O. V. Lopotko  

**EVENLY POSITIVE DEFINITE FUNCTION OF HILBERT SPACE AND SOME ALGEBRAIC RELATIONSHIP**


A generalization of P. A. Minlos, V. V. Sazonov’s theorem is proved in the case of bounded evenly positive definite function given in a Hilbert space. The integral representation is obtained for a family of bounded commutative self-adjoint operators which are connected by algebraic relationship.

This scientific work is devoted to 95-th anniversary of Professor Yu. M. Berezansky

For the first time theorem about integral representation for positive definite (p.d.) functions $k(x)$ ($x \in \mathbb{R}^1$) was obtained in papers of Krein M. G. and Berezansky Yu. M. [1, 5]. Then Berezansky Yu. M. obtained the integral representation for functions $k(x)$ ($x \in \mathbb{R}^n$) [2]. In the next investigations Berezansky Yu. M. used methods of the spectral theory of operators. These methods play an important role in infinite-dimensional analysis. Using these methods Berezansky Yu. M. obtained the integral representation for p.d. functions in the space $(-2l; 2l) \times \mathbb{R}^1 \times \mathbb{R}^1 \times \ldots$ and in the Hilbert space $H_{2l}$ [4]. In particular, for every p.d. continuous in $j$-topology function $k(x)$ ($x \in H_{2l}; l \leq \infty$) the following integral representation is valid

$$k(x) = \int_{H} e^{i(\lambda, x)} d\rho(\lambda), \quad (x \in H_{2l}). \quad (1)$$

Here $d\rho(\lambda)$ is a non-negative finite measure on some $\sigma$-algebra of Borel sets from $H$. Conversely, every integral of form (1) is a p.d. function in $H$ which is continuous at $O$ in $j$-topology. In the case of $l = \infty$, the measure $d\rho(\lambda)$ is uniquely determined by $k(x)$; in the case of $l < \infty$, there is no uniqueness.

The article presents a modification of this theory to the case of bounded evenly p.d. functions $k(x)$ ($x \in H$).

Let $H$ be a real valued separable Hilbert space. A real convex bounded even function $k(x)$ ($x \in H$) is called positive definite (e.p.d.), if for $x^{(1)}, \ldots, x^{(N)} \in H$ and $\xi_1, \ldots, \xi_N \in \mathbb{C}^1$ ($n = N$) the inequality

$$\sum_{j,\kappa=1}^{N} \frac{1}{2} \left[ k(x^{(j)} + x^{(\kappa)}) + k(x^{(j)} - x^{(\kappa)}) \right] \xi_j \overline{\xi_\kappa} \geq 0 \quad (2)$$

2010 Mathematics Subject Classification: 46E20, 47G10.

Keywords: integral representation; bounded evenly positive definite functions; bounded self-adjoint operators.

doi:10.30970/ms.55.1.85-93

© O. V. Lopotko, 2021
Let $\mathcal{A}$ be a nonnegative nuclear operator. A topology given by neighborhoods of the origin of the form $\{x \in H \mid (\mathcal{A}x, x)_H < \varepsilon\}$ ($\varepsilon > 0$) is called the $j$-topology in $H$. A function $k(x)$ ($x \in H$) is said to be continuous at the origin in the $j$-topology if it is continuous at $O$ in the topology inducted by the $j$-topology on $H$.

**Theorem 1.** Let $k(x)$ ($x \in H$) be any bounded e.p.d. function which is continuous at $O$ in the $j$-topology. Then the function admits the representation

$$k(x) = \int_H \cos(\lambda, x) \, d\rho(\lambda) \quad (x \in H),$$

(3)

where $d\rho(\lambda)$ is a non-negative finite measure defined on the $\sigma$-algebra $\mathfrak{B}(H)$ of Borel subsets of $H$. Conversely, each integral of the form (3) is a bounded e.p.d. function in $H$ which is continuous at the origin in the $j$-topology. The measure $d\rho(\lambda)$ is uniquely defined for a given $k$.

We will prove that (2) implies the following inequality:

1) If $\sup |k(x)| = C < +\infty$ then $|k(x)| \leq |k(0)|$, i.e. $C = |k(0)|$.

