm}\|f_\pm(z,x)h\|^2\,dx<\infty.$$

The functions

$$m_\pm(z):=f'_\pm(z,0),\qquad z\in\mathbb{C}\setminus\mathbb{R},$$

are called the Weyl–Titchmarsh m-functions of the equation (2) on the half-lines $\mathbb{R}_\pm$. It is known (see [2]) that the equalities

$$m_\pm(\bar z)=m_\pm^*(z),\qquad z\in\mathbb{C}\setminus\mathbb{R},$$

hold, and the functions m_+ and $-m_-$ are Herglotz functions in the upper half-plane, i.e.,

$$\pm\operatorname{Im} m_\pm(z)\ge 0,\qquad z\in\mathbb{C}_+.$$

Definition 1. Let $q\in C_{b,s}$ and $m_\pm$ be the Weyl–Titchmarsh functions of the equation (2). We call the potential q (the operator T_q) *reflectionless* if the $\mathcal{B}(H)$-valued function

$$n_q(\lambda):=\begin{cases} m_+(\lambda^2), & \operatorname{Im}\lambda>0,\ \operatorname{Re}\lambda\ne 0;\\ m_-(\lambda^2), & \operatorname{Im}\lambda<0,\ \operatorname{Re}\lambda\ne 0,\end{cases}$$

has an analytic continuation to the domain $\mathbb{C}\setminus i\mathbb{R}$. Denote by $\mathcal{Q}$ the set of all reflectionless potentials $q\in C_{b,s}$.

An operator Riccati equation and reflectionless potentials. In this subsection, we briefly summarize the results of [1] that will be used in this paper. We consider the operator Riccati equation

$$S'(x)=KS(x)+S(x)K-2S(x)KS(x),\qquad x\in\mathbb{R}, \tag{3}$$

where $K \in \mathcal{B}_+(H)$, $K > 0$ and $S\colon \mathbb{R} \to \mathcal{B}(H)$. It follows from [1] that if the solution S of the equation (3) satisfies the condition $0 < S(0) < I$, then

$$0 < S(x) < I, \qquad x \in \mathbb{R}.$$

Moreover, S has an analytic continuation in the strip

$$\Pi_K := \left\{ z = x + iy \mid x, y \in \mathbb{R},\ \ |y| < \tfrac{\pi}{2\|K\|} \right\}.$$

This continuation is given by the formula

$$S(z) = e^{zK}(S^{-1}(0) - I + e^{2zK})^{-1} e^{zK}, \qquad z \in \Pi_K, \tag{4}$$

and the estimate

$$\|S(z)\| \le [\cos(y\|K\|)]^{-1}, \qquad z \in \Pi_K, \quad y = \operatorname{Im} z,$$

holds. If, additionally, S satisfies the condition $S'(0) \ge 0$, then $S'(x) \ge 0$ for all $x \in \mathbb{R}$, and

$$0 \le S(x_1) \le S(x_2) \le I, \qquad x_1 \le x_2.$$

Definition 2. Let $\mathscr{S}(K)$ be the set of all solutions S of the equation (3) such that $0 < S(0) < I$, $S'(0) \ge 0$. A function $S \in \mathscr{S}(K)$ is called *regular* if the operators $S(0)$ and $I - S(0)$ belong to $\mathcal{B}_{\mathrm{inv}}(H)$. Denote by $\mathscr{S}_{\mathrm{reg}}(K)$ the set of all regular functions $S \in \mathscr{S}(K)$.

Remark 1. In comparison with [1], we have simplified the notation. Namely, we use the notations $\mathscr{S}(K)$ ($\mathscr{S}_{\mathrm{reg}}(K)$) instead of $\mathscr{S}^+(K)$ ($\mathscr{S}^+_{\mathrm{reg}}(K)$).

For every function $S \in \mathscr{S}(K)$, we associate the operators

$$\Gamma := \Gamma_S := S^{-1}(0) - I, \qquad R := R_S := |S'(0)|^{1/2} S^{-1}(0),$$

and construct the following analytic operator-valued functions in Π_K:

$$\begin{aligned} L(z) &:= L_S(z) := e^{zK}(I - S(z)) + e^{-zK} S(z), \\ \Psi(z) &:= \Psi_S(z) := |S'(0)|^{1/2} L(z), \\ q(z) &:= q_S(z) := -4\Psi_S(z) K \Psi_S^*(\bar{z}). \end{aligned} \tag{5}$$

The main result of [1] is the following theorem.

Theorem 1 ([1])**.** *Let $S \in \mathscr{S}(K)$ and $q = q_S$. Then q is a reflectionless potential and*

$$\|q(z)\| \le \frac{2\|K\|^2}{\cos^2(y\|K\|)}, \quad z \in \Pi_K, \quad y = \operatorname{Im} z.$$

In addition to Theorem 1, we also need other results of [1]. We present them in the form of the following four propositions.

Proposition 1 ([1])**.** *Let $S \in \mathscr{S}(K)$ and*

$$S_\varepsilon(x) = e^{xK}(B_\varepsilon^{-1} - I + e^{2xK})^{-1} e^{xK}, \quad x \in \mathbb{R}, \quad \varepsilon \in (0, 1/2),$$

where $B_\varepsilon := \varepsilon I + (1 - 2\varepsilon) S(0)$. Then $S_\varepsilon \in \mathscr{S}_{\mathrm{reg}}(K)$ for all $\varepsilon \in (0, 1/2)$. Moreover, if $\Psi_\varepsilon = \Psi_{S_\varepsilon}$ and $q_\varepsilon = q_{S_\varepsilon}$, then

$$\|S(z) - S_\varepsilon(z)\| = o(1), \quad \|\Psi(z) - \Psi_\varepsilon(z)\| = o(1), \quad \|q(z) - q_\varepsilon(z)\| = o(1) \tag{6}$$

as $\varepsilon \to 0$ uniformly on compact sets in Π_K.

