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O. V. Lopotko

EVENLY POSITIVE DEFINITE FUNCTION OF HILBERT SPACE AND SOME ALGEBRAIC RELATIONSHIP

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A generalization of P. A. Minlos, V. V. Sazonov's theorem is proved in the case of bounded evenly positive definite function given in a Hilbert space. The integral representation is obtained for a family of bounded commutative self-adjoint operators which are connected by algebraic relationship.

This scientific work is devoted to 95-th anniversary of Professor Yu. M. Berezansky

For the first time theorem about integral representation for positive definite (p.d.) functions $k(x)$ $(x \in \mathbb{R}^1)$ was obtained in papers of Krein M. G. and Berezansky Yu. M. [1, 5]. Then Berezansky Yu. M. obtained the integral representation for functions $k(x)$ $(x \in \mathbb{R}^n)$ [2]. In the next investigations Berezansky Yu. M. used methods of the spectral theory of operators. These methods play an important role in infinite-dimensional analysis. Using these methods Berezansky Yu. M. obtained the integral representation for p.d. functions in the space $(-2l; 2l) \times \mathbb{R}^1 \times \mathbb{R}^1 \times \dots$ and in the Hilbert space H_{2l} [4]. In particular, for every p.d. continuous in j-topology function $k(x)$ $(x \in H_{2l}; l \leq \infty)$ the following integral representation is valid

$$k(x) = \int_H e^{i(\lambda,x)}\, d\rho(\lambda), \quad (x \in H_{2l})\,. \tag{1}$$

Here $d\rho(\lambda)$ is a non-negative finite measure on some σ-algebra of Borel sets from H. Conversely, every integral of form (1) is a p.d. function in H which is continuous at O in j-topology. In the case of $l = \infty$, the measure $d\rho(\lambda)$ is uniquely determined by $k(x)$; in the case of $l < \infty$, there is no uniqueness.

The article presents a modification of this theory to the case of bounded evenly p.d. functions $k(x)$ $(x \in H)$.

Let H be a real valued separable Hilbert space. A real convex bounded even function $k(x)$ $(x \in H)$ is called positive definite (e.p.d.), if for $x^{(1)}, \dots, x^{(N)} \in H$ and $\xi_1, \dots, \xi_N \in \mathbb{C}^1$ $(n = N)$ the inequality

$$\sum_{j,\kappa=1}^{N} \frac{1}{2} \left[k\left(x^{(j)} + x^{(\kappa)}\right) + k\left(x^{(j)} - x^{(\kappa)}\right)\right] \xi_j \overline{\xi_\kappa} \geq 0 \tag{2}$$

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holds.

Let $\mathcal{A}$ be a nonnegative nuclear operator. A topology given by neighborhoods of the origin of the form $\{x \in H \,|(\mathcal{A}x, x)_H < \varepsilon\}$ $(\varepsilon > 0)$ is called the j-topology in H. A function $k(x)$ $(x \in H)$ is said to be continuous at the origin in the j-topology if it is continuous at O in the topology inducted by the j-topology on H.

Theorem 1. *Let $k(x)$ $(x \in H)$ be any bounded e.p.d. function which is continuous at O in the j-topology. Then the function admits the representation*

$$k(x) = \int\limits_H \cos(\lambda, x)\, d\rho(\lambda) \quad (x \in H), \tag{3}$$

where $d\rho(\lambda)$ is a non-negative finite measure defined on the σ-algebra $\mathfrak{B}(H)$ of Borel subsets of H. Conversely, each integral of the form (3) is a bounded e.p.d. function in H which is continuous at the origin in the j-topology. The measure $d\rho(\lambda)$ is uniquely defined for a given k.

We will prove that (2) implies the following inequality:

1) If $\sup|k(x)| = C < +\infty$ then $|k(x)| \le |k(0)|$, i.e. $C = |k(0)|$.

