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ON DISTORTION UNDER MAPPINGS SATISFYING THE INVERSE POLETSKY INEQUALITY

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As it is known, conformal mappings are locally Lipschitz at inner points of a domain, and quasiconformal (quasiregular) mappings are locally Hölder continuous. As for estimates of the distortion of mappings at boundary points of the domain, this problem has not been studied sufficiently even for these classes. We partially fill this gap by considering in this manuscript not even local behavior at the boundary points, but global behavior in the domain of one class of mappings. The paper is devoted to studying mappings with finite distortion. The goal of our investigation is obtaining the distance distortion for mappings at inner and boundary points. Here we study mappings satisfying Poletsky's inequality in the inverse direction. We obtain conditions under which these mappings are either logarithmic Hölder continuous or Hölder continuous in the closure of a domain. We consider several important cases in the manuscript, studying separately bounded convex domains and domains with locally quasiconformal boundaries, as well as domains of more complex structure in which the corresponding distortion estimates must be understood in terms of prime ends. In all the above situations we show that the maps are logarithmically Hölder continuous, which is somewhat weaker than the usual Hölder continuity. However, in the last section we consider the case where the maps are still Hölder continuous in the usual sense. The research technique is associated with the use of the method of moduli and the method of paths liftings. A key role is also played by the lower bounds of the Loewner type for the modulus of families of paths, which are valid only in domains with a special geometry, in particular, bounded convex domains. Another important fact which is also valid for domains of the indicated type, is the possibility of joining pairs of different points in a domain by paths lying (up to a constant) at a distance no closer than a distance between above points.

1. Introduction. The manuscript is devoted to the study of mappings with finite distortion, more precisely, classes of mappings satisfying inverse Poletsky-type moduli inequalities. Such estimates are important because they are part of the definition of quasiconformality by Väisälä (see [1]), and more general classes of maps may also be studied in the context of modulus distortion (see e.g. [2]). In several of our recent papers we have obtained results on the distortion of mappings with controlled upper and lower distortion of the modulus of families of paths. Such estimates have been obtained in various situations, including on the plane, when the mappings are solutions of the Beltrami equation, and in space, when these mappings satisfy the generalized Poletsky inequality in one of two versions, see, for example, [3], [4], [5], [6], [7] and [8]; see also Remark 1 on this occasion. In particular, the

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most common growth estimate that is performed for the mappings in the cited publications is the logarithmic estimate, although in some individual situations the mapping is Hölder continuous, or even Lipschitz continuous, see, for example, [3].

Let us move on to definitions. A Borel function $\rho\colon \mathbb{R}^n \to [0,\infty]$ is called *admissible* for a family Γ of paths γ in $\mathbb{R}^n$ if the relation

$$\int\limits_{\gamma} \rho(x)|dx| \geqslant 1 \tag{1}$$

holds for any locally rectifiable path $\gamma \in \Gamma$. Given $p \geqslant 1$, *p-modulus* of Γ is defined as

$$M_p(\Gamma) = \inf_{\rho \in \mathrm{adm}\,\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)dm(x). \tag{2}$$

Let $M(\Gamma) := M_n(\Gamma)$. Let $Q\colon \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function. Given a domain $D \subset \mathbb{R}^n$, we say that $f\colon D \to \mathbb{R}^n$ *satisfies the inverse Poletsky inequality with respect to p-modulus* if the relation

$$M_p(\Gamma) \leqslant \int\limits_{f(D)} Q(y) \cdot \rho_*^p(y)dm(y) \tag{3}$$

holds for any family of paths Γ in D and any $\rho_* \in \mathrm{adm}\, f(\Gamma)$. Regarding the use of inequality (3) for $p = n$, we may point, for example, to [11, Theorem 3.2], [12, Theorem 6.7.II] and [2, Theorem 8.5]). At the same time, regarding the use of similar inequalities for $p \neq n$ we may point to [9] and [10].

Some upper estimates for $|\overline{f}(x) - \overline{f}(y)|$ were proved in [7], where f is defined in a domain D and satisfies (3) for $p = n$, besides that, $\overline{f}$ denotes the boundary extension of f onto $\overline{D}$, and the points x, y belong to $U \cap \overline{D}$ while U is some neighborhood of a (fixed) point $x_0 \in \partial D$. The purpose of this paper is to obtain similar results for $p \geqslant n$ and any points $x, y \in \overline{D}$, not only $x, y \in U \cap \overline{D}$.

Note that the distortion estimates under non-conformal modulus are useful in the study of composition operators in the Sobolev spaces, see Remark 2. Observe that, the boundary continuous extension of mappings satisfying inequality (3) for $p \geqslant n$, as well as their equicontinuity in the closure of the domain were established in [9] and [10]. However, explicit estimates of the distortion at the boundary points for mappings in (3) have not been obtained. In addition, we show that mappings with the inverse Poletsky inequality are Hölder continuous for $p > n$ whenever the majorant Q in (3) is integrable. This circumstance significantly distinguishes this case from $p = n$, since in this case the mapping is logarithmic Hölder continuous, which is weaker than the usual Hölder continuity.

Let us give some more necessary definitions. A mapping $f\colon D \to \mathbb{R}^n$ is called *discrete* if $\{f^{-1}(y)\}$ consists of isolated points for any $y \in \mathbb{R}^n$, and *open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$. A mapping f between domains D and D' is said to be *closed* if $f(E)$ is closed in D' for any closed set $E \subset D$ (see, e.g., [13, Section 3]).

Given sets $E, F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E,F,D)$ the family of all paths $\gamma\colon [a,b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a,b)$. Due to [14], a domain D in $\mathbb{R}^n$ is called *quasiextremal distance domain* (*QED-domain for short)* if

$$M(\Gamma(E,F,\mathbb{R}^n)) \leqslant A_0 \cdot M(\Gamma(E,F,D)) \tag{4}$$

for some finite number $A_0 \geqslant 1$ and all continua E and F in D. Observe that a half-space or a ball are examples of quasiextremal distance domains, see [15, Lemma 4.3].

In the extended Euclidean space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ we use the *spherical (chordal) metric* $h(x, y) = |\pi(x) - \pi(y)|$, where π is a stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{e_{n+1}}{2}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$, and

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}, \quad h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \quad x \neq \infty, y \neq \infty \tag{5}$$

(see e.g. [1, Definition 12.1]). Given $A, B \subset \overline{\mathbb{R}^n}$ we define

$$h(A, B) = \inf_{x\in A, y\in B} h(x, y), \quad h(A) = \sup_{x,y\in A} h(x, y),$$

where h is a chordal metric in (5). Similarly, put

$$d(A, B) = \inf_{x\in A, y\in B} |x - y|, \quad d(A) = \sup_{x,y\in A} |x - y|.$$

Set

$$B(x_0, r) = \{x \in \mathbb{R}^n \colon |x - x_0| < r\}, \quad \mathbb{B}^n = B(0, 1), \quad S(x_0, r) = \{x \in \mathbb{R}^n \colon |x - x_0| = r\}.$$

A *continuum* is a connected compactum in $\mathbb{R}^n$. The continuum is called *non-degenerate,* if it is not a single point. Given $\delta > 0$, $p \geqslant 1$, domains $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, a non-degenerate continuum $A \subset D'$ and a Lebesgue measurable function $Q \colon D' \to [0, \infty]$ we denote by $\mathfrak{S}^p_{\delta,A,Q}(D, D')$ a family of all open discrete and closed mappings f of D onto D' satisfying the relation (3) such that $h(f^{-1}(A), \partial D) \geqslant \delta$. The following result was formulated and proved for the case $p = n$ in [7] in the "local version", i.e., at the neighborhood of the fixed point.

Theorem 1. *Let $n \geqslant 2$, $p \geqslant n$, and let $Q \in L^1(D')$, let D be a bounded quasiextremal distance domain, and let D' be a convex bounded domain. Then any $f \in \mathfrak{S}^p_{\delta,A,Q}(D, D')$ has a continuous extension $f \colon \overline{D} \to \overline{D'}$ and there exists $C = C(n, p, A, D, D') > 0$ such that*

$$|\overline{f}(x) - \overline{f}(y)| \leqslant C \cdot (\|Q\|_1)^{1/p} \log^{-1/n}\left(1 + \frac{\delta}{|x - y|}\right) \tag{6}$$

for all $x, y \in \overline{D}$ and $f \in \mathfrak{S}^p_{\delta,A,Q}(D, D')$, where $\|Q\|_1$ is a norm of the function Q in $L^1(D')$.