Proposition 2 ([1]). *Let $S \in \mathscr{S}(K)$. Then*

$$-\Psi''(x) + q(x)\Psi(x) = -\Psi(x)K^2, \quad x \in \mathbb{R}, \tag{7}$$
$$S'(x) = \Psi^*(x)\Psi(x), \quad x \in \mathbb{R},$$

and

$$\|\Psi(z)\| \leq \frac{\pi\|K\|^{1/2}}{2\cos(y\|K\|)}, \quad z \in \Pi_K, \quad y = \operatorname{Im} z.$$

If, additionally, $S \in \mathscr{S}_{\mathrm{reg}}(K)$, then $R, \Gamma \in \mathcal{B}(H)$ and

$$\Psi(z) = Re^{-zK}S(z), \quad z \in \Pi_K, \tag{8}$$
$$K\Gamma + \Gamma K = R^*R. \tag{9}$$

For an arbitrary $K \in \mathcal{B}_+(H) \setminus \{0\}$, we introduce the notation

$$\mathcal{O}_\pm(K) := \{\lambda \in \mathbb{C} \mid \pm i\lambda \notin \sigma(K)\}, \quad \mathcal{O}(K) = \mathcal{O}_+(K) \cap \mathcal{O}_-(K).$$

Proposition 3 ([1]). *Let $S \in \mathscr{S}(K)$. Then the formula*

$$M(\lambda) := I - 2\Psi(0)K(K^2 + \lambda^2 I)^{-1}\Psi^*(0) \tag{10}$$

defines an analytic function in $\mathcal{O}(K)$. Moreover, the set

$$\mathcal{O}_M(K) := \{\lambda \in \mathcal{O}(K) \mid M(\lambda) \in \mathcal{B}_{\mathrm{inv}}(H)\}$$

is open, and its complement $\mathbb{C} \setminus \mathcal{O}_M(K)$ is a compact set lying on the imaginary axis.

Proposition 4 ([1]). *Let $S \in \mathscr{S}(K)$, $q = q_S$ and*

$$f(\lambda, x) := e^{i\lambda x}[I - \Psi(x)D(\lambda, x)\Psi^*(0)]M^{-1}(\lambda), \quad \lambda \in \mathcal{O}_M(K),$$

where

$$D(\lambda, x) := K_\lambda e^{-xK} + K_{-\lambda}e^{xK}, \qquad K_\lambda := (K - i\lambda I)^{-1}.$$

Then

$$f(\lambda, \cdot) = \begin{cases} f_+(\lambda^2, \cdot), & \lambda \in \mathbb{C}_+ \setminus i\mathbb{R}; \\ f_-(\lambda^2, \cdot), & \lambda \in \mathbb{C}_- \setminus i\mathbb{R}, \end{cases}$$

where $f_\pm(z, \cdot)$ are the Weyl–Titchmarsh normalized solutions of the equation (2).

Generalization of the results from [1]. In [1], when defining the set $\mathscr{S}(K)$, we required the operator $K \in \mathcal{B}_+(H)$ to satisfy the condition $K > 0$. Within the framework of [1], this was convenient as it simplified the proofs. However, all the above results from [1] remain valid even without this condition.

Hereafter, we assume that $K \in \mathcal{B}_+(H) \setminus \{0\}$, and let P be the orthogonal projector that projects H onto the subspace $H_1 := \overline{\operatorname{ran} K}$. We set $P^\perp := I - P$ and $H_0 := H_1^\perp$.

Lemma 1. *Let $K \in \mathcal{B}_+(H) \setminus \{0\}$, $K_1 := PK|_{H_1}$, and let S be a solution of equation* (3) *satisfying $0 < S(0) < I$ and $S'(0) \geq 0$. Then*

(1) for all $x \in \mathbb{R}$ we have $S(x)P = PS(x)$, $S'(x)P^\perp = P^\perp S'(x) = 0$;

(2) for the function S the following decomposition holds: $S(x) = S_1(x) \oplus S_0(x)$, where $S_1 \in \mathscr{S}(K_1)$, $S_0(x) \equiv S_0(0)$, and $S_j(x) = S(x)|_{H_j}$ $(j \in \{0,1\})$, $K_1 > 0$.

Proof. Let the conditions of Lemma be satisfied. It is easy to check that

$$S(x) = e^{xK}(S^{-1}(0) - I + e^{2xK})^{-1}e^{xK}, \qquad x \in \mathbb{R}. \tag{11}$$

Since $KP^\perp = 0 = P^\perp K$ and (see (3))

$$S'(0) = KS(0) + S(0)K - 2S(0)KS(0),$$

we obtain

$$P^\perp S'(0)P^\perp = -2P^\perp S(0)KS(0)P^\perp.$$

Taking into account that $S'(0) \geq 0$, $S(0) \geq 0$ and $K \geq 0$, we conclude that

$$P^\perp S(0)KS(0)P^\perp = 0 = P^\perp S'(0)P^\perp,$$

and hence,

$$PS(0)P^\perp = 0 = P^\perp S(0)P.$$

Therefore, the equality (11) implies

$$S(x)P = PS(x), \quad S'(x)P^\perp = P^\perp S'(x) = 0, \qquad x \in \mathbb{R}. \tag{12}$$

It follows from the above that H_1 and H_0 are invariant subspaces of $S(x)$. It is easy to see that the function $S_1(x) := S(x)|_{H_1}$ is a solution of the Riccati equation with K_1, i.e.,

$$S_1'(x) = K_1S_1(x) + S_1(x)K_1 - 2S_1(x)K_1S_1(x), \qquad x \in \mathbb{R}.$$

Since $0 < S(0) < I$ and $S'(0) \geq 0$, we conclude that $0 < S_1(0) < I$ and $S_1'(0) \geq 0$. Thus, $S_1 \in \mathscr{S}(K_1)$. It also follows from (12) that the derivative of the function $S_0(x) := S(x)|_{H_0}$ is identically zero, i.e., $S_0(x) \equiv S_0(0)$. □

The above arguments allow us to consider the sets $\mathscr{S}(K)$ and $\mathscr{S}_{\text{reg}}(K)$ for operators $K \in \mathcal{B}_+(H)$ that are not necessarily positive (i.e., omit the requirement that $K > 0$). According to Lemma 1, we have

$$\begin{aligned}
&P\Gamma = \Gamma P, \qquad R = PR = RP,\\
&\Psi_S(z) = P\Psi_S(z) = \Psi_S(z)P = \Psi_{S_1}(z)P, \qquad z \in \Pi_K,\\
&q(z) = Pq_S(z) = q_S(z)P = q_{S_1}(z)P, \qquad z \in \Pi_K.
\end{aligned}$$

From these equalities, we immediately obtain the following two corollaries.