Indeed, put in (2) $N = 2$, $\xi_1 = -1$, $\xi_2 = 1$, $x^{(1)} = x$, $x^{(2)} = 0$. Then $3k(0) + k(2x) - 4k(x) \geq 0$. It follows that $k(2x) + 3k(0) \geq 4k(x)$ and

$$|k(x)| \leq \frac{|k(2x)| + 3|k(0)|}{4} \leq \frac{3|k(0)| + C}{4}, \quad C = \sup |k(x)| \leq \frac{3|k(0)| + C}{4}.$$

It means that $C \leq |k(0)|$, and therefore

$$|k(x)| \leq |k(0)|.$$  

(4)

2) For an arbitrary convex bounded e.p.d. function the following inequality holds

$$|k(x^{(1)}) - k(x^{(2)})|^2 \leq 2k(0)\{k(0) - k(x^{(1)} - x^{(2)})\}.$$

Note that for an arbitrary p.d. kernel $K(x; y)$ we have (see [4, p. 469, Lemma 4.1])

$$|K(x; z) - K(y; z)|^2 \leq K(z; z) \left[ K(x; x) - 2\mathrm{Re} K(x; y) + K(y; y) \right].$$

Put in this inequality $K(x; y) = \frac{1}{2} \left[k \left(x^{(1)} - x^{(2)}\right) + k \left(x^{(1)} + x^{(2)}\right)\right]$ ($z = 0$). We obtain

$$\left|k \left(x^{(1)}\right) - k \left(x^{(2)}\right)\right|^2 \leq k(0) \left[ k(0) + \frac{1}{2} k \left(2x^{(1)}\right) + \frac{1}{2} k \left(2x^{(1)} - k \left(x^{(1)} - x^{(2)}\right) - k \left(x^{(1)} + x^{(2)}\right)\right).$$
But
\[ \frac{1}{2}(k(2x^{(1)}) + k(2x^{(2)})) - k(x^{(1)} + x^{(2)}) \leq 0, \]
because the function \( k(x) \) is convex. Therefore the previous inequality can be rewritten in the form
\[ \left| k \left( x^{(1)} \right) - k \left( x^{(2)} \right) \right|^2 \leq k(0) \left\{ k(0) - k \left( x^{(1)} - x^{(2)} \right) \right\}. \] (5)

**Proof of Theorem 1.** Using (4), (5) (see [4, p.467–469]) we can prove that there exists a non-negative nuclear operator \( \mathcal{A} \) in \( H \) with kernel of zero such that \( k(x) (x \in H) \) is uniformly continuous in the following sense: for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for \( x^{(1)}, x^{(2)} \in H \) satisfying \( (\mathcal{A}(x^{(1)} - x^{(2)}), x^{(1)} - x^{(2)}) < \delta \) and \( x^{(1)} - x^{(2)} \in H \) one has \( |k(x^{(1)}) - k(x^{(2)})| < \varepsilon. \)

We introduce a dot product in \( H \): \( (x, y)_{\mathcal{A}} = (\mathcal{A}x, y)_H \). Let \( H_\mathcal{A} \) be a completion of \( H \) with respect to \((\cdot, \cdot)_{\mathcal{A}}\). Let us describe \( H_\mathcal{A} \) by coordinates. For this purpose we choose an orthonormal basis in \( H \) consisting of the eigen vectors of the operator \( \mathcal{A} \) realizing \( H \) as a space \( l_2 = l_2(\mathbb{R}^1) \) with respect to a decomposition in this basis. Then space \( H_\mathcal{A} \) is realized as a Hilbert space \( l_2(\mathcal{A}) = \{ x \in \mathbb{R}^\infty = \mathbb{R}^1 \times \mathbb{R}^1 \times \ldots \} \) with the scalar product \( (x, y)_{\mathcal{A}} = \sum_{n=1}^\infty a_n x_n y_n \in l_2(\mathcal{A}) \) (\( (a_n \delta_{nm})_{n,m=1}^\infty \) is a matrix of the operator \( \mathcal{A} \) in the considering basis; \( a_n > 0 \)). Let us consider now the standard Gaussian measure \( \gamma_1 = \sqrt{\frac{1}{\pi}} e^{-x^2} \), it is essential that \( \gamma_1(l_2(\mathcal{A})) = 1 \), (see [8]).