The following definition in a slightly modified form was proposed in [16], cf. [17]. The boundary of a domain D is called *locally quasiconformal*, if every point $x_0 \in \partial D$ has a neighborhood U, for which there exists a quasiconformal mapping φ of U onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ such that $\varphi(\partial D \cap U)$ is the intersection of the unit sphere $\mathbb{B}^n$ with a coordinate hyperplane $x_n = 0$, where $x = (x_1, \ldots, x_n)$. Note that, with slight differences in the definition, domains with such boundaries are also called collared domains (see [16]). Let us say a few words about the mapping of φ. Observe that, quasiconformal mappings are locally Hölder continuous (see [12, Theorem 1.11.III]). By the above-mentioned definition, there is $\widetilde{C} > 0$ and some exponent $0 < \alpha \leqslant 1$ such that

$$\left(\widetilde{C}\right)^{-\frac{1}{\alpha}} |x - y|^{\frac{1}{\alpha}} \leqslant |\varphi^{-1}(x) - \varphi^{-1}(y)| \leqslant \widetilde{C} \cdot |x - y|^{\alpha} \qquad \forall x, y \in \mathbb{B}^n. \tag{7}$$

Given $0 < \alpha \leqslant 1$, we say that, the boundary of a domain D is *α-locally quasiconformal,* if ∂D is locally quasiconformal and the mappings φ may be chosen such that (7) holds for some $\widetilde{C} > 0$. The following result holds.

Theorem 2. *Let $n \geqslant 2$, $p \geqslant n$, $0 < \alpha \leqslant 1$, let $Q \in L^1(D')$, let D be a bounded domain with a α-locally quasiconformal boundary, and let D' be a bounded convex domain. Then any $f \in \mathfrak{S}^p_{\delta,A,Q}(D,D')$ has a continuous extension $f\colon \overline{D} \to \overline{D'}$, besides that, there is $C > 0$ such that*

$$|\overline{f}(x) - \overline{f}(y)| \leqslant C \cdot (\|Q\|_1)^{1/p} \log^{-1/n}\left(1 + \frac{\delta}{|x-y|^{\alpha}}\right) \tag{8}$$

for any $x, y \in \overline{D}$ and $f \in \mathfrak{S}^p_{\delta,A,Q}(D,D')$, where $\|Q\|_1$ is a norm of Q in $L^1(D')$.

We will separately consider the case of domains with complex geometry. In this case, distortion estimates should be understood in the terminology of the so-called prime ends. The definitions and notations used below are fully consistent with our previous publication [6], cf. [17]. In particular, a bounded domain D in $\mathbb{R}^n$ is called *regular*, if D can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbb{R}^n$, and, besides that, every prime end in D is regular. Note that the space $\overline{D}_P = D \cup E_D$ is metric, which can be demonstrated as follows. If $g\colon D_0 \to D$ is a quasiconformal mapping of a domain D_0 with a locally quasiconformal boundary onto some domain D, then for $x, y \in \overline{D}_P$ we put

$$\rho(x,y) := |g^{-1}(x) - g^{-1}(y)|, \tag{9}$$

where the element $g^{-1}(x)$, $x \in E_D$, is to be understood as some (single) boundary point of the domain D_0. The specified boundary point is unique and well-defined, see e.g. [18, Theorem 2.1, Remark 2.1], cf. [16, Theorem 4.1]. Given $\widetilde{C} > 0$ and $0 < \alpha \leqslant 1$, we say that, the boundary of a domain D is *α-regular,* if D can be quasiconformally mapped to a domain with a locally α-quasiconformal boundary whose closure is a compact in $\mathbb{R}^n$, and, besides that, every prime end in D is regular.

The following statement holds.

Theorem 3. *Let $n \geqslant 2$, $p \geqslant n$, $\widetilde{C} > 0$, $0 < \alpha \leqslant 1$, let $Q \in L^1(D')$, let D be α-regular domain, and let D' be a bounded convex domain. Then any $f \in \mathfrak{S}^p_{\delta,A,Q}(D,D')$ has a continuous extension $f\colon \overline{D}_P \to \overline{D'}$; in addition, there exists a neighborhood $C = C(n, A, D, D', P_0) > 0$ such that*

$$|\overline{f}(P_1) - \overline{f}(P_2)| \leqslant C \cdot (\|Q\|_1)^{1/p} \log^{-1/n}\left(1 + \frac{\delta}{\rho^{\alpha}(P_1,P_2)}\right) \tag{10}$$

for any $P_1, P_2 \in \overline{D}_P$ and $f \in \mathfrak{S}^p_{\delta,A,Q}(D,D')$, where $\|Q\|_1$ is a norm of Q in $L^1(D')$.

Remark 1. Along with inequality (3), we have often considered the inequality

$$M_p(\Gamma_f(y_0,r_1,r_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y)\cdot \eta^p(|y-y_0|)\,dm(y), \tag{11}$$

where η is arbitrary Lebesgue measurable function $\eta\colon (r_1,r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\,dr \geqslant 1. \tag{12}$$

Observe that inequality (3) is stronger than (11). Indeed, let (3) holds. Now, we put $\rho'(y) := \eta(|y-y_0|)$ for $y \in A(y_0,r_1,r_2) \cap f(D)$, and $\rho'(y) = 0$ otherwise. By the Luzin theorem, we may assume that the function ρ' is Borel measurable (see, e.g., [19, Section 2.3.6]). Then, by [1, Theorem 5.7] we have that

$$\int\limits_{\gamma_*} \rho'(y)|dy| \geqslant \int\limits_{r_1}^{r_2} \eta(r)dr \geqslant 1$$

for any (locally rectifiable) path $\gamma_* \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Substituting the function ρ' in (3), we obtain the desired ratio (11).

Note that for mappings satisfying inequality (11), which is weaker than (3), the problems studied in this manuscript have not yet been investigated.

Remark 2. The condenser in the domain $D \subset \mathbb{R}^n$ is the pair (E, F) of connected closed relatively to D sets $E, F \subset D$. Recall that a continuous function $u \in L^1_p(D)$ is called an admissible function for the condenser (E, F), denoted $u \in W_0(E, F)$, if the set $E \cap D$ is contained in some connected component of the set $\operatorname{Int}\{x\colon u(x) = 0\}$, the set $F \cap D$ is contained in some to the connected component of the set $\operatorname{Int}\{x\colon u(x) = 1\}$. Then we call as a p-capacity of the condenser (E, F) relatively to a domain D the value

$$\operatorname{cap}_p(E, F; \Omega) = \inf \|u | L^1_p(D)\|^p,$$

where the greatest lower bond is taken over all admissible for the condenser $(E, F) \subset D$ functions. If the condenser have no admissible functions we put the capacity is equal to infinity.

Given a Lebesgue measurable function $Q\colon \mathbb{R}^n \to [0, \infty]$, $q < \infty$ and any disjoint nondegenerate compact sets $E, F \subset D$, we set

$$\operatorname{cap}_{q,Q}(E, F, D) = \inf_{u \in W_0(E,F)} \int\limits_D Q(x) \cdot |\nabla u|^q dm(x). \tag{13}$$

Observe that, the classes of homeomorphisms f between domains D and D' generating bounded composition operators on Sobolev spaces can be characterized by the inverse capacity (moduli) Poletsky inequality ([20], [21])

$$\operatorname{cap}_q^{1/q}(f^{-1}(E), f^{-1}(F); D) \leqslant K_{p,q}(f; \Omega) \operatorname{cap}_p^{1/p}(E, F; D'), 1 < q \leqslant p < \infty, \tag{14}$$

for some $0 < K_{p,q} < \infty$. On the other hand, let f be a homeomorphism that satisfies the relation

$$\operatorname{cap}_q(E, F, D) \leqslant \operatorname{cap}_{q,Q}(f(E), f(F), f(D)) \tag{15}$$

for arbitrary compacts (continua) $E, F \subset D$, and

$$\operatorname{cap}_{q,Q}(f(E), f(F), f(D)) = M_{q,Q}(\Gamma(f(E), f(F), f(D))), \tag{16}$$

where $M_{q,Q}(\Gamma(f(E), f(F), f(D))) = \inf\limits_{\rho_* \in \operatorname{adm} \Gamma(f(E), f(F), f(D))} \int_{f(D)} \rho_*^q(y) \cdot Q(y) dm(y)$. Then f satisfies the condition

$$M_q(\Gamma(E, F, D)) \leqslant \operatorname{cap}_{q,Q}(f(E), f(F), f(D)) \leqslant \int\limits_{f(D)} Q(y) \cdot \rho_*^q(y) dm(y) \tag{17}$$

for any function $\rho_* \in \operatorname{adm} f(\Gamma(E, F, D)) = \operatorname{adm} \Gamma(f(E), f(F), f(D))$ (see [9, Theorem 7.2] and relation (7.5) here. By Hesse equality (see [22, Theorem 5.5]),

$$\mathrm{cap}_q(E, F, D) = M_q(\Gamma(E, F, D)),$$

therefore (16) holds, for example, for bounded Q. The general case is unknown.

Then the homeomorphisms f generating bounded composition operators satisfy (14), and, in turn, (14) implies (17) at least for $p = q$ whenever Q satisfies (15) and (16). The general case, when the mappings f are not homeomorphic, p and q are different, and relations (15)–(16) may not hold has not been studied sufficiently.