Indeed, put in (2) $N = 2,\ \xi_1 = -1,\ \xi_2 = 1,\ x^{(1)} = x,\ x^{(2)} = 0$. Then $3k(0) + k(2x) - 4k(x) \ge 0$. It follows that $k(2x) + 3k(0) \ge 4k(x)$ and

$$|k(x)| \le \frac{|k(2x)| + 3|k(0)|}{4} \le \frac{3|k(0)| + C}{4}, \quad C = \sup|k(x)| \le \frac{3|k(0)| + C}{4}.$$

It means that $C \le |k(0)|$, and therefore

$$|k(x)| \le |k(0)|. \tag{4}$$

2) For an arbitrary convex bounded e.p.d. function the following inequality holds

$$|k(x^{(1)}) - k(x^{(2)})|^2 \le 2k(0)\{k(0) - k(x^{(1)} - x^{(2)})\}.$$

Note that for an arbitrary p.d. kernel $K(x;y)$ we have (see [4, p. 469, Lemma 4.1])

$$|K(x;z) - K(y;z)|^2 \le K(z;z)\left[K(x;x) - 2\mathrm{Re}\, K(x;y) + K(y;y)\right].$$

Put in this inequality $K(x;y) = \frac{1}{2}\left[k\left(x^{(1)} - x^{(2)}\right) + k\left(x^{(1)} + x^{(2)}\right)\right]$ $(z = 0)$. We obtain

$$\left|k\left(x^{(1)}\right) - k\left(x^{(2)}\right)\right|^2 \le k(0)\Big[k(0) + \frac{1}{2}k\left(2x^{(1)}\right) + \\ + \frac{1}{2}k\left(2x^{(1)}\right) - k\left(x^{(1)} - x^{(2)}\right) - k\left(x^{(1)} + x^{(2)}\right)\Big].$$

But

$$\frac{1}{2}(k(2x^{(1)}) + k(2x^{(2)})) - k(x^{(1)} + x^{(2)}) \le 0,$$

because the function $k(x)$ is convex. Therefore the previous inequality can be rewritten in the form

$$\left|k\left(x^{(1)}\right) - k\left(x^{(2)}\right)\right|^2 \le k(0)\left\{k(0) - k\left(x^{(1)} - x^{(2)}\right)\right\}. \tag{5}$$

Proof of Theorem 1. Using (4), (5) (see [4, p.467–469]) we can prove that there exists a non-negative nuclear operator $\mathcal{A}$ in H with kernel of zero such that $k(x)$ $(x \in H)$ is uniformly continuous in the following sense: for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $x^{(1)}, x^{(2)} \in H$ satisfying $(\mathcal{A}(x^{(1)} - x^{(2)}), x^{(1)} - x^{(2)}) < \delta$ and $x^{(1)} - x^{(2)} \in H$ one has $|k(x^{(1)}) - k(x^{(2)})| < \varepsilon$.

We introduce a dot product in H: $(x, y)_{\mathcal{A}} = (\mathcal{A}x, y)_H$. Let $H_{\mathcal{A}}$ be a completion of H with respect to $(\cdot,\cdot)_{\mathcal{A}}$. Let us describe $H_{\mathcal{A}}$ by coordinates. For this purpose we choose an orthonormal basis in H consisting of the eigen vectors of the operator $\mathcal{A}$ realizing H as a space $l_2 = l_2(\mathbb{R}^1)$ with respect to a decomposition in this basic. Then space $H_{\mathcal{A}}$ is realized as a Hilbert space $l_2(\mathcal{A}) = \{x \in \mathbb{R}^\infty = \mathbb{R}^1 \times \mathbb{R}^1 \times \ldots | \sum_{n=1}^{\infty} a_n x_n^2 < \infty\}$ with the scalar product $(x, y)_{\mathcal{A}} = \sum_{n=1}^{\infty} a_n x_n y_n \in l_2(\mathcal{A})$ $\left((a_n \delta_{nm})_{n,m=1}^{\infty}\right.$ is a matrix of the operator $\mathcal{A}$ in the considering basis; $\left.a_n > 0\right)$. Let us consider now the standard Gaussian measure $\gamma_1 = \sqrt{\frac{1}{\pi}} e^{-x^2}$, it is essential that $\gamma_1(l_2(\mathcal{A})) = 1$, (see [8]).

Since $k(x)$ $(x \in H)$ is uniformly continuous with respect to the norm $\|\cdot\|_{H_{\mathcal{A}}} = (\cdot,\cdot)_H^{\frac{1}{2}}$ it can be extended by continuity to a uniformly continuous function $k_1(x)$ on the whole space $H_{\mathcal{A}}$. Passing to the coordinate from of $H_{\mathcal{A}}$, we find that $k_1(x)$ is defined on a set $l_2(\mathcal{A}) \subset \mathbb{R}^\infty$ of a full Gaussian measure.