Corollary 1. *Theorem 1 and Propositions 1–4 hold true for an arbitrary $K \in \mathcal{B}_+(H) \setminus \{0\}$.*

Corollary 2. *If $S, \widetilde{S} \in \mathscr{S}(K)$ and $S(0)P = \widetilde{S}(0)P$, then $\Psi_S = \Psi_{\widetilde{S}}$, $q_S = q_{\widetilde{S}}$.*

We define the following subsets of the set $\mathcal{Q}$ of all reflectionless potentials:

$$\begin{aligned}
&\mathcal{Q}_{\text{reg}}(K) := \{q_S \mid S \in \mathscr{S}_{\text{reg}}(K)\}, \qquad &&\mathcal{Q}_\pi(K) := \{q_S \mid S \in \mathscr{S}(K)\},\\
&\mathcal{Q}_{\text{reg}} := \bigcup_K \mathcal{Q}_{\text{reg}}(K), &&\mathcal{Q}_\pi := \bigcup_K \mathcal{Q}_\pi(K).
\end{aligned}$$

It follows from these definitions that

$$\mathcal{Q}_{\mathrm{reg}} \subset \mathcal{Q}_\pi \subset \mathcal{Q}.$$

To understand the structure of the operators T_q with an arbitrary $q \in \mathcal{Q}$, it is first useful to study the properties of the operator T_q with a potential $q \in \mathcal{Q}_\pi$. Although for potentials $q \in \mathcal{Q}_\pi$ there exists a rather simple formula (see (5)), studying the properties of the corresponding operators T_q presents several challenges. On the other hand, the operators T_q with potentials $q \in \mathcal{Q}_{\mathrm{reg}}$ turn out to be much more convenient for analysis. Therefore, we have decided to focus on these operators first. We begin by studying the Jost solutions of the equation (1) with $q \in \mathcal{Q}_{\mathrm{reg}}$, as we consider them to be the key to understanding the corresponding operators T_q.

Note that in the scalar case, the theory of Schrodinger operators with reflectionless potentials is quite well developed (see, for example, [4], [5], [6], [7], [8]), whereas the theory of Schrodinger operators with reflectionless operator-valued potentials is still in its early stages of development. However, the authors hope that within the suggested framework, it will be possible to obtain new results even in the scalar case.

Formulation of the main results. Let $q \in C_{b,s}$. A solution y of the equation (1) is called the right (left) Jost solution if

$$\operatorname*{s-lim}_{x\to+\infty} e^{-i\lambda x} y(x) = I \qquad (\operatorname*{s-lim}_{x\to-\infty} e^{i\lambda x} y(x) = I).$$

For any $\lambda \in \mathbb{C}_+ \setminus i\mathbb{R}$, the Weyl-Titchmarsh solutions of the equation (1) always exist (see [2]). On the other hand, the existence of the Jost solutions can only be guaranteed under additional conditions on the potential q. A classical sufficient condition ensuring the existence of Jost solutions for all $\lambda \in \overline{\mathbb{C}}_+$ is the Marchenko condition [3]

$$\int_{\mathbb{R}} (1+|x|)\|q(x)\|\,dx < \infty.$$

Let $S \in \mathscr{S}_{\mathrm{reg}}(K)$ and set

$$\begin{aligned} e(\lambda,x) &:= e^{i\lambda x}[I - \Psi(x)e^{-xK}K_\lambda R^*], \qquad x\in\mathbb{R}, \qquad \lambda \in \mathcal{O}_+(K), \qquad (13)\\ e_-(\lambda,x) &:= e^{-i\lambda x}[I - \Psi(x)e^{xK}K_\lambda \Gamma^{-1} R^*], \qquad x\in\mathbb{R}, \qquad \lambda \in \mathcal{O}_+(K),\\ \theta(\lambda) &:= I - R\Gamma^{-1}K_\lambda R^*, \qquad \lambda \in \mathcal{O}_+(K). \end{aligned}$$

The main result of this paper is the following two theorems.

Theorem 2. *Let $S \in \mathscr{S}_{\mathrm{reg}}(K)$ and $q = q_S$. Then for an arbitrary $\lambda \in \mathcal{O}_+(K)$ the function $e(\lambda,\cdot)$ (respectively, $e_-(\lambda,\cdot)$) is the right (left) Jost solution of the equation* (1). *Moreover,*

(1) for each $\lambda \in \mathcal{O}(K)$ the following relation holds:

$$e(\lambda,x) = e_-(-\lambda,x)\,\theta(\lambda), \qquad x \in \mathbb{R}; \qquad (14)$$

(2) for each $\lambda \in \mathcal{O}(K)$ the functions $e(\lambda,\cdot)$ and $e_-(\lambda,\cdot)$ form a fundamental system of solutions to the equation (1), *which also means that every $\mathcal{B}(H)$-valued solution has the form*

$$y(x) = e(\lambda,x)a + e_-(\lambda,x)b, \qquad x\in\mathbb{R},$$

where the operators $a, b \in \mathcal{B}(H)$ are uniquely determined by the solution y;

(3) for all $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$ the following estimates hold:

$$\|e^{-i\lambda x} e(\lambda, x)\| \le 1, \quad \|e^{i\lambda x} e_-(\lambda, x)\| \le 1, \quad \|\theta(\lambda)\| \le 1 \qquad (x \in \mathbb{R}).$$

Let us agree to call the functions

$$e(\lambda) := e(\lambda, 0), \qquad e_-(\lambda) := e_-(\lambda, 0) \qquad (\lambda \in \mathcal{O}_+(K))$$

the right and left Jost functions of the operator T_q, respectively. These functions carry important information and therefore deserve special attention.

Theorem 3. *Let $S \in \mathscr{S}_{\text{reg}}(K), q = q_S$. Then (see* (10)*)*

(1) for all $\lambda \in \mathcal{O}(K)$ the operators $e(\lambda)$, $e_-(\lambda)$ and $\theta(\lambda)$ belong to $\mathcal{B}_{\text{inv}}(H)$, and the following equality hold:

$$e(\lambda)e^*(\bar{\lambda}) = M(\lambda) = e_-(\lambda)e_-^*(\bar{\lambda}), \qquad e_-^{-1}(\lambda)e(\lambda) = \theta(\lambda);$$

(2) for all $\lambda \in \mathbb{C}_+ \setminus i\mathbb{R}$

$$e(\lambda, x) = f_+(\lambda^2, x)e(\lambda), \qquad e_-(\lambda, x) f_-(\lambda^2, x) e_-(\lambda), \qquad x \in \mathbb{R}. \tag{15}$$

2. Proof of Theorems 2 and 3. The proof of the main theorems is divided into parts, each of which is formulated as a separate lemma.