2. Auxiliary lemmas. Let D, D' be domains in $\mathbb{R}^n$. For given numbers $n \leqslant p < \infty$, $\delta > 0$, a continuum $A \subset D'$ and an arbitrary Lebesgue measurable function $Q\colon D' \to [0, \infty]$, we denote by $\mathfrak{P}^p_{\delta,A,Q}(D, D')$ a family of all open discrete and closed mappings f of D onto D' satisfying the condition

$$M_p(\Gamma(E, F, D)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y) \cdot \eta^p(|y - y_0|)dm(y) \tag{18}$$

for any $y_0 \in D'$, any compacts

$$E \subset f^{-1}(\overline{B(y_0, r_1)}),\ F \subset f^{-1}(D' \setminus B(y_0, r_2)), 0 < r_1 < r_2 < r_0 = \sup\limits_{y\in D'} |y - y_0|,$$

and any Lebesgue measurable function $\eta\colon (r_1, r_2) \to [0, \infty]$ with the condition

$$\int\limits_{r_1}^{r_2} \eta(r)dr \geqslant 1, \tag{19}$$

such that $h(f^{-1}(A), \partial D) \geqslant \ \delta$.

Let ∂D be a boundary of the domain $D \subset \mathbb{R}^n$. Then the boundary ∂D is called *weakly flat* at the point $x_0 \in \partial D$, if for each $P > 0$ and for any neighborhood U of this point there is a neighborhood $V \subset U$ of the same point such that $M(\Gamma(E, F, D)) > P$ for any continua E, $F \subset D$ that intersect ∂U and ∂V. The boundary of a domain D is called weakly flat if the corresponding property holds at any point of ∂D. The following statement holds (see Theorem 1.2 in [9].

Proposition 1. *Let $D \subset \mathbb{R}^n$ be a bounded domain with a weakly flat boundary. Suppose that, $Q \in L^1(D')$. If D' is locally connected on its boundary, then any $f \in \mathfrak{P}^p_{\delta,A,Q}(D, D')$ has a continuous extension $\overline{f}\colon \overline{D} \to \overline{D'}$, $\overline{f}(\overline{D}) = \overline{D'}$, and the family $\mathfrak{P}^p_{\delta,A,Q}(\overline{D}, \overline{D'})$, which consists of all extended mappings $\overline{f}\colon \overline{D} \to \overline{D'}$, is equicontinuous in $\overline{D}$.*

Remark 3. Observe that, $\mathfrak{S}^p_{\delta,A,Q}(D, D') \subset \mathfrak{P}^p_{\delta,A,Q}(D, D')$, because the relation (3) obviously implies

$$M_p(\Gamma(E, F, D)) \leqslant \int\limits_{f(D)} Q(y) \cdot \rho_*^p(y)dm(y) \qquad \forall\ \rho_* \in \mathrm{adm}(f(\Gamma(E, F, D))). \tag{20}$$

In addition, (20) implies (18) for the corresponding E and F by Theorem 7.1 in [9].

The following lemma was proved in several different situations in [23] and [7]. In particular, in [23] we considered the case when the family $\mathfrak{S}^p_{\delta,A,Q}$ consists of mappings of the unit ball onto itself and satisfies condition (3) for $p = n$. Besides that, [7] deals for the case $p = n$ while the conditions on D and D' a similar to the mentioned below.

Lemma 1. *Let D and D' be domains satisfying the conditions of Theorem 1, and let E be a continuum in D', $Q \in L^1(D')$. Then there exists $\delta_1 > 0$ such that $\mathfrak{S}^p_{\delta,A,Q} \subset \mathfrak{S}^p_{\delta_1,E,Q}$. In other words, if f is an open discrete and closed mapping of D onto D' satisfying the condition (3) such that $h(f^{-1}(A), \partial D) \geqslant \delta$, then there exists $\delta_1 > 0$, which does not depend on f, such that $h(f^{-1}(E), \partial D) \geqslant \delta_1$.*

Proof. Let us prove Lemma 1 from the opposite. Suppose that its conclusion is not true. Then, there are sequences $y_m \in E$, $f_m \in \mathfrak{S}^p_{\delta,A,Q}$ and $x_m \in D$ such that $f_m(x_m) = y_m$ and $h(x_m, \partial D) \to 0$ as $m \to \infty$. Without loss of generality, we may assume that $x_m \to x_0$ as $m \to \infty$, where x_0 may be equal to ∞ if D is unbounded. Observe that the quasiextremal distance domains have weakly flat boundaries (see Lemma 2(ii) in [24]). In addition, convex domains are locally connected on the boundary, which follows directly from the definition of a convex domain. Now, by Proposition 1 and Remark 3 the family $\{f_m\}_{m=1}^{\infty}$ is equicontinuous at x_0. Then, for any $\varepsilon > 0$ there is $m_0 \in \mathbb{N}$ such that $h(f_m(x_m), f_m(x_0)) < \varepsilon$ for $m \geqslant m_0$. On the other hand, since f_m is closed, $f_m(x_0) \in \partial D'$. Due to the compactness of the space $\overline{\mathbb{R}^n}$ and the closure of $\partial D'$, we may assume that $f_m(x_0)$ converges to some $B \in \partial D'$ as $m \to \infty$. Therefore, by the triangle inequality,

$$h(f_m(x_m), f_m(x_0)) \geqslant h(f_m(x_m), B) - h(B, f_m(x_0)) \geqslant \frac{1}{2} \cdot h(E, \partial D')$$

for sufficiently large $m \in \mathbb{N}$. Finally, we have a contradiction: $h(f_m(x_m), f_m(x_0)) \geqslant \delta_0$, $\delta_0 := \frac{1}{2} \cdot h(E, \partial D')$ and, at the same time, $h(f_m(x_m), f_m(x_0)) < \varepsilon$ for $m \geqslant m_0$. The resulting contradiction refutes the original assumption. □

The following lemma was proved in [25], cf. the proof of Theorem 1.1 in [23].

Lemma 2. *Let D' be a bounded convex domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $B(y_*, \delta_*/2)$ be a ball centered at the point $y_* \in D'$, where $\delta_* := d(y_*, \partial D')$. Let $z_0 \in \partial D'$. Then for any points $A, B \in B(z_0, \delta_*/8) \cap D'$ there are points $C, D \in \overline{B(y_*, \delta_*/2)}$, for which the segments $[A, C]$ and $[B, D]$ are such that*

$$d([A, C], [B, D]) \geqslant C_0 \cdot |A - C|, \tag{21}$$

where $C_0 > 0$ is some constant that depends only on δ_* and $d(D')$.

3. Proof of Theorem 1. A path $\alpha\colon [a, b) \to D$ is called a *total f-lifting* of β starting at x, if (1) $\alpha(a) = x$; (2) $(f \circ \alpha)(t) = \beta(t)$ for any $t \in [a, b)$. We have the following, see [13, Lemma 3.7].

Proposition 2. *Let $f\colon D \to \mathbb{R}^n$ be a discrete open and closed (boundary preserving) mapping, $\beta\colon [a, b) \to f(D)$ be a path, and $x \inf^{-1}(\beta(a))$. Then β has a total f-lifting starting at x.*

Before proceeding to the proof of Theorem 1, let us first prove its local version, cf. [7, Theorem 1].

Lemma 3. *Let $n \geqslant 2$, $p \geqslant n$, and let $Q \in L^1(D')$, let D be a bounded quasiextremal distance domain, and let D' be a convex bounded domain. Then any $f \in \mathfrak{S}^p_{\delta,A,Q}(D, D')$ has a continuous extension $f\colon \overline{D} \to \overline{D'}$ and for any $x_0 \in \partial D$, $x_0 \neq \infty$, there exists a neighborhood U and $C = C(n, p, A, D, D') > 0$ such that*

$$|\overline{f}(x) - \overline{f}(y)| \leqslant C \cdot (\|Q\|_1)^{1/p} \log^{-1/n}\left(1 + \frac{\delta}{|x - y|}\right) \tag{22}$$

for every $x, y \in U \cap \overline{D}$ and $f \in \mathfrak{S}^p_{\delta,A,Q}(D, D')$, where $\|Q\|_1$ is a norm of the function Q in $L^1(D')$.

Proof. We will mainly use the scheme of proof of [7, Theorem 1]. The possibility of a continuous extension of the mapping f to the boundary of the domain D follows by Theorems 1.2 and 7.1 in [9] due to the comments made after the formulation of Lemma 1.