Using $k_1(x)$ we introduce a quasi-scalar product

$$\langle \varphi, \psi \rangle = \int\limits_{l_2(\mathcal{A})} \int\limits_{l_2(\mathcal{A})} \frac{1}{2}\left[k_1(x+y) + k_1(x-y)\right] \varphi(y) \overline{\psi(x)}\, d\gamma_1(x)\, d\gamma_1(y), \quad \left(\varphi, \psi \in C_{b,cyl}\left(\mathbb{R}^\infty\right)\right). \tag{6}$$

Then, H_κ be a Hilbert space obtained as a result of a completion $(\cdot,\cdot)_{H_\kappa} = \langle\cdot,\cdot\rangle$. Since the kernel $\frac{1}{2}\left[k_1(x+y) + k_1(x-y)\right]$ is even on x, y, we consider (6) on even functions. Now we will construct a family of unitary operators, acting in H_κ. For this goal we will construct a rigging (chain)

$$H_0 \supset H_+ \supset \mathcal{D}.$$

The role of the space H_0 will be played just by H_κ, namely $H_0 = H_\kappa$ and $H_+ = L_2\left(l_2(\mathcal{A}), \mathfrak{B}(l_2(\mathcal{A}); \gamma_1)\right)$ and $\mathcal{D}$, which consists of the restrictions of functions from the space $\mathcal{A}(\mathbb{R}^\infty) = \bigcup_{n=1}^{\infty} \mathcal{A}(\mathbb{R}^n)$, where $\mathcal{A}(\mathbb{R}^n) = \mathcal{A}(\mathbb{R}^1) \otimes \ldots \otimes \mathcal{A}(\mathbb{R}^1)$ (n times) and $\mathcal{A}(\mathbb{R}^1) = l_2(\mathbb{R}^1, \gamma_1)$ on the cylindrical functions. For details of the space $\mathcal{A}(\mathbb{R}^\infty)$ see [3, p. 110–113]. The properties of the space $\mathcal{A}(\mathbb{R}^\infty)$ imply the properties of the space $\mathcal{D}$ and the embedding $\mathcal{D} \subset L_2 = H_+$. This embedding is quasi-nuclear. Indeed, the scalar product (6) has the form $(\varphi, \psi)_{H_\kappa} = (K\varphi, \psi)_{L_2}$, where K is an integral operator in L_2 generated by the continuous kernel $K(x, y) = \frac{1}{2}\left[k(x+y) + k(x-y)\right]$. Since

$$\int\limits_{l_2(\mathcal{A})} K(x, x)\, d\gamma_1(x) \le k(0) < \infty,$$

this operator is nuclear. On the other hand, for the chain $H_- \supset H_0 \supset H_+$, we have $(\varphi, \psi)_{H_0} = (\mathrm{I}\varphi, \psi)_{H_+}$ $(\varphi, \psi \in H_+)$; therefore $K = O(\mathrm{I} \restriction H_+)$, and the nuclearity of K yields the quasinuclearity of the embedding $H_+ \to H_0$. Hence, we have constructed the chain

$$H_\kappa = H_0 \supset H_+ = L_2 \supset \mathcal{D} = \mathcal{A}(\mathbb{R}^\infty). \tag{7}$$

Let us construct a family of unitary operators $\mathcal{A}_x$ $(x \in L_2 = H)$ acting in the space H_κ. They are the shift operators of functions by t with a certain factor. The factor appears because the Gaussian measure γ_1 is not invariant under such shifts. Let us first define the operator $\mathcal{A}_t$ on the function $\varphi \in C^\infty_{b,cyl}(\mathbb{R}^\infty) = \bigcup_{n=1}^\infty C_b^\infty(\mathbb{R}^n)$ by the equality

$$(\mathcal{A}_x t)(x) = \frac{1}{2}\left[\varphi(x-t)\left(\frac{d\gamma_{1,-t}}{d\gamma_1}\right)(x) + \varphi(x+t)\left(\frac{d\gamma_{1,t}}{d\gamma_1}\right)(x)\right], (t \in L_2; x \in l_2(\mathcal{A})) \tag{8}$$

where derivative is

$$\left(\frac{d\gamma_{1,t}}{d\gamma_1}\right)(\cdot) = \rho_{\gamma_1}(t, \cdot) = \exp\left(-\|t\|^2_{L_2} - 2(t, \cdot)_{L_2}\right)$$