Lemma 2. *Let $S \in \mathscr{S}_{\text{reg}}(K)$ and $q = q_S$. Then for each $\lambda \in \mathcal{O}_+(K)$ the function $e(\lambda, \cdot)$ is a solution of the equation* (1) *and*

$$\begin{aligned} &\operatorname*{s-lim}_{x\to+\infty} e^{-i\lambda x} e(\lambda, x) = I, && \operatorname*{s-lim}_{x\to+\infty} e^{-i\lambda x} e'(\lambda, x) = i\lambda I, \\ &\operatorname*{s-lim}_{x\to-\infty} e^{-i\lambda x} e(\lambda, x) = \theta(\lambda), && \operatorname*{s-lim}_{x\to-\infty} e^{-i\lambda x} e'(\lambda, x) = i\lambda\theta(\lambda). \end{aligned} \tag{16}$$

Moreover, for an arbitrary $\lambda \in \mathcal{O}_-(K)$

$$\begin{aligned} &\operatorname*{s-lim}_{x\to+\infty} e^{i\lambda x} e^*(\bar{\lambda}, x) = I, && \operatorname*{s-lim}_{x\to+\infty} e^{i\lambda x} (e^*)'(\bar{\lambda}, x) = -i\lambda I, \\ &\operatorname*{s-lim}_{x\to-\infty} e^{i\lambda x} e(\bar{\lambda}, x) = \theta^*(\bar{\lambda}), && \operatorname*{s-lim}_{x\to-\infty} e^{i\lambda x} e'(\bar{\lambda}, x) = -i\lambda\theta^*(\bar{\lambda}). \end{aligned} \tag{17}$$

Proof. Let the conditions of Lemma be satisfied and

$$\Phi(x) := Re^{-xK}, \qquad x \in \mathbb{R}. \tag{18}$$

Then the formula (13) can be rewritten as

$$e(\lambda, x) = e^{i\lambda x}[I - \Psi(x)K_\lambda\Phi^*(x)], \qquad x \in \mathbb{R}. \tag{19}$$

Thus,

$$e^{-i\lambda x}[-e''(\lambda, x) + q(x)e(\lambda, x) - \lambda^2 e(\lambda, x)] = [2i\lambda(\Psi K_\lambda \Phi^*)' + (\Psi K_\lambda \Phi^*)'' - q\Psi K_\lambda \Phi^* + q](x).$$

Using the equality (7), we obtain

$$\begin{gathered}2i\lambda(\Psi K_\lambda\Phi^*)' + (\Psi K_\lambda\Phi^*)'' - q\Psi K_\lambda\Phi^* + q = \\ = 2i\lambda\Psi' K_\lambda\Phi^* - 2i\lambda\Psi K K_\lambda\Phi^* + (\Psi'' - q\Psi + \Psi K^2)K_\lambda\Phi^* - 2\Psi' K K_\lambda\Phi^* + q = \\ = -2\Psi'(K - i\lambda I)K_\lambda\Phi^* - 2i\lambda\Psi K K_\lambda\Phi^* + 2\Psi K^2 K_\lambda\Phi^* + q = \\ = -2\Psi'\Phi^* + 2\Psi K(K - i\lambda I)K_\lambda\Phi^* + q = -2(\Psi\Phi^*)' + q.\end{gathered}$$

Therefore, it suffices to show that

$$2(\Psi\Phi^*)' = q.$$

It follows from (8) that

$$\Psi(x)\Phi^*(x) = \Phi(x)S(x)\Phi^*(x),$$

and hence, taking into account (3), we obtain

$$2(\Psi\Phi^*)' = 2(\Phi S\Phi^*)' = 2\Phi(-KS - SK + S')\Phi^* = -4\Phi SKS\Phi^* = -4\Psi K\Psi^* = q.$$

Thus, for $\lambda \in \mathcal{O}_+(K)$ the function $e(\lambda, \cdot)$ is a solution of the equation (1).

In view of (4), (8) and (18), the formula (13) can be rewritten as

$$e(\lambda, x) = e^{i\lambda x}[I - R(\Gamma + e^{2xK})^{-1}K_\lambda R^*], \qquad x \in \mathbb{R}, \quad \lambda \in \mathcal{O}_+(K). \tag{20}$$

Since

$$R = RP, \quad P\Gamma = \Gamma P, \quad PK = KP,$$

then

$$e(\lambda, x) = e^{i\lambda x}[I - RP(\Gamma + Pe^{2xK}P)^{-1}PK_\lambda R^*], \qquad x \in \mathbb{R}.$$

Taking into account that $\operatorname*{s-lim}_{x\to+\infty} e^{-xK}P = 0$, we get the equalities

$$\operatorname*{s-lim}_{x\to+\infty} P(\Gamma + Pe^{2xK}P)^{-1}P = \operatorname*{s-lim}_{x\to+\infty} e^{-xK}P(e^{-xK}\Gamma e^{-xK} + I)^{-1}e^{-xK}P = 0,$$

$$\operatorname*{s-lim}_{x\to-\infty}(\Gamma + Pe^{2xK}P)^{-1} = \Gamma^{-1}.$$

Therefore, by applying (20), the relations (16) and (17) can be easily derived. □

Lemma 3. *Let $S \in \mathscr{S}_{\mathrm{reg}}(K)$ and $q = q_S$. Then for each $\lambda \in \mathcal{O}_+(K)$ the function $e_-(\lambda, \cdot)$ is a solution of the equation* (1) *and*

$$\begin{aligned}&\operatorname*{s-lim}_{x\to-\infty} e^{i\lambda x}e_-(\lambda, x) = I, &&\operatorname*{s-lim}_{x\to-\infty} e^{i\lambda x}e'_-(\lambda, x) = i\lambda I,\\ &\operatorname*{s-lim}_{x\to+\infty} e^{i\lambda x}e_-(\lambda, x) = \theta^*(-\bar\lambda), &&\operatorname*{s-lim}_{x\to+\infty} e^{i\lambda x}e'_-(\lambda, x) = i\lambda\theta^*(-\bar\lambda).\end{aligned} \tag{21}$$

Proof. Consider the function $S^\circ(x) := I - S(-x)$. It is easy to check that the function S° is a solution of the Riccati equation (3) and $S^\circ \in \mathscr{S}_{\mathrm{reg}}(K)$. Let us define

$$\Psi^\circ := \Psi_{S^\circ}, \quad q^\circ := q_{S^\circ}, \quad R^\circ := R_{S^\circ}, \quad \Gamma^\circ := \Gamma_{S^\circ}.$$

Simple calculations show that

$$R^\circ = R\Gamma^{-1}, \quad \Gamma^\circ = \Gamma^{-1}, \quad \Psi^\circ(x) = \Psi(-x), \quad q^\circ(x) = q(-x). \tag{22}$$