Let us prove the relation (22). Fix $x_0 \in \partial D$, and let $y_* \in D'$. Let $\delta_* := d(y_*, \partial D')$ and let $E = \overline{B(y_*, \delta_*/2)} \subset D'$. By Lemma 1 we may find $\delta_1 > 0$ such that $h(f^{-1}(E), \partial D) \geqslant \delta_1$ for every $f \in \mathfrak{S}^p_{\delta,A,Q}$. By Theorems 1.2 and 7.1 in [9] the family $\mathfrak{S}^p_{\delta,A,Q}$ is equicontinuous in $\overline{D}$. Thus, for $\delta_*/8$ there exists a neighborhood $U \subset B(x_0, \delta_1/2)$ of x_0 such that $|f(x) - f(x_0)| < \delta_*/8$ for every $x \in U \cap D$ and all $f \in \mathfrak{S}^p_{\delta,A,Q}$. Let $x, y \in U \cap D$ and let

$$\varepsilon_0 := |f(x) - f(y)| < \delta_0 := \delta_*/4.$$

Now we apply Lemma 2 for $A = f(x)$, $B = f(y)$ and $z_0 = f(x_0)$. Due to this lemma, there are segments $I \ni A$ and $J \ni B$ in D' such that $I \cap E \neq \varnothing \neq J \cap E$, and

$$d(I, J) \geqslant C_0 \cdot |f(x) - f(y)|, \tag{23}$$

where C_0 is some constant depending only on E and D'.

Let α_1 and β_1 be total f-liftings of paths I and J starting at the points x and y, respectively (they exist by Proposition 2).

By definition, $|\alpha_1| \cap f^{-1}(E) \neq \varnothing \neq |\beta_1| \cap f^{-1}(E)$. Since $h(f^{-1}(E), \partial D) \geqslant \delta_1$ and $x, y \in B(x_0, \delta_1/2)$, then

$$d(\alpha_1) \geqslant \delta_1/2, \quad d(\beta_1) \geqslant \delta_1/2. \tag{24}$$

Let $\Gamma := \Gamma(\alpha_1, \beta_1, D)$. Then, by (4),

$$M(\Gamma) \geqslant (1/A_0) \cdot M(\Gamma(\alpha_1, \beta_1, \mathbb{R}^n)), \tag{25}$$

and on the other hand, by [26, Lemma 7.38],

$$M(\Gamma(\alpha_1, \beta_1, \mathbb{R}^n)) \geqslant c_n \cdot \log\left(1 + \frac{1}{m}\right), \tag{26}$$

where $c_n > 0$ is some constant depending only on n, and

$$m = \frac{d(\alpha_1, \beta_1)}{\min\{d(\alpha_1), d(\beta_1)\}}.$$

By Hölder inequality, for any function $\rho \in \operatorname{adm}\Gamma$,

$$M(\Gamma) \leqslant \int\limits_D \rho^n(x)dm(x) \leqslant \left(\int\limits_D \rho^p(x)dm(x)\right)^{\frac{n}{p}} \cdot m^{\frac{p-n}{n}}(D). \tag{27}$$

Letting (27) to inf over all $\rho \in \operatorname{adm}\Gamma$, we obtain that

$$M(\Gamma) \leqslant \int\limits_D \rho^n(x)dm(x) \leqslant (M_p(\Gamma))^{\frac{n}{p}} \cdot m^{\frac{p-n}{n}}(D). \tag{28}$$

Here we take into account that D is bounded, so that $m^{\frac{p-n}{n}}(D) < \infty$. Now, by (27) and (28) we obtain that

$$M_p(\Gamma(\alpha_1, \beta_1, \mathbb{R}^n)) \geqslant c_n^{\frac{p}{n}} m^{-\frac{n-p}{n}}(D) \cdot \log^{\frac{p}{n}}\left(1 + \frac{1}{m}\right). \tag{29}$$

Now, we put $\rho(y)=\begin{cases}\frac{1}{C_0\varepsilon_0}, & y\in D';\\ 0, & y\notin D'.\end{cases}$ Observe that ρ satisfies (1) for $f(\Gamma)$, see (21). By the definition of $\mathfrak{S}^p_{\delta,A,Q}$, we obtain that

$$M_p(\Gamma)\leqslant\frac{1}{C_0^p\varepsilon_0^p}\int\limits_{D'}Q(y)dm(y)=C_0^{-p}\cdot\frac{\|Q\|_1}{|f(x)-f(y)|^p}. \tag{30}$$

By (29) and (30), it follows that

$$c_n^{\frac{p}{n}}m^{-\frac{n-p}{n}}(D)\cdot\log^{\frac{p}{n}}\left(1+\frac{\delta_1}{2|x-y|}\right)\leqslant C_0^{-p}\cdot\frac{\|Q\|_1}{|f(x)-f(y)|^p}.$$

The desired inequality (22) follows from the last relation, where $C:=C_0^{-1}\cdot\widetilde{c_n}^{-1/n}m^{\frac{n-p}{np}}(D)$, taking into account that, according to L'Hospital's rule, $\log\left(1+\frac{1}{nt}\right)\sim\log\left(1+\frac{1}{kt}\right)$ as $t\to+0$ for any different $k,n>0$.

We have proved Lemma 3 for the inner points $x,y\in U\cap D$. For the points $x,y\in U\cap\overline{D}$, this statement follows by passing to the limit $\overline{x}\to x$ and $\overline{y}\to y$, $\overline{x},\overline{y}\in D$. □

Proof of Theorem 1. The possibility of a continuous extension $f\colon\overline{D}\to\overline{D'}$ for $f\in\mathfrak{S}^p_{\delta,A,Q}(D,D')$ was established in Lemma 3. It remains to prove the relation (6) for any $x,y\in\overline{D}$. Let us prove by contradiction. Assume that the conclusion of Theorem 1 does not hold. Now, for any $m\in\mathbb{N}$ there exists $\overline{f_m}\colon\overline{D}\to\overline{D'}$, $f_m\in\mathfrak{S}^p_{\delta,A,Q}(D,D')$, and $x_m,y_m\in\overline{D}$ such that

$$|\overline{f_m}(x_m)-\overline{f_m}(y_m)|\geqslant m\cdot(\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta}{|x_m-y_m|}\right). \tag{31}$$

Since f_m has a continuous extension to ∂D, we may assume that $x_m,y_m\in D$. Since D is bounded, we may find subsequences x_{m_k},y_{m_k}, $k\in\{1,2,\ldots\}$, and points $x_0,y_0\in\overline{D}$ such that $x_{m_k}\to x_0$ and $y_{m_k}\to y_0$ as $k\to\infty$. There are two cases: 1) $x_0\neq y_0$, 2) $x_0=y_0$. In the first case, when $x_0\neq y_0$, observe that, there exists $M\in\mathbb{N}$ such that $|\overline{f_m}(x_m)-\overline{f_m}(y_m)|\leqslant M$ for any $m\in\mathbb{N}$. Indeed, by the assumption D' is bounded, so that by the triangle inequality $|\overline{f_m}(x_m)-\overline{f_m}(y_m)|\leqslant|\overline{f_m}(x_m)|+|\overline{f_m}(y_m)|\leqslant 2\sup\limits_{x\in\overline{D}}|f_m(x)|\leqslant 2\cdot d(D')$. In this case, we may set $M:=2\cdot d(D')$. In turn,

$$(\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta}{|x_m-y_m|}\right)\ \to\ (\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta}{|x_0-y_0|}\right):=C_1$$

as $m\to\infty$. Thus, for sufficiently large $m\in\mathbb{N}$,

$$|\overline{f_m}(x_m)-\overline{f_m}(y_m)|\leqslant\frac{2M}{C_1}\cdot(\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta}{|x_m-y_m|}\right).$$

The latter contradicts with (31).

In the second case, when $x_0=y_0$, the relation contradicts with (22) whenever $x_0\in\partial D$, and with Theorems 1.1, 7.1 in [9] whenever $x_0\in D$. □

4. Proof of Theorem 2. Just as in the previous section, we formulate a "local version" of Theorem 2, cf. [7, Theorem 2].

Lemma 4. *Let $n\geqslant 2$, $p\geqslant n$, and let $Q\in L^1(D')$, let D be a bounded domain with a locally quasiconformal boundary, and let D' be a bounded convex domain. Then any*

$f\in\mathfrak{S}^p_{\delta,A,Q}(D,D')$ has a continuous extension $f\colon\overline{D}\rightarrow\overline{D^{\,\prime}}$, while, for any $x_0\in\partial D$, $x_0\ne\infty$, there exists a neighborhood V of x_0 and $C=C(n,p,A,D,D')>0$ such that

$$|\overline{f}(x)-\overline{f}(y)|\leqslant C\cdot(\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta}{|x-y|^{\alpha}}\right) \tag{32}$$

for any $x,y\in V\cap\overline{D}$ and $f\in\mathfrak{S}^p_{\delta,A,Q}(D,D')$, where α is a number from the definition of quasiconformal boundary in (7) and $\|Q\|_1$ is a norm of Q in $L^1(D')$.