Since \( k(x) (x \in H) \) is uniformly continuous with respect to the norm \( \| \cdot \|_{H_\mathcal{A}} = (\cdot, \cdot)^{\frac{1}{2}}_H \) it can be extended by continuity to a uniformly continuous function \( k_1(x) \) on the whole space \( H_\mathcal{A} \). Passing to the coordinate from of \( H_\mathcal{A} \), we find that \( k_1(x) \) is defined on a set \( l_2(\mathcal{A}) \subset \mathbb{R}^\infty \) of a full Gaussian measure.

Using \( k_1(x) \) we introduce a quasi-scalar product
\[ \langle \varphi, \psi \rangle = \int_{l_2(\mathcal{A})} \int_{l_2(\mathcal{A})} \frac{1}{2} \left[ k_1(x + y) + k_1(x - y) \right] \varphi(y) \overline{\psi(x)} \, d\gamma_1(x) \, d\gamma_1(y), \quad (\varphi, \psi \in C_{b,cyl} (\mathbb{R}^\infty)) \]. (6)

Then, \( H_\kappa \) be a Hilbert space obtained as a result of a completion \( (\cdot, \cdot)_{H_\kappa} = (\cdot, \cdot) \). Since the kernel \( \frac{1}{2} [k_1(x + y) + k_1(x - y)] \) is even on \( x, y \), we consider (6) on even functions. Now we will construct a family of unitary operators, acting in \( H_\kappa \). For this goal we will construct a rigging (chain)

\[ H_0 \supset H_+ \supset \mathcal{D}. \]

The role of the space \( H_0 \) will be played just by \( H_\kappa \), namely \( H_0 = H_\kappa \) and \( H_+ = L_2(l_2(\mathcal{A}), \mathcal{B}(l_2(\mathcal{A}); \gamma_1)) \) and \( \mathcal{D} \), which consists of the restrictions of functions from the space \( \mathcal{A}(\mathbb{R}^\infty) = \bigcup_{n=1}^\infty \mathcal{A}(\mathbb{R}^n) \), where \( \mathcal{A}(\mathbb{R}^n) = \mathcal{A}(\mathbb{R}^1) \otimes \ldots \otimes \mathcal{A}(\mathbb{R}^1) \) (\( n \) times) and \( \mathcal{A}(\mathbb{R}^1) = l_2(\mathbb{R}^1, \gamma_1) \) on the cylindrical functions. For details of the space \( \mathcal{A}(\mathbb{R}^\infty) \) see [3, p. 110–113]. The properties of the space \( \mathcal{A}(\mathbb{R}^\infty) \) imply the properties of the space \( \mathcal{D} \) and the embedding \( \mathcal{D} \subset L_2 = H_+ \). This embedding is quasi-nuclear. Indeed, the scalar product (6) has the form \( (\varphi, \psi)_{H_\kappa} = (K \varphi, \psi)_{L_2} \), where \( K \) is an integral operator in \( L_2 \) generated by the continuous kernel \( K(x, y) = \frac{1}{2} [k(x + y) + k(x - y)] \). Since
\[ \int_{l_2(\mathcal{A})} K(x, x) \, d\gamma_1(x) \leq k(0) < \infty, \]
this operator is nuclear. On the other hand, for the chain $H_- \supset H_0 \supset H_+$, we have 
$$(\varphi, \psi)_{H_0} = (1, \varphi, \psi)_{H_+}, \quad (\varphi, \psi \in H_+);$$
therefore $K = O(I \upharpoonright H_+)$, and the nuclearity of $K$ yields 
the quasinuclearity of the embedding $H_+ \to H_0$. Hence, we have constructed the chain

$$H_\kappa = H_0 \supset H_+ = L_2 \supset \mathcal{D} = A(\mathbb{R}^\infty).$$ (7)