Since for an even kernel $K(x,y)$ and even φ, ψ expression (6) has the form

$$\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} \frac{1}{2}\left[k(x+y) + k(x-y)\right] \varphi(y)\overline{\psi(x)}\, d\gamma_1(x)\, d\gamma_1(y) =$$

$$= \int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y-x)\varphi(y)\overline{\psi(x)}\, d\gamma_1(x)\, d\gamma_1(y) = \int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(x+y)\varphi(y)\overline{\psi(x)}\, d\gamma_1(x)\, d\gamma_1(y),$$

by a simple change of variables we get for any $t \in L_2$

$$(\mathcal{A}_t\varphi, \mathcal{A}_t\psi)_{H_\kappa} = (\varphi, \psi)_{H_\kappa} \quad \left(\varphi, \psi \in C^\infty_{b,cyb}(\mathbb{R}^\infty)\right).$$

Indeed,

$$\frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y-x)\varphi(x-t)\frac{d\gamma_{1,-t}}{d\gamma_1}(x)\overline{\psi(y-t)}\frac{d\gamma_{1,t}}{d\gamma_1}(y)\, d\gamma_1(x)\, d\gamma_1(y) = \begin{bmatrix} x = x' + t; \\ y = y' + t \end{bmatrix} =$$

$$= \frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y'-x')\varphi(x')\frac{d\gamma_1}{d\gamma_{1,t}}(x')\overline{\psi(y')}\frac{d\gamma_1}{d\gamma_{1,t}}(y')\, d\gamma_{1,t}(x')\, d\gamma_{1,t}(y') =$$

$$= \frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y'-x')\varphi(x')\overline{\psi(y')}\, d\gamma_1(x')\, d\gamma_1(y'), \tag{9}$$

$$\frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y-x)\left[\varphi(x-t)\frac{d\gamma_{1,t}}{d\gamma_1}(x)\overline{\psi(y+t)}\frac{d\gamma_{1,t}}{d\gamma_1}(y) + \right.$$

$$\left. + \varphi(x+t)\frac{d\gamma_{1,t}}{d\gamma_1}(x)\overline{\psi(y-t)}\frac{d\gamma_{1,-t}}{d\gamma_1}(y)\right] d\gamma_1(x)\, d\gamma_1(y) =$$

$$= \frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(x+y)\varphi(x-t)\frac{d\gamma_{1,-t}}{d\gamma_1}(x)\overline{\psi(y+t)}\frac{d\gamma_{1,t}}{d\gamma_1}(y)\, d\gamma_1(x)\, d\gamma_1(y) +$$

$$+ \frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(x+y)\varphi(x+t)\frac{d\gamma_{1,t}}{d\gamma_1}(x)\overline{\psi(y-t)}\frac{d\gamma_{1,-t}}{d\gamma_1}(y)\, d\gamma_1(x)\, d\gamma_1(y).$$

But

$$\frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(x+y)\varphi(x-t)\frac{d\gamma_{1,-t}}{d\gamma_1}(x)\overline{\psi(y+t)}\frac{d\gamma_{1,t}}{d\gamma_1}(y)\,d\gamma_1(x)\,d\gamma_1(y) = \begin{bmatrix} x = x'-t; \\ y = y'+t \end{bmatrix} =$$

$$= \frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(x'+y')\varphi(x')\frac{d\gamma_1}{d\gamma_{1,-t}}(x')\overline{\psi(y')}\frac{d\gamma_1}{d\gamma_{1,t}}(y')\,d\gamma_{1,-t}(x')\,d\gamma_{1,t}(y') =$$

$$= \frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(x'+y')\varphi(x')\overline{\psi(y')}\,d\gamma_1(x')\,d\gamma_1(y'), \tag{10}$$

$$\frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(x+y)\varphi(x-t)\frac{d\gamma_{1,-t}}{d\gamma_1}(x)\overline{\psi(y+t)}\frac{d\gamma_{1,t}}{d\gamma_1}(y)\,d\gamma_1(x)\,d\gamma_1(y) = \begin{bmatrix} x = x'+t; \\ y = y'-t \end{bmatrix} =$$