Proof. Observe that the locally quasiconformal boundaries are weakly flat (see [27, Proposition 2.2], see also [1, Theorem 17.10]). Now, the possibility of a continuous extension of the mapping f to the boundary of the domain D follows by Theorems 1.2 and 7.1 in [9] due to the comments made after the formulation of Lemma 1.

The further proof comes down to modifying the scheme of the proof of Theorem 2 in [7]. Put $x_0\in\partial D$. Let $y_*\in D'$ be an arbitrary point of D', $\delta_*:=d(y_*,\partial D')$ and $E=\overline{B(y_*,\delta_*/2)}\subset D'$. By Lemma 1, there exists $\delta_1>0$ such that $h(f^{-1}(E),\partial D)\geqslant\delta_1$ for all $f\in\mathfrak{S}^p_{\delta,A,Q}$. Then $d(f^{-1}(E),\partial D)\geqslant\delta_1$ for any $f\in\mathfrak{S}^p_{\delta,A,Q}$. In addition, since by Theorems 1.2 and 7.1 in [9] the family $\mathfrak{S}^p_{\delta,A,Q}$ is equicontinuous at $\overline{D}$, for the number $\delta_*/8$ there is a neighborhood $U\subset B(x_0,\delta_1/4)$ of x_0 such that $|f(x)-f(x_0)|<\delta_*/8$ for any $x,y\in U\cap D$ and all $f\in\mathfrak{S}^p_{\delta,A,Q}$.

By the definition of a locally quasiconformal boundary, there exist a neighborhood U^* of the point x_0 and a quasiconformal mapping $\varphi\colon U^*\rightarrow\mathbb{B}^n$, $\varphi(U^*)=\mathbb{B}^n$, such that $\varphi(D\cap U^*)=\mathbb{B}^n_+$, where $\mathbb{B}^n_+=\{x\in\mathbb{B}^n\colon x=(x_1,\ldots,x_n),x_n>0\}$ is a half-ball.
We may assume that $\varphi(x_0)=0$ and $\overline{U^*}\subset U$ (see the proof of Theorem 17.10 in [1]). Let V be any neighborhood in U^* such that $\overline{V}\subset U^*$, and let

$$\delta_2:=d(\partial V,\partial U^*). \tag{33}$$

Consider the auxiliary mapping

$$F(w):=f(\varphi^{-1}(w)),\qquad F\colon\mathbb{B}^n_+\rightarrow U^*. \tag{34}$$

Let $x,y\in V\cap D$ and

$$\varepsilon_0:=|f(x)-f(y)|<\delta_0:=\delta_*/4. \tag{35}$$

Now, we apply Lemma 2 for the points $A=f(x)$, $B=f(y)$ and $z_0=f(x_0)$. According to this lemma, there exist segments $I\ni A$ and $J\ni B$ in D' such that $I\cap E\ne\varnothing\ne J\cap E$, moreover

$$d(I,J)\geqslant C_0\cdot|f(x)-f(y)|, \tag{36}$$

where C_0 depends only on E and $d(D')$.

By Proposition 2 there exist whole f-liftings α_1 and β_1 of the paths I and J starting at the points x and y. Now, by the definition, $|\alpha_1|\cap f^{-1}(E)\ne\varnothing\ne|\beta_1|\cap f^{-1}(E)$. Then $|\alpha_1|\cap U\ne\varnothing\ne|\alpha_1|\cap(\mathbb{R}^n\setminus U)$ and $|\beta_1|\cap U\ne\varnothing\ne|\beta_1|\cap(\mathbb{R}^n\setminus U)$. By [28, Theorem 1.I.5.46]

$$|\alpha_1|\cap\partial U\ne\varnothing,\quad|\beta_1|\cap\partial U\ne\varnothing. \tag{37}$$

Similarly,

$$|\alpha_1|\cap\partial V\ne\varnothing,\quad|\beta_1|\cap\partial V\ne\varnothing. \tag{38}$$

Due to (37), α_1 and β_1 contain subpaths α_1^* and β_1^* with origins at the points x and y which belong entirely in U^* and have end points at ∂U^*. Due to (33), (37) and (38)

$$d(\alpha_1^*)\geqslant\delta_2,\quad d(\beta_1^*)\geqslant\delta_2. \tag{39}$$

Consider the paths $\varphi(\alpha_1^*)$ and $\varphi(\beta_1^*)$. Let $\overline{x},\overline{y}\in U^*$ be such that $d(\alpha_1^*)=|\overline{x}-\overline{y}|$. We put $x^*=\varphi(\overline{x})$ and $y^*=\varphi(\overline{y})$. Then by (7), $\widetilde{C}\cdot|x^*-y^*|^{\alpha}\geqslant|\overline{x}-\overline{y}|=d(\alpha_1^*)\geqslant\delta_2$, or

$$|x^*-y^*|\geqslant\left(\frac{1}{\widetilde{C}}\delta_2\right)^{1/\alpha}. \tag{40}$$

From (40), we obtain that $d(\varphi(\alpha_1^*))\geqslant\left(\frac{1}{\widetilde{C}}\delta_2\right)^{1/\alpha}$. Similarly, $d(\varphi(\beta_1^*))\geqslant\left(\frac{1}{\widetilde{C}}\delta_2\right)^{1/\alpha}$. Let

$$\Gamma:=\Gamma(\varphi(\alpha_1^*),\varphi(\beta_1^*),\mathbb{B}^n_+).$$

Observe that, $\mathbb{B}^n_+$ is also a QED-domain with some $A_0^*<\infty$ in (4) (see [15, Lemma 4.3]). Then, on the one hand, by (4)

$$M(\Gamma)\geqslant(1/A_0^*)\cdot M(\Gamma(\varphi(\alpha_1^*),\varphi(\beta_1^*),\mathbb{R}^n)), \tag{41}$$

and on the other hand, by [26, Lemma 7.38]

$$M(\Gamma(\varphi(\alpha_1^*),\varphi(\beta_1^*),\mathbb{R}^n))\geqslant c_n\cdot\log\left(1+\frac{1}{m}\right), \tag{42}$$

where $c_n>0$ is some constant that depends only on n,

$$m=\frac{d(\varphi(\alpha_1^*),\varphi(\beta_1^*))}{\min\{d(\varphi(\alpha_1^*)),d(\varphi(\beta_1^*))\}}.$$

Then, combining (41) and (42) and taking into account that $d(\varphi(\alpha_1^*),\varphi(\beta_1^*))\leqslant|\varphi(x)-\varphi(y)|$, we obtain that

$$M(\Gamma)\geqslant\widetilde{c_n}\cdot\log\left(1+\frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{1/\alpha}d(\varphi(\alpha_1^*),\varphi(\beta_1^*))}\right)\geqslant\widetilde{c_n}\cdot\log\left(1+\frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{1/\alpha}|\varphi(x)-\varphi(y)|}\right), \tag{43}$$

where $\widetilde{c_n}>0$ is some constant that depends only on n and A_0^* from the definition of QED-domain.

By Hölder inequality, for any function $\rho\in\operatorname{adm}\Gamma$,

$$M(\Gamma)\leqslant\int\limits_D\rho^n(x)dm(x)\leqslant\left(\int\limits_D\rho^p(x)dm(x)\right)^{\frac{n}{p}}\cdot m^{\frac{p-n}{n}}(D). \tag{44}$$

Letting (44) to inf over all $\rho\in\operatorname{adm}\Gamma$, we obtain that

$$M(\Gamma)\leqslant\int\limits_D\rho^n(x)dm(x)\leqslant(M_p(\Gamma))^{\frac{n}{p}}\cdot m^{\frac{p-n}{n}}(D). \tag{45}$$

Here we take into account that D is bounded, so that $m^{\frac{p-n}{n}}(D)<\infty$. Now, by (43) and (45) we obtain that

$$M_p(\Gamma(\alpha_1,\beta_1,\mathbb{R}^n))\geqslant\widetilde{c_n}^{\frac{p}{n}}m^{-\frac{n-p}{n}}(D)\cdot\log^{\frac{p}{n}}\left(1+\frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{1/\alpha}|\varphi(x)-\varphi(y)|}\right). \tag{46}$$

Let us now establish an upper bound for $M_p(\Gamma)$. Note that, F in (34) satisfies the relation (3) with the function $\widetilde{Q}(x)=K_0\cdot Q(x)$ instead of Q, where $K_0\geqslant 1$ is the constant of a quasiconformality of φ^{-1}. Let us put

$$\rho(y)=\begin{cases}\frac{1}{C_0\varepsilon_0}, & y\in D';\\ 0, & y\notin D',\end{cases}$$

where C_0 is the universal constant in inequality (36) and ε_0 is defined in (35). Note that ρ satisfies the relation (1) for $F(\Gamma)$ due to the relation (21). Then, by the definition of $\mathfrak{S}^p_{\delta,A,Q}$, due to the definition of F in (34), we obtain that

$$M_p(\Gamma)\leqslant\frac{1}{C_0^p\varepsilon_0^p}\int\limits_{D'}K_0Q(y)dm(y)=C_0^{-p}K_0\cdot\frac{\|Q\|_1}{|f(x)-f(y)|^p}. \tag{47}$$