Let us construct a family of unitary operators $\mathcal{A}_x (x \in L_2 = H)$ acting in the space $H_\kappa$. They are the shift operators of functions by $t$ with a certain factor. The factor appears because the Gaussian measure $\gamma_1$ is not invariant under such shifts. Let us first define the operator $\mathcal{A}_t$ on the function $\varphi \in C^{\infty}_{b,cyl}(\mathbb{R}^\infty) = \bigcup_{n=1}^{\infty} C^{\infty}_b(\mathbb{R}^n)$ by the equality

$$(\mathcal{A}_t \varphi)(x) = \frac{1}{2} \left[ \varphi(x-t) \left( \frac{d\gamma_1}{d\gamma_1}(x) \right) + \varphi(x+t) \left( \frac{d\gamma_1}{d\gamma_1}(x) \right) \right], \quad (t \in L_2; x \in l_2(\mathcal{A}))$$ (8)

where derivative is

$$\left( \frac{d\gamma_1}{d\gamma_1}(x) \right) = \rho_{\gamma_1}(t, \cdot) = \exp \left( -\|t\|_{L_2}^2 - 2(t, \cdot)_{L_2} \right)$$

Since for an even kernel $K(x, y)$ and even $\varphi, \psi$ expression (6) has the form

$$\int_{l_2(\mathcal{A})} \int_{l_2(\mathcal{A})} \frac{1}{2} \left[ k(x+y) + k(x-y) \right] \varphi(y) \psi(x) d\gamma_1(x) d\gamma_1(y) =$$

$$= \int_{l_2(\mathcal{A})} \int_{l_2(\mathcal{A})} k(y-x) \varphi(y) \psi(x) d\gamma_1(x) d\gamma_1(y) = \int_{l_2(\mathcal{A})} \int_{l_2(\mathcal{A})} k(x+y) \varphi(y) \psi(x) d\gamma_1(x) d\gamma_1(y),$$

by a simple change of variables we get for any $t \in L_2$

$$(\mathcal{A}_t \varphi, \mathcal{A}_t \psi)_{H_\kappa} = (\varphi, \psi)_{H_\kappa}, \quad (\varphi, \psi \in C^{\infty}_{b,cyl}(\mathbb{R}^\infty)).$$

Indeed,

$$\frac{1}{4} \int_{l_2(\mathcal{A})} \int_{l_2(\mathcal{A})} k(y-x) \varphi(x-t) \frac{d\gamma_1}{d\gamma_1}(x) \psi(y-t) \frac{d\gamma_1}{d\gamma_1}(y) d\gamma_1(x) d\gamma_1(y) = \left[ x = x' + t; \quad y = y' + t \right] =$$

$$= \frac{1}{4} \int_{l_2(\mathcal{A})} \int_{l_2(\mathcal{A})} k(y'-x') \varphi(x') \frac{d\gamma_1}{d\gamma_1}(x') \psi(y') \frac{d\gamma_1}{d\gamma_1}(y') \frac{d\gamma_1}{d\gamma_1}(x') d\gamma_1(y') =$$

$$= \frac{1}{4} \int_{l_2(\mathcal{A})} \int_{l_2(\mathcal{A})} k(y'-x') \varphi(x') \psi(y') d\gamma_1(x') d\gamma_1(y'),$$ (9)

$$+ \varphi(x+t) \frac{d\gamma_1}{d\gamma_1}(x) \psi(y-t) \frac{d\gamma_1}{d\gamma_1}(y) d\gamma_1(x) d\gamma_1(y) =$$

$$= \frac{1}{4} \int_{l_2(\mathcal{A})} \int_{l_2(\mathcal{A})} k(x+y) \varphi(x-t) \frac{d\gamma_1}{d\gamma_1}(x) \psi(y-t) \frac{d\gamma_1}{d\gamma_1}(y) d\gamma_1(x) d\gamma_1(y) +$$

$$+ \frac{1}{4} \int_{l_2(\mathcal{A})} \int_{l_2(\mathcal{A})} k(x+y) \varphi(x+t) \frac{d\gamma_1}{d\gamma_1}(x) \psi(y+t) \frac{d\gamma_1}{d\gamma_1}(y) d\gamma_1(x) d\gamma_1(y).$$
But

\[
\frac{1}{4} \int \int_{I_2(A) I_2(A)} k(x + y) \varphi(x - t) \frac{d\gamma_{1,-t}}{d\gamma_1}(x) \overline{\psi(y + t)} \frac{d\gamma_{1,t}}{d\gamma_1}(y) \, d\gamma_1(x) \, d\gamma_1(y) = \begin{cases} x = x' - t; \\ y = y' + t \end{cases} = \\
= \frac{1}{4} \int \int_{I_2(A) I_2(A)} k(x' + y') \varphi(x') \frac{d\gamma_{1,-t}}{d\gamma_1}(x') \overline{\psi(y') \frac{d\gamma_{1,t}}{d\gamma_1}(y')} \, d\gamma_1_t(x') \, d\gamma_1(y'),
\]