In view of Lemma 2, the function

$$e^\circ(\lambda,x)=e^{i\lambda x}[I-\Psi^\circ(x)e^{-xK}K_\lambda(R^\circ)^*],\qquad x\in\mathbb{R},\quad \lambda\in\mathcal{O}_+(K), \tag{23}$$

is a solution of the equation $-y''(x)+q(-x)y(x)=\lambda^2y(x)$. Moreover,

$$\begin{aligned}&\operatorname*{s-lim}_{x\to+\infty} e^{-i\lambda x}e^\circ(\lambda,x)=I, &&\operatorname*{s-lim}_{x\to+\infty} e^{-i\lambda x}(e^\circ)'(\lambda,x)=i\lambda I,\\ &\operatorname*{s-lim}_{x\to-\infty} e^{-i\lambda x}e^\circ(\lambda,x)=\theta^\circ(\lambda), &&\operatorname*{s-lim}_{x\to-\infty} e^{-i\lambda x}(e^\circ)'(\lambda,x)=i\lambda\theta^\circ(\lambda),\end{aligned} \tag{24}$$

where

$$\theta^\circ(\lambda)=I-R^\circ(\Gamma^\circ)^{-1}K_\lambda(R^\circ)^*,\qquad \lambda\in\mathcal{O}_+(K).$$

This implies that the function $x\mapsto e^\circ(\lambda,-x)$ is a solution of the equation $-y''+qy=\lambda^2y$. Taking into account that (22) and (23), we obtain that

$$e^\circ(\lambda,-x)=e^{-i\lambda x}[I-\Psi(x)e^{xK}K_\lambda\Gamma^{-1}R^*]=e_-(\lambda,x),\qquad x\in\mathbb{R},\quad \lambda\in\mathcal{O}_+(K),$$

and

$$\theta^\circ(\lambda)=I-RK_\lambda\Gamma^{-1}R^*=\theta^*(-\bar\lambda).$$

Therefore, using the asymptotics (24), we get the statement of Lemma. □

Lemma 4. *Let $S\in\mathscr{S}_{\mathrm{reg}}(K)$. Then for each $\lambda\in\mathcal{O}(K)$ the functions $e(\lambda,\cdot)$ and $e(-\lambda,\cdot)$ form a fundamental system of solutions to the equation* (1)*, i.e., every $\mathcal{B}(H)$-valued solution of the equation* (1) *has the form*

$$y(x)=e(\lambda,x)a+e(-\lambda,x)b,\qquad x\in\mathbb{R},$$

where the operators $a,b\in\mathcal{B}(H)$ are uniquely determined by the solution y.

Proof. For an abitrary $\lambda\in\mathcal{O}(K)$, denote by $E(\lambda,\cdot)$ and $E_*(\lambda,\cdot)$ the functions acting from $\mathbb{R}$ to $\mathcal{B}(H\times H)$ by the formulas

$$E(\lambda,x):=\begin{pmatrix}e(\lambda,x) & e(-\lambda,x)\\ e'(\lambda,x) & e'(-\lambda,x)\end{pmatrix},\quad E_*(\lambda,x):=\begin{pmatrix}(e^*)'(\bar\lambda,x) & -e^*(\bar\lambda,x)\\ -(e^*)'(-\bar\lambda,x) & e^*(-\bar\lambda,x)\end{pmatrix}.$$

Let us show that for all $x\in\mathbb{R}$ and $\lambda\in\mathcal{O}(K)$ the equalities

$$E_*(\lambda,x)E(\lambda,x)=-2i\lambda\mathbb{I},\qquad E(\lambda,x)E_*(\lambda,x)=-2i\lambda\mathbb{I},\qquad x\in\mathbb{R}, \tag{25}$$

hold, where $\mathbb{I}$ is the identity operator in the algebra $\mathcal{B}(H\times H)$, i.e., $\mathbb{I}:=\begin{pmatrix}I&0\\0&I\end{pmatrix}$. For a fixed $x\in\mathbb{R}$, the left-hand and right-hand sides of the equalities (25) are analytic functions in $\mathcal{O}(K)$. Therefore, it is sufficient to prove that the equalities (25) hold for each $x\in\mathbb{R}$ and for large $\lambda\in\mathbb{R}$.

Fix $\lambda\in\mathbb{R}\setminus\{0\}$, and consider the Wronskians

$$\begin{aligned}W_1(x)&:=(e^*)'(\lambda,x)e(\lambda,x)-e^*(\lambda,x)e'(\lambda,x),\\ W_2(x)&:=(e^*)'(-\lambda,x)e(\lambda,x)-e^*(-\lambda,x)e'(\lambda,x).\end{aligned}$$

It is easy to check that $W_j'(x) \equiv 0$. Taking into account that (16) and (17), we obtain

$$W_1(x) \equiv \operatorname*{s-lim}_{x\to+\infty}[(e^*)'(\lambda,x)e(\lambda,x) - e^*(\lambda,x)e'(\lambda,x)] = -2i\lambda I,$$
$$W_2(x) \equiv \operatorname*{s-lim}_{x\to+\infty}[(e^*)'(-\lambda,x)e(\lambda,x) - e^*(-\lambda,)e'(\lambda,x)] = 0.$$

It follows that

$$E_*(\lambda,x)E(\lambda,x) = -2i\lambda\mathbb{I}, \qquad x\in\mathbb{R}, \qquad \lambda\in\mathbb{R}\setminus\{0\}.$$

By the definition of the functions $e(\lambda,\cdot)$, it is easy to conclude that for a fixed $x\in\mathbb{R}$

$$\left\| E(\lambda,x) - \begin{pmatrix} e^{i\lambda x}I & e^{-i\lambda x}I \\ i\lambda e^{i\lambda x}I & -i\lambda e^{-i\lambda x}I \end{pmatrix} \right\| = o(1), \qquad \mathbb{R}\ni\lambda\to\infty.$$

Therefore, for each $x\in\mathbb{R}$, there exists $r_x>0$ such that the operators $E(\lambda,x)$ are invertible in the algebra $\mathcal{B}(H\times H)$ for $\lambda>r_x$. From the above, it follows that

$$E(\lambda,x)E_*(\lambda,x) = E_*(\lambda,x)E(\lambda,x) = -2i\lambda\mathbb{I}, \qquad \lambda>r_x.$$

Thus, the equalities (25) are established.