$$= \frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(x'+y')\varphi(x')\frac{d\gamma_1}{d\gamma_{1,t}}(x')\overline{\psi(y')}\frac{d\gamma_1}{d\gamma_{1,-t}}(y')\,d\gamma_{1,t}(x')\,d\gamma_{1,-t}(y') =$$

$$= \frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(x'+y')\varphi(x')\overline{\psi(y')}\,d\gamma_1(x')\,d\gamma_1(y'), \tag{11}$$

$$\frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y-x)\varphi(x+t)\frac{d\gamma_{1,t}}{d\gamma_1}(x)\overline{\psi(y+t)}\frac{d\gamma_{1,t}}{d\gamma_1}(y)\,d\gamma_1(x)\,d\gamma_1(y) = \begin{bmatrix} x = x'-t; \\ y = y'-t \end{bmatrix} =$$

$$= \frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y'-x')\varphi(x')\frac{d\gamma_1}{d\gamma_{1,-t}}(x')\overline{\psi(y')}\frac{d\gamma_1}{d\gamma_{1,-t}}(y')\,d\gamma_{1,-t}(x')\,d\gamma_{1,-t}(y') =$$

$$= \frac{1}{4}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y'-x')\varphi(x')\overline{\psi(y')}\,d\gamma_1(x')\,d\gamma_1(y'). \tag{12}$$

Summing up (9)–(12), we obtain that $(\mathcal{A}_t\varphi, \mathcal{A}_t\psi)_{H_\kappa} = (\varphi,\psi)_{H_\kappa}$ $(\varphi,\psi \in C_{b,cyl}(\mathbb{R}^\infty))$. Thus, each operator $\mathcal{A}_t$ is isometric and therefore it may be extended to an isometric operator on the whole of H_κ by continuity; we preserve the notation $\mathcal{A}_t$ for the latter.

Now, we can verify the equality

$$(\mathcal{A}_t\varphi, \psi)_{H_\kappa} = (\varphi, \mathcal{A}_t\psi)_{H_\kappa}, (\varphi,\psi \in C_{b,cyl}(\mathbb{R}^\infty)) \tag{13}$$

Indeed,

$$(\mathcal{A}_t\varphi, \psi)_{H_\kappa} = \frac{1}{2}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y-x)\varphi(x-t)\left(\frac{d\gamma_{1,-t}}{d\gamma_1}\right)(x)\overline{\psi(y)}\,d\gamma_1(x)\,d\gamma_1(y) +$$

$$+\frac{1}{2}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y-x)\varphi(x+t)\left(\frac{d\gamma_{1,t}}{d\gamma_1}\right)(x)\overline{\psi(y)}\,d\gamma_1(x)\,d\gamma_1(y) = \begin{bmatrix} x = x'+t; \\ y = y'+t \end{bmatrix} =$$

$$= \frac{1}{2}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y'-x')\varphi(x')\frac{d\gamma_1}{d\gamma_{1,t}}(x')\overline{\psi(y'+t)}\,d\gamma_{1,t}(x')\,d\gamma_{1,t}(y') + \begin{bmatrix} x = x'-t; \\ y = y'-t \end{bmatrix} +$$

$$+\frac{1}{2}\int\limits_{l_2(\mathcal{A})}\int\limits_{l_2(\mathcal{A})} k(y'-x')\varphi(x')\frac{d\gamma_1}{d\gamma_{1,-t}}(x')\overline{\psi(y'-t)}\,d\gamma_{1,-t}(x')\,d\gamma_{1,-t}(y') =$$

$$= \frac{1}{2} \int\limits_{l_2(\mathcal{A})} \int\limits_{l_2(\mathcal{A})} k(y'-x')\varphi(x')\overline{\psi(y'+t)}\frac{d\gamma_{1,t}}{d\gamma_1}(y')\,d\gamma_1(x')\,d\gamma_1(y')+$$

$$+\frac{1}{2} \int\limits_{l_2(\mathcal{A})} \int\limits_{l_2(\mathcal{A})} k(y'-x')\varphi(x')\overline{\psi(y'-t)}\frac{d\gamma_{1,-t}}{d\gamma_1}(y')\,d\gamma_1(x')\,d\gamma_1(y') = (\varphi, \mathcal{A}_{-t}\psi)_{H_\kappa}.$$