(10)

\[
\frac{1}{4} \int \int_{I_2(A) I_2(A)} k(x + y) \varphi(x - t) \frac{d\gamma_{1,-t}}{d\gamma_1}(x) \overline{\psi(y + t)} \frac{d\gamma_{1,t}}{d\gamma_1}(y) \, d\gamma_1(x) \, d\gamma_1(y) = \begin{cases} x = x' + t; \\ y = y' - t \end{cases} = \\
= \frac{1}{4} \int \int_{I_2(A) I_2(A)} k(x' + y') \varphi(x') \frac{d\gamma_{1,-t}}{d\gamma_1}(x') \overline{\psi(y')} \frac{d\gamma_{1,t}}{d\gamma_1}(y') \, d\gamma_1_t(x') \, d\gamma_1(y'),
\]

(11)

\[
\frac{1}{4} \int \int_{I_2(A) I_2(A)} k(y - x) \varphi(x + t) \frac{d\gamma_{1,t}}{d\gamma_1}(x) \overline{\psi(y + t)} \frac{d\gamma_{1,t}}{d\gamma_1}(y) \, d\gamma_1(x) \, d\gamma_1(y) = \begin{cases} x = x' - t; \\ y = y' - t \end{cases} = \\
= \frac{1}{4} \int \int_{I_2(A) I_2(A)} k(y' - x') \varphi(x') \frac{d\gamma_{1,-t}}{d\gamma_1}(x') \overline{\psi(y')} \frac{d\gamma_{1,-t}}{d\gamma_1}(y') \, d\gamma_1_t(x') \, d\gamma_1(y'),
\]

(12)

Summing up (9)–(12), we obtain that \((\mathcal{A}_t \varphi, \mathcal{A}_t \psi)_{H_\kappa} = (\varphi, \psi)_{H_\kappa} (\varphi, \psi \in C_{b,cyl}(\mathbb{R}^\infty))\). Thus, each operator \(\mathcal{A}_t\) is isometric and therefore it may be extended to an isometric operator on the whole of \(H_\kappa\) by continuity; we preserve the notation \(\mathcal{A}_t\) for the latter.

Now, we can verify the equality

\[
(\mathcal{A}_t \varphi, \psi)_{H_\kappa} = (\varphi, \mathcal{A}_t \psi)_{H_\kappa}, (\varphi, \psi \in C_{b,cyl}(\mathbb{R}^\infty))
\]

Indeed,

\[
(\mathcal{A}_t \varphi, \psi)_{H_\kappa} = \frac{1}{2} \int \int_{I_2(A) I_2(A)} k(y - x) \varphi(x - t) \left( \frac{d\gamma_{1,-t}}{d\gamma_1}(x) \overline{\psi(y)} \right) d\gamma_1(x) \, d\gamma_1(y) + \\
+ \frac{1}{2} \int \int_{I_2(A) I_2(A)} k(y - x) \varphi(x + t) \left( \frac{d\gamma_{1,t}}{d\gamma_1}(x) \overline{\psi(y)} \right) d\gamma_1(x) \, d\gamma_1(y) = \begin{cases} x = x' + t; \\ y = y' + t \end{cases} = \\
= \frac{1}{2} \int \int_{I_2(A) I_2(A)} k(y' - x') \varphi(x') \frac{d\gamma_{1,-t}}{d\gamma_1}(x') \overline{\psi(y' + t)} \, d\gamma_1_t(x') \, d\gamma_1_t(y') + \begin{cases} x = x' - t; \\ y = y' - t \end{cases} + \\
+ \frac{1}{2} \int \int_{I_2(A) I_2(A)} k(y' - x') \varphi(x') \frac{d\gamma_{1,-t}}{d\gamma_1}(x') \overline{\psi(y' - t)} \, d\gamma_1_t(x') \, d\gamma_1_t(y') =
\]
Similarly, we can verify the equality
\[(A_t \varphi, A_s \psi)_{H_\kappa} = \frac{1}{2} \left[ (A_{t-s} \varphi, \psi)_{H_\kappa} + (A_{t+s} \varphi, \psi)_{H_\kappa} \right]. \tag{14}\]