Let y be an arbitrary $\mathcal{B}(H)$-valued solution of (1). Consider the solution

$$u(x) = e(\lambda,x)a + e(-\lambda,x)b - y(x), \qquad x\in\mathbb{R},$$

where

$$\begin{pmatrix} a \\ b \end{pmatrix} = E^{-1}(\lambda,0)\begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}.$$

It is easy to see that $u(0)=u'(0)=0$, i.e., u is a solution of the Cauchy problem for the equation (1) with zero initial conditions. By the uniqueness theorem, $u\equiv 0$ and

$$y(x) = e(\lambda,x)a + e(-\lambda,x)b, \qquad x\in\mathbb{R}.$$

Obviously, in this case, the operators a and b are uniquely determined. □

Lemma 5. *Let $S\in\mathscr{S}_{\mathrm{reg}}(K)$, $R=R_S,\Gamma=\Gamma_S$. For all $\lambda\in\mathcal{O}(K)$ and $x\in\mathbb{R}$, the following equalities hold:*

$$(\Gamma+e^{2xK})^{-1}K_\lambda R^*RK_{-\lambda}(\Gamma+e^{2xK})^{-1} = K_{-\lambda}(\Gamma+e^{2xK})^{-1} + (\Gamma+e^{2xK})^{-1}K_\lambda - $$
$$-2(\Gamma+e^{2xK})^{-1}e^{xK}K(K^2+\lambda^2 I)^{-1}e^{xK}(\Gamma+e^{2xK})^{-1}, \tag{26}$$
$$\Gamma^{-1}K_\lambda R^*RK_{-\lambda}\Gamma^{-1} = \Gamma^{-1}K_\lambda + K_{-\lambda}\Gamma^{-1}, \quad K_{-\lambda}\Gamma^{-1}R^*R\Gamma^{-1}K_\lambda = K_{-\lambda}\Gamma^{-1} + \Gamma^{-1}K_\lambda. \tag{27}$$

Proof. In view of (9), for an arbitrary $x\in\mathbb{R},\ \lambda\in\mathcal{O}(K)$

$$(K-i\lambda I)(\Gamma+e^{2xK}) + (\Gamma+e^{2xK})(K+i\lambda I) = R^*R + 2Ke^{2xK}. \tag{28}$$

Multiplying the equality (28) on the left by the operator $(\Gamma+e^{2xK})^{-1}K_\lambda$ and on the right by $K_{-\lambda}(\Gamma+e^{2xK})^{-1}$, we get

$$K_{-\lambda}(\Gamma+e^{2xK})^{-1} + (\Gamma+e^{2xK})^{-1}K_\lambda = (\Gamma+e^{2xK})^{-1}K_\lambda R^*RK_{-\lambda}(\Gamma+e^{2xK})^{-1} +$$
$$+2(\Gamma+e^{2xK})^{-1}e^{xK}K(K+\lambda^2 I)^{-1}e^{xK}(\Gamma+e^{2xK})^{-1}.$$

Therefore, the equality (26) is proved. It also follows from (9) that

$$(K - i\lambda I)\Gamma + \Gamma(K + i\lambda I) = R^*R. \tag{29}$$

Multiplying the equality (29) on the left by the operator $\Gamma^{-1}K_\lambda$ and on the right by $K_{-\lambda}\Gamma^{-1}$, we get the first equality in (27). Multiplying the equality (29) on the left by the operator $K_{-\lambda}\Gamma^{-1}$ and on the right by $\Gamma^{-1}K_\lambda$, we obtain the second equality in (27). □

Lemma 6. *Let $S \in \mathscr{S}_{\mathrm{reg}}(K)$, $q = q_S$. For all $\lambda \in \mathcal{O}(K)$ and $x \in \mathbb{R}$, the following equalities hold:*

$$e(\lambda, x)e^*(\bar\lambda, x) = I - 2\Psi(x)K(K^2 + \lambda^2 I)^{-1}\Psi^*(x), \tag{30}$$

$$e(\lambda, x) = e_-(-\lambda, x)\theta(\lambda), \tag{31}$$

$$\theta(\lambda)\theta^*(\bar\lambda) = \theta^*(\bar\lambda)\theta(\lambda) = I. \tag{32}$$

Proof. Let the conditions of Lemma be satisfied. Multiplying the equality (26) on the left by R and on the right by R^*, we get

$$e(\lambda, x)e^*(\bar\lambda, x) = I - 2R(\Gamma + e^{2xK})^{-1}e^{xK}K(K + \lambda^2 I)^{-1}e^{xK}(\Gamma + e^{2xK})^{-1}R^*.$$

Taking into account that (see (4) and (8))

$$R(\Gamma + e^{2xK})^{-1}e^{xK} = Re^{-xK}S(x) = \Psi(x),$$

we obtain (30).

Let us prove (31). From the definitions, it follows that

$$e_-(-\lambda, x)\theta(\lambda) = e^{i\lambda x}[I - \Psi(x)e^{xK}K_{-\lambda}\Gamma^{-1}R^*][I - R\Gamma^{-1}K_\lambda R^*].$$

Using (27), we get

$$\Psi(x)e^{xK}K_{-\lambda}\Gamma^{-1}R^*R\Gamma^{-1}K_\lambda R^* = \Psi(x)e^{xK}K_{-\lambda}\Gamma^{-1}R^* + \Psi(x)e^{xK}\Gamma^{-1}K_\lambda R^*.$$

Thus,

$$e_-(-\lambda, x)\theta(\lambda) = e^{i\lambda x}[I + \Psi(x)e^{xK}\Gamma^{-1}K_\lambda R^* - R\Gamma^{-1}K_\lambda R^*].$$

Since (see (4) and (8))

$$\begin{aligned}\Psi(x)e^{xK} - R = Re^{-xK}S(x)e^{xK} - R = R(\Gamma + e^{2xK})^{-1}e^{2xK} - R = \\ = -R(\Gamma + e^{2xK})^{-1}\Gamma = -\Psi(x)e^{-xK}\Gamma,\end{aligned}$$

then

$$e_-(-\lambda, x)\theta(\lambda) = e^{i\lambda x}[I - \Psi(x)e^{-xK}K_\lambda R^*] = e(\lambda, x).$$

It remains to prove (32). Multiplying the first equality in (27) on the left by R and on the right by R^*, we obtain an equality equivalent to $\theta(\lambda)\theta^*(\bar\lambda) = I$. Similarly, multiplying the second equality in (27) on the left by R and on the right by R^*, we get an equality equivalent to $\theta^*(\bar\lambda)\theta(\lambda) = I$. □

Lemma 7. *Let $S \in \mathscr{S}_{\mathrm{reg}}(K)$. Then*

$$\|e^{-i\lambda x}e(\lambda, x)\| \le 1, \quad \|e^{i\lambda x}e_-(\lambda, x)\| \le 1, \quad x \in \mathbb{R}, \quad \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}, \tag{33}$$

$$\|\theta(\lambda)\| \le 1, \qquad \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}. \tag{34}$$