Similarly, we can verify the equality

$$(\mathcal{A}_t\varphi, \mathcal{A}_s\psi)_{H_\kappa} = \frac{1}{2}\left[(\mathcal{A}_{t-s}\varphi, \psi)_{H_\kappa} + (\mathcal{A}_{t+s}\varphi, \psi)_{H_\kappa}\right]. \tag{14}$$

It implies that $\mathcal{A}_t = \mathcal{A}_t^* = \mathcal{A}_{-t}$, and $\mathcal{A}_t \cdot \mathcal{A}_s = \frac{1}{2}\left[\mathcal{A}_{t+s} + \mathcal{A}_{t-s}\right]$ $(t, s \in l_2)$. Hence, we have constructed the group of unitary operators $(\mathcal{A}_t)_{t\in l_2}$. These operators and the chain (6) are connected by the conditions 1, 2 of Theorem 1.2 [4, Ch. 4]. Therefore, we can write the representation

$$\mathcal{A}_x = \int\limits_X e^{i(\lambda,x)}\,dE(\lambda), \quad (x \in X),$$

for the introduced operators $\mathcal{A}_x$ in the space H_κ. Here, E is a decomposition of unity on $\mathfrak{B}(X)$.

But so as $\mathcal{A}_x = \mathcal{A}_{-x}$, we can write such representation

$$\mathcal{A}_x = \int\limits_H \cos(\lambda, x)\,dE(\lambda), \quad (x \in H). \tag{15}$$

For any $x \in X_{\mathcal{A}}$, the space H_κ includes δ-function δ_x, concentrated at the point x, and $(\delta_x, \delta_y)_{H_\kappa} = \frac{1}{2}\left[k_1(x-y) + k_1(x+y)\right]$ $(x, y \in X_{\mathcal{A}})$. More precisely, for a given negative space H_κ and zero space L_2, we can construct a chain $H_\kappa \supset L_2 \supset H_+$. As a result, we obtain the space H_+ consisting of continuous bounded functions on $X_{\mathcal{A}}$. The space H_κ includes vectors δ_x such that $(\delta_x, \varphi)_{L_2} = \overline{\varphi(x)}$ $(x \in X_{\mathcal{A}}, \varphi \in H_+)$ and the indicated equality holds.

It is easy to understand that $\mathcal{A}_t\delta_0 = \delta_t$ for any $t \in X$. Indeed, let $(\chi_n(\cdot))_{n=1}^\infty$ be a sequence of functions from L_2 which converges to δ_0 in H_κ, and let each of these functions be equal to the characteristic function of the ball centered at O in $X_{\mathcal{A}}$ with radius n^{-1} divided by the measure γ_1 of this ball. Clearly, the action of the operator $\mathcal{A}_t$ on χ_n is given by (7).

For $\varphi \in H_+$ we have

$$(\mathcal{A}_t\delta_0, \varphi)_{L_2} = \lim_{n\to\infty}(\mathcal{A}_t\chi_n, \varphi)_{L_2} = \lim_{n\to\infty}\left[\int\limits_{H_{\mathcal{A}}} \frac{1}{2}\chi_n(x-t)\frac{d\gamma_{1,-t}}{d\gamma_1}(x)\overline{\varphi(x)}\,d\gamma_1(x)+\right.$$

$$\left.+\int\limits_{H_{\mathcal{A}}} \frac{1}{2}\chi_n(x+t)\frac{d\gamma_{1,t}}{d\gamma_1}(x)\overline{\varphi(x)}\,d\gamma_1(x)\right] =$$

$$= \lim_{n\to\infty}\left[\int\limits_{H_{\mathcal{A}}} \frac{1}{2}\chi_n(x-t)\overline{\varphi(x)}\,d\gamma_{1,-t}(x) + \int\limits_{H_{\mathcal{A}}} \frac{1}{2}\chi_n(x+t)\overline{\varphi(x)}\,d\gamma_{1,t}(x)\right] =$$

$$= \lim_{n\to\infty}\left[\int\limits_{H_{\mathcal{A}}} \frac{1}{2}\chi_n(x)\overline{\varphi(x+t)}\,d\gamma_1(x) + \int\limits_{H_{\mathcal{A}}} \frac{1}{2}\chi_n(x)\overline{\varphi(x-t)}\,d\gamma_1(x)\right] = \overline{\varphi(t)} = (\delta_t, \varphi)_{L_2},$$

and this yields the equality $\mathcal{A}_t\delta_0 = \delta_t$.