It implies that \(A_t = A_t^* = A_{-t}\), and \(A_t \cdot A_s = \frac{1}{2} [A_{t+s} + A_{t-s}] (t, s \in l_2).\) Hence, we have constructed the group of unitary operators \((A_t)_{t \in l_2}\). These operators and the chain \((6)\) are connected by the conditions 1, 2 of Theorem 1.2 [4, Ch. 4]. Therefore, we can write the representation
\[A_x = \int_X e^{i(\lambda, x)} dE(\lambda), \quad (x \in X),\]

for the introduced operators \(A_x\) in the space \(H_\kappa\). Here, \(E\) is a decomposition of unity on \(\mathfrak{B}(X)\).

But so as \(A_x = A_{-x}\), we can write such representation
\[A_x = \int_H \cos(\lambda, x) dE(\lambda), \quad (x \in H). \tag{15}\]

For any \(x \in X_A\), the space \(H_\kappa\) includes \(\delta\)-function \(\delta_x\), concentrated at the point \(x\), and \((\delta_x, \delta_y)_{H_\kappa} = \frac{1}{2} [k_1(x-y) + k_1(x+y)] (x, y \in X_A)\). More precisely, for a given negative space \(H_\kappa\) and zero space \(L_2\), we can construct a chain \(H_\kappa \supset L_2 \supset H_+\). As a result, we obtain the space \(H_+\) consisting of continuous bounded functions on \(X_A\). The space \(H_\kappa\) includes vectors \(\delta_x\) such that \((\delta_x, \varphi)_{L_2} = \varphi(x) (x \in X_A, \varphi \in H_+)\) and the indicated equality holds.

It is easy to understand that \(A_t \delta_0 = \delta_t\) for any \(t \in X\). Indeed, let \((\chi_n(\cdot))_{n=1}^\infty\) be a sequence of functions from \(L_2\) which converges to \(\delta_0\) in \(H_\kappa\), and let each of these functions be equal to the characteristic function of the ball centered at \(0\) in \(X_A\) with radius \(n^{-1}\) divided by the measure \(\gamma_1\) of this ball. Clearly, the action of the operator \(A_t\) on \(\chi_n\) is given by \((7)\).

For \(\varphi \in H_+\) we have
\[
(A_t \delta_0, \varphi)_{L_2} = \lim_{n \to \infty} (A_t \chi_n, \varphi)_{L_2} = \lim_{n \to \infty} \left[ \int_{H_A} \frac{1}{2} \chi_n(x-t) \frac{d\gamma_{1,-t}(x)\varphi(x)}{d\gamma_1} d\gamma_1(x) + \int_{H_A} \frac{1}{2} \chi_n(x+t) \frac{d\gamma_{1,t}(x)\varphi(x)}{d\gamma_1} d\gamma_1(x) \right] =
\]
\[
= \lim_{n \to \infty} \left[ \int_{H_A} \frac{1}{2} \chi_n(x-t) \varphi(x) d\gamma_{1,-t}(x) + \int_{H_A} \frac{1}{2} \chi_n(x+t) \varphi(x) d\gamma_{1,t}(x) \right] =
\]
\[
= \lim_{n \to \infty} \left[ \int_{H_A} \frac{1}{2} \chi_n(x) \varphi(x+t) d\gamma_1(x) + \int_{H_A} \frac{1}{2} \chi_n(x) \varphi(x-t) d\gamma_1(x) \right] = \varphi(t) = (\delta_t, \varphi)_{L_2},
\]
and this yields the equality \( A_t \delta_0 = \delta_t \).

After these remarks, representation (3) immediately follows from (15). One should apply the last equality to \( \delta_0 \) and multiply by a scalar \( \delta_0 \) in \( H_\kappa \). Here \( \sigma(\alpha) = (E(\alpha)\delta_0, \delta_0)_{H_\kappa} \) \( (\alpha \in \mathcal{B}(X)) \). Theorem 1 is proved.