Proof. First, we will prove Lemma in the case $K \in \mathcal{B}_{\rm inv}(H)$ and for the functions $e(\lambda,\cdot)$ and θ. For a fixed $x \in \mathbb{R}$, the function

$$g(\lambda) := e^{-i\lambda x} e(\lambda, x)$$

is analytic in the domain $\mathcal{O}_+(K)$, which contains $\overline{\mathbb{C}}_+$. It follows from (30) that

$$g(\xi)g^*(\xi) \le I, \qquad \xi \in \mathbb{R},$$

and hence,

$$\|g(\xi)\| \le 1, \qquad \xi \in \mathbb{R}. \tag{35}$$

Moreover (see (19)),

$$\|g(\lambda)\| = \|I - \Psi(x)K_\lambda \Phi^*(x)\| = 1 + o(1), \qquad \lambda \to \infty.$$

Denote by γ_r ($r > 1$) a simple closed curve in $\mathbb{C}$ consisting of the interval $[-r, r]$ and the semicircle $\{z \in \mathbb{C}_+ \mid |z| = r\}$. Let $h_1, h_2 \in H$. Then the scalar function

$$\varphi(\lambda) := (g(\lambda)h_1 \mid h_2), \qquad \lambda \in \overline{\mathbb{C}}_+,$$

is analytic in $\mathbb{C}_+$ and continuous in the closure of the domain Ω_r bounded by the curve γ_r. From the above, it follows that

$$\max_{\lambda \in \gamma_r} |\varphi(\lambda)| \le \|h_1\|\|h_2\|(1 + o(1)), \qquad r \to \infty.$$

Therefore, according to the maximum principle, we have

$$|\varphi(\lambda)| \le \|h_1\|\|h_2\|, \qquad \lambda \in \overline{\mathbb{C}}_+,$$

thus,

$$|(g(\lambda)h_1 \mid h_2)| \le \|h_1\|\|h_2\|, \qquad h_1, h_2 \in H.$$

This implies that

$$\|e^{-i\lambda x} e(\lambda, x)\| = \|g(\lambda)\| \le 1, \qquad \lambda \in \overline{\mathbb{C}}_+.$$

From (32), we have

$$\|\theta(\xi)\| = 1, \qquad \xi \in \mathbb{R}.$$

Moreover, the function θ is analytic in $\mathcal{O}_+(K) \supset \overline{\mathbb{C}}_+$ and

$$\|\theta(\lambda)\| = \|I - R\Gamma^{-1}K_\lambda R^*\| = 1 + o(1), \qquad \lambda \to \infty.$$

Repeating the same reasoning as in the case of the function g, we obtain the estimate

$$\|\theta(\lambda)\| \le 1, \qquad \lambda \in \overline{\mathbb{C}}_+.$$

Let us consider the general case. For this, we will apply the previously established results for $K \in \mathcal{B}_{\rm inv}(H)$ and pass to the limit to cover a general non-zero $K \in \mathcal{B}_+(H)$. For an arbitrary $\eta \in (0, 1)$, put by definition

$$K_{\{\eta\}} := K + \eta I, \quad S_{\{\eta\}}(x) := e^{x(K+\eta I)}[S^{-1}(0) - I + e^{2x(K+\eta I)}]^{-1} e^{x(K+\eta I)}.$$

It is easy to check that $S_{\{\eta\}} \in \mathscr{S}_{\rm reg}(K_{\{\eta\}})$ and

$$S_{\{\eta\}}(0) = S(0), \qquad S'_{\{\eta\}}(0) = S'(0) + 2\eta S(0)(I - S(0)). \tag{36}$$

Let $\Gamma_{\{\eta\}} := \Gamma_{S_{\{\eta\}}}$ and $R_{\{\eta\}} := R_{S_{\{\eta\}}}$. It follows from (36) that

$$\Gamma_{\{\eta\}} = \Gamma, \qquad \lim_{\eta\to 0} \|S'(0) - S'_{\{\eta\}}(0)\| = 0, \qquad \lim_{\eta\to 0} \|R - R_{\{\eta\}}\| = 0. \tag{37}$$

Denote by $e_{\{\eta\}}(\lambda,\cdot)$ and $\theta_{\{\eta\}}$ the functions $e(\lambda,\cdot)$ and θ corresponding to the function $S_{\{\eta\}}$. Since $K_{\{\eta\}} \in \mathcal{B}_{\rm inv}(H)$, it follows from the previously proven results that

$$\|e^{-i\lambda x} e_{\{\eta\}}(\lambda, x)\| \le 1, \quad \|\theta_{\{\eta\}}(\lambda)\| \le 1, \quad x \in \mathbb{R}, \quad \lambda \in \overline{\mathbb{C}}_+, \quad \eta \in (0,1). \tag{38}$$

From (20), we obtain that for all $x \in \mathbb{R}$ and $\lambda \in \mathcal{O}_+(K)$

$$e_{\{\eta\}}(\lambda, x) = e^{i\lambda x}[I - R_{\{\eta\}}(\Gamma + e^{2x(K+\eta I)})^{-1}(K + \eta I - i\lambda I)^{-1} R^*_{\{\eta\}}].$$

Moreover, by definition, we have

$$\theta_{\{\eta\}}(\lambda) = I - R_{\{\eta\}}\Gamma^{-1}(K + \eta I - i\lambda I)^{-1} R^*_{\{\eta\}}, \qquad \lambda \in \mathcal{O}_+(K).$$

From this, taking into account (37), we obtain that in the uniform operator topology

$$\lim_{\eta\to 0} e_{\{\eta\}}(\lambda, x) = e(\lambda, x), \qquad \lim_{\eta\to 0} \theta_{\{\eta\}}(\lambda) = \theta(\lambda), \qquad x \in \mathbb{R}, \quad \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}.$$

Passing to the limit in (38) as $\eta \to 0$, we get

$$\|e^{-i\lambda x} e(\lambda, x)\| \le 1, \quad \|\theta(\lambda)\| \le 1, \qquad x \in \mathbb{R}, \quad \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}.$$

From the proven result, it follows that (see the proof of Lemma 3)

$$\|e^{-i\lambda x} e^{\circ}(\lambda, x)\| \le 1, \qquad \lambda \in \overline{\mathbb{C}}_+.$$

Since $e_-(\lambda, x) = e^{\circ}(\lambda, -x)$, we conclude that

$$\|e^{i\lambda x} e_-(\lambda, x)\| \le 1, \qquad \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}.$$