After these remarks, representation (3) immediately follows from (15). One should apply the last equality to δ_0 and multiply by a scalar δ_0 in H_κ. Here $\sigma(\alpha) = (E(\alpha)\delta_0, \delta_0)_{H_\kappa}$ $(\alpha \in \mathfrak{B}(X))$. Theorem 1 is proved. □

Applying Theorem 1 we can prove the following theorem.

Theorem 2. *In order that a family $(\mathcal{A}_t)$ $(t \in H)$ of self-adjoint bounded operators in the Hilbert space H admits a representation*

$$\mathcal{A}_t = \int\limits_H \cos(\lambda, t)\, dE(\lambda), \quad (\lambda \in H), \tag{16}$$

where E is some decomposition of identity of $\mathfrak{B}(H)$, it is necessary and sufficient that the operators satisfy the following conditions:

1) $\frac{1}{2}\left[\mathcal{A}_{t+s} + \mathcal{A}_{t-s}\right] = \mathcal{A}_t\mathcal{A}_s, \quad \mathcal{A}_{(t_1,\dots,t_n,\dots)} = \mathcal{A}_{(t_1,\dots,-t_n,\dots)}; \quad \mathcal{A}_0 = \mathrm{I}$;
2) $\mathcal{A}_t$ *strongly continuously depends on* t;
3) $\|\mathcal{A}_t\| \le b < \infty$.

Proof. Sufficiency. Let operators $\mathcal{A}_t$ commute accordingly to 1) and have a common simple spectrum. Then there exists a cycle vector Ω such that a closed linear hull $\{\mathcal{A}_t, \Omega\} = H$. Put $(\mathcal{A}_t\Omega, \Omega) = k(t)$. Then $k(t)$ is continuous and even in every variable and e.p.d., bounded function on H. Indeed, the function $k(t)$ is e.p.d.

$$\sum_{i,j=1}^{n} \frac{1}{2}\left[k\left(x^{(i)} + x^{(j)}\right) + k\left(x^{(i)} - x^{(j)}\right)\right]\xi_i\overline{\xi_j} = \sum_{i,j=1}^{n} \frac{1}{2}\left[(\mathcal{A}_{t^{(i)}+t^{(j)}}\Omega, \Omega) + (\mathcal{A}_{t^{(i)}-t^{(j)}}\Omega, \Omega)\right]\xi_i\overline{\xi_j} =$$

$$= \sum_{i,j=1}^{n} \frac{1}{2}\left[(\mathcal{A}_{t^{(i)}}\mathcal{A}_{t^{(j)}}\Omega, \Omega) + (\mathcal{A}_{t^{(j)}}\Omega, \mathcal{A}_{t^{(i)}}\Omega)\right]\xi_i\overline{\xi_j} = \left\|\sum_{i=1}^{n} \xi_i\mathcal{A}_{t^{(i)}}\Omega\right\|_H^2 \ge 0.$$

The continuity, evenness in every variable and boundedness of the function $k(t)$ follow from assumptions of Theorem 2. Therefore by Theorem 1, we obtain an integral representation for the function $k(t)$

$$k(t) = \int\limits_H \cos(\lambda, t)\, d\rho(\lambda), \quad (t \in H),$$

where $d\rho(\lambda)$ is an even non-negative finite measure, which is defined on some σ-algebra $\mathfrak{B}(H)$ of Borel sets from H. We construct a correspondence: every vector $\sum_{\kappa=1}^{n} c_\kappa\mathcal{A}_{t^{(\kappa)}}\Omega \in H$ is paired with the function

$$\sum_{\kappa=1}^{n} c_\kappa \cos\left(\lambda, t^{(\kappa)}\right) \in l_2\left(H, d\rho(\lambda)\right), \quad (n = 1, 2, \dots).$$