Applying Theorem 1 we can prove the following theorem.

**Theorem 2.** In order that a family \( (A_t) (t \in H) \) of self-adjoint bounded operators in the Hilbert space \( H \) admits a representation

\[
A_t = \int_H \cos(\lambda, t) \, dE(\lambda), \quad (\lambda \in H),
\]

where \( E \) is some decomposition of identity of \( \mathcal{B}(H) \), it is necessary and sufficient that the operators satisfy the following conditions:

1) \( \frac{1}{2} [A_{t+s} + A_{t-s}] = A_t A_s, \quad A_{(t_1,...,t_n,...)} = A_{(t_1,...,-t_n,...)}; \quad A_0 = I; \)

2) \( A_t \) strongly continuously depends on \( t; \)

3) \( \|A_t\| \leq b < \infty. \)

**Proof.** Sufficiency. Let operators \( A_t \) commute accordingly to 1) and have a common simple spectrum. Then there exists a cycle vector \( \Omega \) such that a closed linear hull \( \{A_t, \Omega\} = H. \)

Put \( (A_0 \Omega, \Omega) = k(t) \). Then \( k(t) \) is continuous and even in every variable and e.p.d., bounded function on \( H \). Indeed, the function \( k(t) \) is e.p.d.

\[
\sum_{i,j=1}^n \frac{1}{2} [k(x^{(i)} + x^{(j)}) + k(x^{(i)} - x^{(j)})] \xi_i \xi_j = \sum_{i,j=1}^n \frac{1}{2} [(A_{(t^{(i)} \pm t^{(j)})}\Omega, \Omega) + (A_{(t^{(i)} \pm t^{(j)})}\Omega, \Omega)] \xi_i \xi_j = \sum_{i,j=1}^n \frac{1}{2} [(A_{(t^{(i)})\Omega, \Omega} + (A_{(t^{(i)})\Omega, A_{(t^{(i)})\Omega}})] \xi_i \xi_j = \left\| \sum_{i=1}^n \xi_i A_{(t^{(i)})\Omega} \right\|_H^2 \geq 0.
\]

The continuity, evenness in every variable and boundedness of the function \( k(t) \) follow from assumptions of Theorem 2. Therefore by Theorem 1, we obtain an integral representation for the function \( k(t) \)

\[
k(t) = \int_H \cos(\lambda, t) \, d\rho(\lambda), \quad (t \in H),
\]

where \( d\rho(\lambda) \) is an even non-negative finite measure, which is defined on some \( \sigma \)-algebra \( \mathcal{B}(H) \) of Borel sets from \( H \). We construct a correspondence: every vector \( \sum_{\kappa=1}^n c_{\kappa} A_{t^{(\kappa)}} \Omega \in H \) is paired with the function

\[
\sum_{\kappa=1}^n c_{\kappa} \cos \left( \lambda, t^{(\kappa)} \right) \in l_2 (H, d\rho(\lambda)), \quad (n = 1, 2, \ldots).
\]

This correspondence is isometric because

\[
\left\| \sum_{\kappa=1}^n c_{\kappa} A_{t^{(\kappa)}} \Omega \right\|_H^2 = \sum_{i,j=1}^n c_i c_j \left( A_{t^{(i)}} \Omega, A_{t^{(j)}} \Omega \right)_H = \sum_{i,j=1}^n c_i c_j \frac{1}{2} \left[ (A_{t^{(i)} + t^{(j)}} \Omega, \Omega)_H + (A_{t^{(i)} - t^{(j)}} \Omega, \Omega)_H \right] = \frac{1}{2} \sum_{i,j=1}^n c_i c_j \left( A_{t^{(i)} + t^{(j)}} \Omega, \Omega \right)_H + \frac{1}{2} \sum_{i,j=1}^n c_i c_j \left( A_{t^{(i)} - t^{(j)}} \Omega, \Omega \right)_H = \sum_{i,j=1}^n \frac{1}{2} c_i c_j \left( (A_{t^{(i)} + t^{(j)}} \Omega, \Omega)_H + (A_{t^{(i)} - t^{(j)}} \Omega, \Omega)_H \right) = \sum_{i,j=1}^n \frac{1}{2} c_i c_j \left( k(t^{(i)} + t^{(j)}) + k(t^{(i)} - t^{(j)}) \right) \right\|_H^2 \geq 0.
\]
\[
\sum_{i,j=1}^{n} c_i c_j \int_{H} \frac{1}{2} \left[ \cos \left( \lambda, t^{(i)} + t^{(j)} \right) + \cos \left( \lambda, t^{(i)} - t^{(j)} \right) \right] d\rho(\lambda) = \\
= \left\| \sum_{\kappa=1}^{n} c_\kappa \cos \left( \lambda, t^{(\kappa)} \right) \right\|_{l_2(H,d\rho(\lambda))}^2.
\]