□

Lemma 8. *Let $S \in \mathscr{S}_{\rm reg}(K)$, $q = q_S$. Then*

(1) for all $\lambda \in \mathcal{O}(K)$ the operators $\theta(\lambda)$ belong to $\mathcal{B}_{\rm inv}(H)$, and for $\lambda \in \mathbb{R}\setminus\{0\}$ the operators $\theta(\lambda)$ are unitary;

(2) for all $\lambda \in \mathcal{O}_M(K)$ the operators $e(\lambda)$ and $e_-(\lambda)$ belong to $\mathcal{B}_{\rm inv}(H)$, and the following equalities hold:

$$e(\lambda)e^*(\bar\lambda) = M(\lambda) = e_-(\lambda)e_-^*(\bar\lambda), \quad e(\lambda) = e_-(\lambda)\theta(\lambda). \tag{39}$$

Proof. Let $\lambda \in \mathcal{O}(K)$. It follows from (32) that $\theta(\lambda) \in \mathcal{B}_{\mathrm{inv}}(H)$ and $\theta^{-1}(\lambda) = \theta^*(\bar{\lambda})$. Therefore, if $\lambda \in \mathbb{R} \setminus \{0\}$, then $\theta(\lambda)$ is unitary.

The equalities (30) imply that for all $\lambda \in \mathcal{O}_M(K)$

$$e(\lambda)e^*(\bar{\lambda}) = I - 2\Psi(0)K(K^2 + \lambda^2 I)^{-1}\Psi^*(0) = M(\lambda),$$

and, in view of (31),

$$e(\lambda) = e_-(-\lambda)\theta(\lambda), \qquad \lambda \in \mathcal{O}(K). \tag{40}$$

Therefore, taking into account (32), we get

$$M(\lambda) = e(\lambda)e^*(\bar{\lambda}) = e_-(-\lambda)\theta(\lambda)\theta^*(\bar{\lambda})e_-^*(-\bar{\lambda}) = e_-(-\lambda)e_-^*(-\bar{\lambda}).$$

Since $-\lambda \in \mathcal{O}_M(K)$ and $M(-\lambda) = M(\lambda)$, we have $e_-(\lambda)e_-^*(\bar{\lambda}) = M(\lambda)$.

It remains to verify that for all $\lambda \in \mathcal{O}_M(K)$ the operators $e(\lambda)$ and $e_-(\lambda)$ belong to $\mathcal{B}_{\mathrm{inv}}(H)$. Since $\theta(\lambda) \in \mathcal{B}_{\mathrm{inv}}(H)$, in view of the equality (40), it suffices to establish that $e(\lambda) \in \mathcal{B}_{\mathrm{inv}}(H)$ for all $\lambda \in \mathcal{O}_M(K)$. Let us fix $\lambda \in \mathcal{O}_M(K)$. By the definition of the set $\mathcal{O}_M(K)$, the operator $M(\lambda)$ belongs to $\mathcal{B}_{\mathrm{inv}}(H)$. Therefore, it follows from (39) that $\operatorname{ran} e(\lambda) = H$. And hence, by the bounded inverse theorem, $e(\lambda) \in \mathcal{B}_{\mathrm{inv}}(H)$ if $\ker e(\lambda) = \{0\}$. Assume there exists a nonzero $h \in H$ such that $e(\lambda)h = 0$. In the proof of Lemma 4, we established

$$E_*(\lambda, x)E(\lambda, x) = -2i\lambda\mathbb{I}, \qquad x \in \mathbb{R}.$$

This implies that for all $x \in \mathbb{R}$

$$W_1(x) = (e^*)'(\lambda, x)e(\lambda, x) - e^*(\lambda, x)e'(\lambda, x) = -2i\lambda I.$$

And hence,

$$W_1(0)h = -e^*(\lambda)e'(\lambda, 0)h = -2i\lambda h. \tag{41}$$

Then, taking into account (39), we obtain

$$0 = -2i\lambda e(\lambda)h = -e(\lambda)e^*(\lambda)e'(\lambda, 0) = -M(\lambda)e'(\lambda, 0)h.$$

Since $M(\lambda) \in \mathcal{B}_{\mathrm{inv}}(H)$, it follows that $e'(\lambda, 0)h = 0$. Hence, by (41), we conclude that $h = 0$, which leads to a contradiction. Therefore, $\ker e(\lambda) = \{0\}$ and $e(\lambda) \in \mathcal{B}_{\mathrm{inv}}(H)$. □

Remark 2. The function $\mathcal{O}(K) \ni \lambda \mapsto \theta(\lambda) \in \mathcal{B}(H)$ is an analogue of the Blaschke product in the upper half-plane $\mathbb{C}_+$. Moreover, all its singularities and the singularities of the function $\lambda \mapsto \theta^{-1}(\lambda)$ are concentrated in the set $\mathbb{C} \setminus \mathcal{O}(K)$, which is compact on the imaginary axis. These singularities are not necessarily poles.

With all the above results at hand, the proofs of Theorems 2 and 3 can be completed as follows.

Proof of Theorem 2. According to Lemmas 2 and 3, for all $\lambda \in \mathcal{O}_+(K)$ the function $e(\lambda, \cdot)$ is the right Jost solution and the function $e_-(\lambda, \cdot)$ is the left Jost solution of the equation (1). The equality (14) was proven in Lemma 6. It follows from Lemma 4 and the equalities (14), (32) that for each $\lambda \in \mathcal{O}(K)$ the functions $e(\lambda, \cdot)$ and $e_-(\lambda, \cdot)$ form a fundamental system of solutions to the equation (1). Finally, the estimates in Part (3) were established in Lemma 7. □

Proof of Theorem 3. Part (1) of Theorem 3 was proven in Lemma 8. It remains to prove the equalities (15). Fix $\lambda \in \mathbb{C}_+ \setminus i\mathbb{R}$, and put

$$y_+(x) := e(\lambda, x)e^{-1}(\lambda), \qquad y_-(x) := e_-(\lambda, x)e_-^{-1}(\lambda), \qquad x \in \mathbb{R}.$$

Then $y_+(0) = y_-(0) = I$, and using (16) and (21), for each $h \in H$, we have

$$\int_{\mathbb{R}_\pm} \|y_\pm(x)h\|^2\, dx < \infty.$$

By the uniqueness of the Weyl-Titchmarsh solutions, we conclude that $y_\pm = f_\pm(\lambda, \cdot)$. Therefore,

$$e(\lambda, x) = f_+(\lambda^2, x)e(\lambda), \qquad e_-(\lambda, x)f_-(\lambda^2, x)e_-(\lambda), \qquad x \in \mathbb{R}.$$

□

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