This correspondence is isometric because

$$\left\|\sum_{\kappa=1}^{n} c_\kappa\mathcal{A}_{t_\kappa}\Omega\right\|_H^2 = \sum_{i,j=1}^{n} c_i\overline{c_j}\left(\mathcal{A}_{t_i}\Omega, \mathcal{A}_{t_j}\Omega\right)_H = \sum_{i,j=1}^{n} c_i\overline{c_j}\frac{1}{2}\left[\left(\mathcal{A}_{t_i+t_j}\Omega, \Omega\right)_H + \left(\mathcal{A}_{t_i-t_j}\Omega, \Omega\right)_H\right] =$$

$$= \sum_{i,j=1}^{n} c_i \overline{c_j} \int\limits_H \frac{1}{2} \left[\cos\left(\lambda, t^{(i)} + t^{(j)}\right) + \cos\left(\lambda, t^{(i)} - t^{(j)}\right) \right] \, d\rho(\lambda) =$$

$$= \left\| \sum_{\kappa=1}^{n} c_\kappa \cos\left(\lambda, t^{(\kappa)}\right) \right\|^2_{l_2(H, d\rho(\lambda))} .$$

Then it can be extended to the unitary map $U\colon H \to l_2\left(H, d\rho(\lambda)\right)$. Furthermore, the operator $\mathcal{A}_t$ corresponds to the operator of multiplication in $l_2\left(H, d\rho(\lambda)\right)$ on the function $\cos(\lambda, t)$.

Since

$$\mathcal{A}_t \left(\sum_{\kappa=1}^{n} c_\kappa \mathcal{A}_{t^{(\kappa)}} \Omega \right) = \sum_{\kappa=1}^{n} c_\kappa \left(\frac{\mathcal{A}_{t+t^{(\kappa)}} + \mathcal{A}_{t-t^{(\kappa)}}}{2} \right) \Omega =$$

$$= U^* \left(\sum_{\kappa=1}^{n} c_\kappa \frac{\cos\left(\lambda, t + t^{(\kappa)}\right) + \cos\left(\lambda, t - t^{(\kappa)}\right)}{2} \right) =$$

$$= U^* \cos\left(\lambda, t\right) U \left(\sum_{\kappa=1}^{n} c_\kappa \cos\left(\lambda, t^{(\kappa)}\right) \right) = U^* \cos\left(\lambda, t\right) U \left(\sum_{\kappa=1}^{n} c_\kappa \mathcal{A}_{t^{(\kappa)}} \Omega \right),$$

that is $\mathcal{A}_t = U^* Q_{\cos(\lambda,t)} U$ ($Q_{\cos(\lambda,t)}$ denotes the operator of multiplication on $\cos\left(\lambda, t\right)$ in $l_2\left(H, d\rho(\lambda)\right)$).

Since for a family $Q_{\cos(\lambda,t)}$ the integral representation is true

$$Q_{\cos(\lambda,t)} = \int\limits_H \cos\left(\lambda, t\right) \, dQ_{\chi_\Delta(\cdot)},$$

we have $\mathcal{A}_t = \int_H \cos\left(\lambda, t\right) \, dE(\lambda)$, where $E(\Delta) = U^* Q_{\chi_\Delta(\cdot)} U$. $\chi_\Delta(\cdot)$ is the characteristic function of an interval $\Delta \in H$. So the integral representation (16) is proved.

The uniqueness of decomposition of identity follows from the uniqueness of the measure $d\rho(\lambda)$. If the cyclic vector Ω is missed then for the family of commuted self-adjoint operators $\mathcal{A}_t$ $(t \in H)$ in H there exists a sequence of measures on σ-algebra of cylindric sets in H with a Borel basis $\rho_1, \ldots, \rho_\kappa, \ldots$ and a unitary operator

$$U\colon H \to \oplus \sum_{\kappa=1}^{\infty} l_2\left(H, d\rho_\kappa(\lambda)\right)$$

such that in every space $l_2\left(H, d\rho_\kappa(\lambda)\right)$ one has $\mathcal{A}_t = U^* Q_{\cos(\lambda,t)} U$, where $Q_{\cos(\lambda,t)}$ denotes the operator of multiplication on $\cos\left(\lambda, t\right)$ in $l_2\left(H, d\rho_\kappa(\lambda)\right)$.

The necessity of the conditions 1), 2), 3) follows from integral representation (16). $\square$

An integral representation for the family of bounded self-adjoint operators in one-dimensional case, which are connected by an algebraic relationship, was proved in [6].

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National Forestry and Wood Technology University of Ukraine
Lviv, Ukraine
Lopotko1304@gmail.com

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