It follows by (43) and (47) that

$$\widetilde{c_n}^{\frac{p}{n}}m^{-\frac{n-p}{n}}(D)\cdot\log^{\frac{p}{n}}\left(1+\frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{1/\alpha}|\varphi(x)-\varphi(y)|}\right)\leqslant C_0^{-p}K_0\cdot\frac{\|Q\|_1}{|f(x)-f(y)|^p}.$$

Now, from the latter relation, it follows that

$$|f(x)-f(y)|\leqslant C_0^{-1}\widetilde{c_n}^{-\frac{1}{n}}K_0^{\frac{1}{p}}\cdot(\|Q\|_1)^{\frac{1}{p}}\log^{-\frac{1}{n}}\left(1+\frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{1/\alpha}|\varphi(x)-\varphi(y)|}\right)\leqslant$$

$$\leqslant C_0^{-1}\widetilde{c_n}^{-\frac{1}{n}}K_0^{\frac{1}{p}}\cdot(\|Q\|_1)^{\frac{1}{p}}\log^{-\frac{1}{n}}\left(1+\frac{\delta_2^{1/\alpha}}{(\widetilde{C})^{(1/\alpha)+1}|x-y|^\alpha}\right),$$

which is the desired inequality (8), where $C:=C_0^{-1}\cdot\widetilde{c_n}^{-1/n}\cdot K_0$ and $r_0=\delta_2^{1/\alpha}/(\widetilde{C})^{1/\alpha+1}$ instead of δ. However, we may replace r_0 by δ here, because, by L'Hospital's rule, $\log\left(1+\frac{1}{nt}\right)\sim\log\left(1+\frac{1}{kt}\right)$ as $t\to+0$ for any different $k,n>0$.

We proved Lemma 4 for the inner points $x,y\in V\cap D$. For $x,y\in V\cap\overline{D}$, this statement follows by means of the transition to the limit $\overline{x}\to x$ and $\overline{y}\to y$, $\overline{x},\overline{y}\in D$. □

Proof of Theorem 2. is similar to the proof of Theorem 1, however, we will carry it out completely. The possibility of a continuous extension $f\colon\overline{D}\to\overline{D'}$ for $f\in\mathfrak{S}^p_{\delta,A,Q}(D,D')$ was established in Lemma 4. It remains to prove the relation (6) for any $x,y\in\overline{D}$. Let us prove by contradiction. Assume that the conclusion of Theorem 2 does not hold. Now, for any $m\in\mathbb{N}$ there exists $\overline{f_m}\colon\overline{D}\to\overline{D'}$, $f_m\in\mathfrak{S}^p_{\delta,A,Q}(D,D')$, and $x_m,y_m\in\overline{D}$ such that

$$|\overline{f_m}(x_m)-\overline{f_m}(y_m)|\geqslant m\cdot(\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta}{|x_m-y_m|^\alpha}\right). \tag{48}$$

Since f_m has a continuous extension to ∂D, we may assume that $x_m,y_m\in D$. Since D is bounded, we may find subsequences x_{m_k},y_{m_k}, $k\in\{1,2,\ldots\}$, and points $x_0,y_0\in\overline{D}$ such that $x_{m_k}\to x_0$ and $y_{m_k}\to y_0$ as $k\to\infty$. There are two cases: 1) $x_0\ne y_0$, 2) $x_0=y_0$. In the first case, when $x_0\ne y_0$, observe that, there exists $M\in\mathbb{N}$ such that $|\overline{f_m}(x_m)-\overline{f_m}(y_m)|\leqslant M$ for any $m\in\mathbb{N}$. Indeed, by the assumption D' is bounded, so that by the Triangle inequality $|\overline{f_m}(x_m)-\overline{f_m}(y_m)|\leqslant|\overline{f_m}(x_m)|+|\overline{f_m}(y_m)|\leqslant 2\sup\limits_{x\in\overline{D}}|f_m(x)|\leqslant 2\cdot d(D')$. In this case, we may set $M:=2\cdot d(D')$. In turn,

$$(\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta}{|x_m-y_m|^\alpha}\right)\ \to\ (\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta}{|x_0-y_0|^\alpha}\right):=C_1$$

as $m\to\infty$. Thus, for sufficiently large $m\in\mathbb{N}$,

$$|\overline{f_m}(x_m)-\overline{f_m}(y_m)|\leqslant\frac{2M}{C_1}\cdot(\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta}{|x_m-y_m|^{\alpha}}\right).$$

The latter contradicts with (48).

In the second case, when $x_0=y_0$, the relation contradicts with (22) whenever $x_0\in\partial D$. If $x_0=y_0\in D$, then by Theorems 1.1 and 4.1 in [9], since $0<\alpha\leqslant 1$ and $|x_m-y_m|<1$ for sufficiently large m we obtain that

$$|\overline{f_m}(x_m)-\overline{f_m}(x_0)|\leqslant\frac{C\cdot(\|Q\|_1)^{1/p}}{\log^{1/n}\left(1+\frac{\delta}{|x_m-y_m|}\right)}\leqslant\frac{C\cdot(\|Q\|_1)^{1/p}}{\log^{1/n}\left(1+\frac{\delta}{|x_m-y_m|^{\alpha}}\right)}.$$

The latter contradicts with (48). □

5. Proof of Theorem 3.

Proof. The proof of Theorem 3 almost completely coincides with the proof of a similar statement for the conformal modulus (see Proof of Theorem 3 in [7]), let us demonstrate this. Let $f\in\mathfrak{S}^p_{\delta,A,Q}(D,D')$. Since D is a regular domain, there exists a quasiconformal mapping g^{-1} of the domain D onto a domain D_0 with a locally quasiconformal boundary, and, by the definition of the metric ρ in the prime ends space, one has

$$\rho(P_1,P_2):=|g^{-1}(P_1)-g^{-1}(P_2)|. \tag{49}$$

Consider the auxiliary mapping

$$F(x)=(f\circ g)(x),\quad x\in D_0. \tag{50}$$

Since g^{-1} is quasiconformal, there is a constant $1\leqslant K_1<\infty$ such that

$$\frac{1}{K_1}\cdot M(\Gamma)\leqslant M(g(\Gamma))\leqslant K_1\cdot M(\Gamma) \tag{51}$$

for any family of paths Γ in D_0. Considering inequalities (51) and taking into account that f satisfies the relation (3), we obtain that also F satisfies the relation (3) with a new function $\widetilde{Q}(x):=K_1\cdot Q(x)$. In addition, since g is a fixed homeomorphism, then $h(F^{-1}(A),\partial D_0)\geqslant\delta_0>0$, where $\delta_0>0$ is some fixed number. Then Theorem 2 may be applied to the map F. Applying this theorem, we obtain that,

$$|F(x)-F(y)|\leqslant C^*K_1^{\frac{1}{p}}\cdot(\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta_0}{|x-y|^{\alpha}}\right) \tag{52}$$

for all $x,y\in D_0$, where $\|Q\|_1$ is the norm of the function Q in $L^1(D')$. If $P_1,P_2\in D_P\setminus E_D$, then $P_1=g(x)$ and $P_2=g(y)$ for some $x,y\in D_0$. Taking into account the relation (52) and using the relation $|x-y|=|g^{-1}(P_1)-g^{-1}(P_2)|=\rho(P_1,P_2)$, we obtain that

$$|F(g^{-1}(P_1))-F(g^{-1}(P_2))|\leqslant C^*K_1^{\frac{1}{p}}\cdot(\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta_0}{\rho^{\alpha}(P_1,P_2)}\right),$$

or, due to (50),

$$|f(P_1)-f(P_2)|\leqslant C^*K_1^{\frac{1}{p}}\cdot(\|Q\|_1)^{1/p}\log^{-1/n}\left(1+\frac{\delta_0}{\rho^{\alpha}(P_1,P_2)}\right).$$

The last ratio is desired if we put here $C:=C^*K_1^{\frac{1}{p}}$. Here we also take into account that, by L'Hospital's rule, $\log\left(1+\frac{1}{nt}\right)\sim\log\left(1+\frac{1}{kt}\right)$ as $t\to+0$ for any different $k,n>0$. Thus, in the last relation, we may write δ instead δ_0.