Then it can be extended to the unitary map \( U : H \rightarrow l_2(H,d\rho(\lambda)) \). Furthermore, the operator \( A_t \) corresponds to the operator of multiplication in \( l_2 \left( H, d\rho(\lambda) \right) \) on the function \( \cos(\lambda, t) \).

Since

\[
A_t \left( \sum_{\kappa=1}^{n} c_\kappa A_{t^{(\kappa)}} \Omega \right) = \sum_{\kappa=1}^{n} c_\kappa \left( \frac{A_{t^{(\kappa)}} + A_{t^{(\kappa)}}}{2} \right) \Omega = \\
= U^* \left( \sum_{\kappa=1}^{n} c_\kappa \cos \left( \lambda, t^{(\kappa)} \right) + \cos \left( \lambda, t^{(\kappa)} \right) \right) = \\
= U^* \cos(\lambda, t) U \left( \sum_{\kappa=1}^{n} c_\kappa \cos \left( \lambda, t^{(\kappa)} \right) \right) = U^* \cos(\lambda, t) U \left( \sum_{\kappa=1}^{n} c_\kappa A_{t^{(\kappa)}} \Omega \right),
\]

that is \( A_t = U^* Q_{\cos(\lambda,t)} U \) (\( Q_{\cos(\lambda,t)} \) denotes the operator of multiplication on \( \cos(\lambda, t) \) in \( l_2 \left( H, d\rho(\lambda) \right) \)).

Since for a family \( Q_{\cos(\lambda,t)} \) the integral representation is true

\[
Q_{\cos(\lambda,t)} = \int_{H} \cos(\lambda, t) \ dQ_{\chi_{\Delta}(\cdot)},
\]

we have \( A_t = \int_{H} \cos(\lambda, t) \ dE(\lambda) \), where \( E(\Delta) = U^* Q_{\chi_{\Delta}(\cdot)} U \). \( \chi_{\Delta}(\cdot) \) is the characteristic function of an interval \( \Delta \in H \). So the integral representation (16) is proved.

The uniqueness of decomposition of identity follows from the uniqueness of the measure \( d\rho(\lambda) \). If the cyclic vector \( \Omega \) is missed then for the family of commuted self-adjoint operators \( A_t \ (t \in H) \) in \( H \) there exists a sequence of measures on \( \sigma \)-algebra of cylindric sets in \( H \) with a Borel basis \( \rho_1, \ldots, \rho_\kappa, \ldots \) and a unitary operator

\[
U : H \rightarrow \bigoplus_{\kappa=1}^{\infty} l_2(H,d\rho_\kappa(\lambda))
\]

such that in every space \( l_2 \left( H, d\rho_\kappa(\lambda) \right) \) one has \( A_t = U^* Q_{\cos(\lambda,t)} U \), where \( Q_{\cos(\lambda,t)} \) denotes the operator of multiplication on \( \cos(\lambda, t) \) in \( l_2 \left( H, d\rho_\kappa(\lambda) \right) \).

The necessity of the conditions 1), 2), 3) follows from integral representation (16). \( \square \)

An integral representation for the family of bounded self-adjoint operators in one-dimensional case, which are connected by an algebraic relationship, was proved in [6].
REFERENCES


National Forestry and Wood Technology University of Ukraine
Lviv, Ukraine
Lopotko1304@gmail.com

Received 24.04.2020
Revised 18.12.2020