Observe that, every $f \in \mathfrak{S}^p_{\delta,A,Q}(D,D')$ has a continuous extension to $E_D := \overline{D}_P \setminus D$. Indeed, g^{-1} is continuous in $\overline{D}$, see the comments made after relation (9). Also, F has a continuous extension to $\overline{D_0}$. Now, $f(x) = (F \circ g^{-1})(x)$ has a continuous extension to $\overline{D}_P$, as required. If $P_1, P_2 \in E_D$, we obtain the desired inequality in (10) taking the limit in $|\overline{f}(x_m) - \overline{f}(y_m)| \leqslant C \cdot (\|Q\|_1)^{1/p} \log^{-1/n}\left(1 + \frac{\delta}{\rho^{\alpha}(x_m,y_m)}\right)$ as $m \to \infty$, where $x_m \to P_1$ and $y_m \to P_2$ as $m \to \infty$, $x_m, y_m \in D$, $m \in \{1, 2, \ldots\}$. □

6. The case $p > n$. A metric space (X, d, μ) is called $\widetilde{Q}$-*Ahlfors-regular* for some $\widetilde{Q} \geqslant 1$ if, for any $x_0 \in X$ and some constant $C \geqslant 1$,

$$\frac{1}{C} R^{\widetilde{Q}} \leqslant \mu(B(x_0, R)) \leqslant C R^{\widetilde{Q}}.$$

The Ahlfors α-regular spaces have Hausdorff dimension α (see [29, p. 61–62]). Let (X, d, μ) be a metric measure space with metric d and a locally finite Borel measure μ. Following [29], §7.22, a Borel function $\rho\colon X \to [0, \infty]$ is said to be an *upper gradient* of a function $u\colon X \to \mathbb{R}$ if

$$|u(x) - u(y)| \leqslant \int\limits_{\gamma} \rho ds$$

for any rectifiable path γ connecting the points x and $y \in X$, where, as usual, $\int_\gamma \rho ds$ denotes the linear integral of the function ρ over the path γ. Such a space X will be said to admit the $(1; p)$-Poincaré inequality if there exist constants $C \geqslant 1$ and $\tau > 0$ such that

$$\frac{1}{\mu(B)} \int\limits_{B} |u - u_B| d\mu(x) \leqslant C(\operatorname{diam} B) \left(\frac{1}{\mu(\tau B)} \int\limits_{\tau B} \rho^p d\mu(x) \right)^{1/p}$$

for any ball $B \subset X$ and arbitrary locally bounded continuous function $u\colon X \to \mathbb{R}$ and any upper gradient ρ of u, where

$$u_B := \frac{1}{\mu(B)} \int\limits_{B} u d\mu(x).$$

For distinct points $x, y \in X$ denote by Γ_{xy} the collection of all compact rectifiable paths in X connecting x to y. The following result holds (see [30, Proposition 4.1]).

Proposition 3. *Let X be a Q-Ahlfors regular metric measure space that $(1; p)$-Poincaré inequality holds for some $p > Q$. Then $M_p(\Gamma_{xy}) \geqslant C \cdot (d(x, y))^{Q-p}$.*

Let $f\colon D \to \mathbb{R}^n$, $n \geqslant 2$, and let $Q\colon \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function such that $Q(y) \equiv 0$ for $y \in \mathbb{R}^n \setminus f(D)$. Let $A = A(y_0, r_1, r_2)$. Let $\Gamma_f(y_0, r_1, r_2)$ denotes the family of all paths $\gamma\colon [a, b] \to D$ such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$, i.e., $f(\gamma(a)) \in S(y_0, r_1)$, $f(\gamma(b)) \in S(y_0, r_2)$, and $f(\gamma(t)) \in A(y_0, r_1, r_2)$ for any $a < t < b$. Given $p \geqslant 1$, we say that f *satisfies the inverse Poletsky inequality* at $y_0 \in f(D)$ with respect to p-modulus, if the relation

$$M_p(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A} Q(y) \cdot \eta^p(|y - y_0|) dm(y) \tag{53}$$

holds for any $0 < r_1 < r_2 < r_0 := \sup\limits_{y\in D'} |y - y_0|$ and any Lebesgue measurable function $\eta\colon (r_1, r_2) \to [0, \infty]$ such that $\int_{r_1}^{r_2} \eta(r)dr \geqslant 1$.

Let $f\colon D \to D'$ be a mapping satisfying the relation (53) for some $p \geqslant n$, $n \geqslant 2$, and let Q is integrable in D'. Assume that D is n-Ahlfors regular domain (as a metric measure space with the Euclidean metric $d(x, y) = |x - y|$) that $(1; p)$-Poincaré inequality holds for some $p > n$. Also, let D' be a domain in $\mathbb{R}^n$. Let $f(x) \neq f(y)$. Set

$$\varepsilon_0 := |f(x) - f(y)|. \tag{54}$$

Now, by Proposition 3

$$M_p(\Gamma_{xy}) \geqslant C \cdot |x - y|^{n-p}. \tag{55}$$

Draw through the points $f(x)$ and $f(y)$ a line $r = r(t) = f(x) + (f(x) - f(y))t$, $-\infty < t < \infty$. Let $t_0 < 0$ be an arbitrary point such that $z^1 := r(t_0) \in D'$. Now, $f(\Gamma_{xy}) \subset \Gamma(S(z^1, \varepsilon^1), S(z^1, \varepsilon^2), A(z^1, \varepsilon^1, \varepsilon^2))$, where $\varepsilon^1 := |f(x) - z^1|$ and $\varepsilon^2 := |f(y) - z^1|$. Now, $\Gamma_{xy} \subset \Gamma_f(z^1, \varepsilon^1, \varepsilon^2)$. Observe that

$$|f(y) - f(x)| + \varepsilon^1 = |f(y) - f(x)| + |f(x) - z^1| = |z^1 - f(y)| = \varepsilon^2, \tag{56}$$

1 and, thus, $\varepsilon^1 < \varepsilon^2$. Now let us prove the upper bound for $M_p(\Gamma_{xy})$. We set

$$\eta(t) = \begin{cases} \frac{1}{\varepsilon_0}, & t \in [\varepsilon^1, \varepsilon^2];\\ 0, & t \notin [\varepsilon^1, \varepsilon^2], \end{cases}$$

here ε_0 is a number from (54). Note that η satisfies the relation (19) for $r_1 = \varepsilon^1$ and $r_2 = \varepsilon^2$. Indeed, it follows from (54) and (56) that $r_1 - r_2 = \varepsilon^2 - \varepsilon^1 = |f(y) - z^1| - |f(x) - z^1| = |f(x) - f(y)| = \varepsilon_0$. Then $\int_{\varepsilon^1}^{\varepsilon^2} \eta(t)dt = (1/\varepsilon_0) \cdot (\varepsilon^2 - \varepsilon^1) \geqslant 1$. By the inequality (9) and the relation (53) applied at the point z^1, we obtain that

$$M_p(\Gamma_{xy}) \leqslant M_p(\Gamma_f(z^1, \varepsilon^1, \varepsilon^2)) \leqslant \frac{1}{\varepsilon_0^p} \int\limits_{D'} Q(z)dm(z) = \frac{\|Q\|_1}{|f(x) - f(y)|^p}. \tag{57}$$

By (55) and (57), we obtain that $C \cdot |x - y|^{n-p} \leqslant \|Q\|_1/|f(x) - f(y)|^p$. From the latter ratio, we obtain that

$$|f(x) - f(y)| \leqslant C^{-1/p} \cdot (\|Q\|_1)^{1/p} |x - y|^{\frac{p-n}{p}}. \tag{58}$$

Thus, the following statement holds.

Theorem 4. *Let $f\colon D \to D'$ be a mapping satisfying the relation (53) for some $p \geqslant n$, $n \geqslant 2$, and let Q is integrable in D'. Assume that D is n-Ahlfors regular domain (as a metric measure space with the Euclidean metric $d(x, y) = |x - y|$) that $(1; p)$-Poincaré inequality holds for some $p > n$. Also, let D' be a domain in $\mathbb{R}^n$. Then the relation (58) holds for every $x, y \in D$, where $C > 0$ is some constant which does not depend on f. Moreover, f has a continuous extension $f\colon \overline{D} \to \overline{D'}$ and the relation (58) holds for every $x, y \in \overline{D}$, as well.*

Proof. For $x, y \in D$, the relation (58) was established before the formulation of Theorem 4. Let us to prove the possibility of a continuous boundary extension of f to ∂D. Assume the contrary. Then there are $x_0 \in \partial D$ and sequences $x_m, y_m \in D$, $m \in \{1, 2, \ldots\}$, such that $x_m, y_m \to x_0$ as $m \to \infty$, however, $|f(x_m) - f(y_m)| \geqslant \varepsilon_1 > 0$ for some $\varepsilon_1 > 0$. The latter contradicts the relation (58), because it follows from it that, $|f(x_m) - f(y_m)| \leqslant C^{-1/p} \cdot (\|Q\|_1)^{1/p} |x_m - y_m|^{\frac{p-n}{p}} \to 0$ as $m \to \infty$. Finally, the relation (58) may be obtained for $x, y \in \overline{D}$ by means of a limit transition operation. □

Example 1. In the unit disk $\mathbb{D}\subset\mathbb{C}$, let us consider the family of fractional linear transformations $f_n(z)=\frac{z-\frac{n-1}{n}}{1-z\frac{n-1}{n}}$, $n\in\{1,2,\ldots\}$, $z=x+iy\in\mathbb{D}$, $i^2=-1$. Note that, every f_n map the unit disk onto itself homeomorphically. Observe that, $f_n\in C^1(\mathbb{D})$ and $f_n^{-1}\in C^1(\mathbb{D})$, so that $f_n\in W^{1,p}_{\rm loc}(\mathbb{D})$ and $f_n^{-1}\in W^{1,p}_{\rm loc}(\mathbb{D})$ for every $n\in\mathbb{N}$ and for every $p>1$. Thus, f_n satisfies the relations (53)–(19) for every $n\in\mathbb{N}$ with $Q(z)=K_{O,p}(z,f)$, where

$$K_{O,p}(z,f)\ =\ \begin{cases}\frac{\|f'(z)\|^p}{|J(z,f)|}, & J(z,f)\neq 0;\\ 1, & f'(z)=0;\\ \infty, & \text{otherwise},\end{cases}$$

$$\|f'(z)\|=\max_{h\in\mathbb{R}^n\setminus\{0\}}\frac{|f'(z)h|}{|h|}=|f'(z)|,\quad J(z,f)=\det f'(z)=|f'(z)|^2, \tag{59}$$

see e.g. [31, Theorem 1.1]. Observe that, the $(1;p)$-Poincare inequality holds in $\mathbb{D}$ for any $p>1$ (see [32, Theorem 10.5]). Observe that, $\mathbb{D}$ is a Loewner space (see [29, Theorem 8.2 and Example 8.24(a)]). Now, $\mathbb{D}$ is Ahlfors regular, see [29, Proposition 8.19]. Now, we may apply Theorem 4. By this theorem one has

$$|f(x)-f(y)|\leqslant C^{-1/p}\cdot\left(\int\limits_{\mathbb{D}}|f_n'(z)|^{p-2}dm(z)\right)^{1/p}|x-y|^{\frac{p-n}{p}},\qquad \forall\ x,y\in\mathbb{D},$$

while, by the direct calculation, $|f_n'(z)|=1-\frac{(n-1)^2}{n}/\left(1-z\frac{n-1}{n}\right)^2$, $n\in\{1,2,\ldots\}$. We see from the latter inequality that f_n is Hölder continuous for every fixed $n\in\{1,2,\ldots\}$. However, the integral $I_n:=\int_{\mathbb{D}}|f_n'(z)|^{p-2}dm(z)$ is unbounded over $n\in\mathbb{N}$; otherwise, the family $\{f_n\}_{n=1}^{\infty}$ must be equicontinuous, but this is not true (for instance, we may consider the sequence $z_n=\frac{n-1}{n}$, $n\in\{1,2,\ldots\}$. For this sequence, $f_n(z_n)=0$, while $f_n(1)=1$ and $|z_n-1|\to 0$ as $n\to\infty$. At the same time, $|f_n(z_n)-f_n(1)|\not\to 0$ as $n\to\infty$). The reason for this phenomenon is that the sequence of mappings f_n does not have a single integrable function Q in (53) that is common to all mappings f_n, $n\in\{1,2,\ldots\}$.

REFERENCES

1. J. Väisälä, *Lectures on n-Dimensional Quasiconformal Mappings*, Lecture Notes in Math., V.229, Springer–Verlag, Berlin etc., 1971.
2. O. Martio, V. Ryazanov, U. Srebro, E. Yakubov, Moduli in Modern Mapping Theory, Springer Monographs in Mathematics, Springer, New York etc., 2009.
3. V. Ryazanov, R. Salimov, E. Sevost'yanov, *On the Hölder property of mappings in domains and on boundaries*, Journal of Mathematical Sciences, **246** (2020), №1, 60–74.
4. E. Sevost'yanov, *The Boundary Hölder Continuity of Mappings with the Poletsky Condition*, Journal of Mathematical Sciences, **281** (2024), №5, 818–835.
5. O.P. Dovhopiatyi, E.A. Sevost'yanov, *On homeomorphisms with a fixed point onto domains with a Poincaré inequality*, Journal of Mathematical Sciences, **282** (2024), №1, 28–43.
6. O.P. Dovhopiatyi, N.S. Ilkevych, E.O. Sevost'yanov, A.L. Targonskii, *On prime end distortion estimates of mappings with the Poletsky condition in domains with the Poincaré inequality*, Mat. Stud., **61** (2024), №2, 148–159.
7. O. Dovhopiatyi, E. Sevost'yanov, *On boundary Hölder logarithmic continuity of mappings in some domains*, Acta Mathematica Hungarica **172** (2024), №2, 499–512.
8. E. Sevost'yanov, O. Dovhopiatyi, N. Ilkevych, M. Androschuk, *On the behavior of one class of mappings acting upon domains with locally quasiconformal boundary*, Ukr. Math. J., **76** (2024), 843–858.

9. V. Gol'dshtein, E. Sevost'yanov E., Ukhlov A., *On the theory of generalized quasiconformal mappings*, Complex Analysis and its Synergies, **10** (2024), Article number 2.
10. N. Ilkevych, E. Sevost'yanov, A. Ukhlov, *On the Equicontinuity of Generalized Quasiconformal Mappings by Prime Ends*, Complex Analysis and Operator Theory, **18** (2024), №4, Article number 102.
11. O. Martio, S. Rickman, J. Väisälä, *Distortion and singularities of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A1, **465** (1970), 1–13.
12. S. Rickman, Quasiregular mappings, Springer-Verlag, Berlin, 1993.
13. M. Vuorinen, *Exceptional sets and boundary behavior of quasiregular mappings in n-space,* Ann. Acad. Sci. Fenn. Ser. A 1. Math. Dissertationes, **11** (1976), 1–44.
14. F.W. Gehring, O. Martio, *Quasiextremal distance domains and extension of quasiconformal mappings,* J. Anal. Math., **45** (1985), 181–206.
15. M. Vuorinen, *On the existence of angular limits of n-dimensional quasiconformal mappings,* Ark. Math., **18** (1980), 157–180.
16. R. Näkki, *Prime ends and quasiconformal mappings*, Journal d'Analyse Mathématique, **35** (1979), 13–40.
17. D.A. Kovtonyuk, V.I. Ryazanov, *On the theory of prime ends for space mappings*, Ukrainain Mathematical Journal, **67** (2015), №4, 528–541.
18. D.P. Ilyutko, E.A. Sevost'yanov, *On prime ends on Riemannian manifolds*, J. Math. Sci., **241** (2019), 47–63.
19. H. Federer, Geometric Measure Theory, Springer, Berlin etc., 1969.
20. A. Ukhlov, *On mappings, which induce embeddings of Sobolev spaces,* Siberian Math. J., **34** (1993), 185–192.
21. V. Gol'dshtein, E. Sevost'yanov E., Ukhlov A., *Composition operators on Sobolev spaces and weighted moduli inequalitites*, Mathematical Reports, **26(76)** (2024), №2, 101–113.
22. J. Hesse, *A p-extremal length and p-capacity equality,* Ark. Mat., **13** (1975), 131–144.
23. E. Sevost'yanov, *On logarithmic Hölder continuity of mappings on the boundary*, Annales Fennici Mathematici, **47** (2022), 251–259.
24. E. Sevost'yanov, S. Skvortsov, *On the convergence of mappings in metric spaces with direct and inverse modulus conditions*, Ukrainain Mathematical Journal, **70** (2018), №7, 1097–1114.
25. O. Dovhopiatyi, *On the possibility of joining two pairs of points in convex domains using paths,* Proc. Inst. Appl. Math. and Mech., **47** (2023), 3–12.
26. M. Vuorinen, Conformal Geometry and Quasiregular Mappings, Lecture Notes in Math., V.1319, Springer–Verlag, Berlin etc., 1988.
27. N.S. Ilkevych, E.A. Sevost'yanov, S.A. Skvortsov, *On the global behavior of inverse mappings in terms of prime ends,* Annales Fennici Mathematici, **46** (2021), 371–388.
28. K. Kuratowski, Topology, V.2, Academic Press, New York–London, 1968.
29. J. Heinonen, *Lectures on analysis on metric spaces,* Springer Science+Business Media, New York, 2001.
30. T. Adamowicz, N. Shanmugalingam, *Non-conformal Loewner type estimates for modulus of curve families,* Ann. Acad. Sci. Fenn. Math., **35** (2010), 609–626.
31. R.R. Salimov, E.A. Sevost'yanov, *The Poletskii and Väisälä inequalities for the mappings with (p, q)-distortion,* Complex Variables and Elliptic Equations, **59** (2014), №2, 217–231.
32. P. Hajlasz, P. Koskela P., *Sobolev met Poincare,* Mem. Amer. Math. Soc., **145** (2000), №688, 1–101.

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