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BOUNDARY VALUE PROBLEMS WITH NONLOCAL CONDITIONS FOR HYPERBOLIC SYSTEMS OF EQUATIONS WITH TWO INDEPENDENT VARIABLES

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Nonlocal boundary value problems for arbitrary order hyperbolic systems with one spatial variable are considered. A priori estimates for general nonlocal mixed problems for systems with smooth and piecewise smooth coefficients are obtained. The correct solvability of such problems is proved. Examples of additional conditions necessity are provided.

1. Introduction. Theory of boundary-value problems for hyperbolic partial differential equations with two independent variables is almost half a century old. However, even today, it offers new problems solving of which require various methods of modern analysis. Such interest is caused by the fact that these equations arise in the studying of important processes of science and technology

Not limited to the well-known examples of string oscillations, electromagnetic perturbations, etc., let us point to more complex processes that lead to equations and systems of hyperbolic type.

A basic evolution model of age-structured population ([38, 65]) leads to the Lotka–MacKendrick system

$$\partial_t u + \partial_x u = -\lambda(x)u, \quad u(x,0) = \nu(x), \quad u(0,t) = v(t)\int_0^L \beta(x)u(x,t)\,dx,$$

where λ, ν, L, β are standard biological parameters, and the birth rate $v(t)$ is often used to control the population (v is the control).

Similar problems arise in solid-state physics ([15]). The peculiarity of these problems is that the characteristics of hyperbolic system are both inclined and horizontal. From a physics point of view, this means that some of perturbations in the medium propagate with a finite velocity and some with unbounded one. The mathematical model of such a problem takes the form

$$\frac{\partial u}{\partial t} + \operatorname{sh} E\frac{\partial u}{\partial x} = -u^2 \operatorname{ch} E, \quad \frac{\partial E}{\partial x} = pu, \quad \frac{\partial \varphi}{\partial x} = -qE.$$

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Here u is the concentration of impurity sputtering of an electroluminophore; E is the electric field strength; φ is the potential; p, q are some constants.

The initial and boundary conditions for this system are set as follows

$$u(x,0)=u_0(x),\quad u(0,t)-\frac{\partial u}{\partial x}(0,t)=0,\quad u(l(t),t)=g(l(t)),$$
$$E(l(t),t)=0,\quad \varphi(0,t)=\varphi_0,\quad \varphi(l(t),t)=0,$$

where $l(t)$ is a currently unknown curve describing the position of the area right boundary and satisfying the relation

$$l'(t)l(t)g(l(t))-b\int\limits_0^{l(t)}u(x,t)\operatorname{sh}E\,dx=0$$

and the initial condition

$$\int\limits_0^{l(0)}xg(x)\,dx=-\frac{\varphi_0}{pq}.$$

Many applications, such as the theory of service networks with complex routing, require that the solvability conditions of some infinite differential equation systems must be find out [25]. In the hyperbolic case, the general formulation of the mixed problem for a countable system takes the form [17]

$$\frac{\partial u_i}{\partial t}+\lambda_i(x,t)\frac{\partial u_i}{\partial x}=\sum_{j=1}^{\infty}a_{ij}(x,t)u_j+f_i(x,t),\quad i\in\mathbb{N},$$
$$u_i(x,0)=g_i(x),\quad x\in[0,l],\quad i\in\mathbb{N},$$
$$u_i(0,t)=\sum_{i\in I^-}\alpha_{ij}(t)u_j(0,t)+h_i(t),\ t\in[0,T],\ i\in I^+,$$
$$u_i(l,t)=\sum_{i\in I^+}\beta_{ij}(t)u_j(l,t)+h_i(t),\ t\in[0,T],\ i\in I^-.$$

Here $I^+=\{2k-1\colon k\in\mathbb{N}\}$, and $I^-=\mathbb{N}\setminus I^+$.

The analysis of elastic oscillations of a piezoelectric converter [67] of thin flat ring form with impulse voltage $V(t)$ applied between the outer and inner radii is based on the study of solutions to the problem

$$\frac{\partial^2 u}{\partial t^2}=v^2\Big(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{1}{r}u\Big),\quad r_1<r<r_2,\ t>0,$$
$$u(r,0)=u'_t(r,0)=0,\quad u(r_2,t)=0,$$
$$h\int_{r_1}^{r_2}\Big(\frac{\partial u}{\partial r}-\frac{1}{r}u\Big)\,dr=V(t).$$

The colliding process of ropes having a uniform connection and moving at the same speed [11] leads to various boundary-value problems for following systems of equations

$$\frac{\partial^2 u_i}{\partial x\partial y}=f_i(x,y,u_1,u_2,\dots,u_n),\quad i=1,\dots,n.$$

Small vibrations of a beam, taking into account the information about displacements and inertia of rotation, are described by the system [23]

$$\frac{\partial^2 u}{\partial x^2} - \frac{k\rho}{G}\frac{\partial^2 u}{\partial t^2} = \frac{\partial v}{\partial x} + f(x,t), \quad \frac{\partial^2 v}{\partial x^2} - \frac{\rho}{E}\frac{\partial^2 v}{\partial t^2} = \frac{GF}{KEl}\Big(\frac{\partial u}{\partial x} - v\Big).$$

Here u are the transverse displacements of the beam points; v is the average rotation of the normal section; G, E are the elastic characteristics of the beam; ρ is the linear density; F is the cross-sectional area; K is the geometric characteristic of the system; l is the beam length.

The above examples indicate to the need for the theory of boundary-value problems of differential equations and systems of hyperbolic type with two independent variables such that it would cover both classical problems. For instance, Cauchy problem, mixed problem in rectangular domain, Goursat problem, and new more complex problems, in particular, problems with non-local (discrete or distributed) conditions, problems without initial conditions, problems with moving known or unknown boundaries, etc.

This paper proposes a unified approach to solving nonlocal boundary value problems for linear hyperbolic equations and systems of general form with one spatial variable. It is based on the experience of many scientists who have employed various methods in the study of hyperbolic equations and systems.

The method of characteristics is chosen as a basis of this approach. It allows to reduce the problems under consideration to the corresponding systems of Volterra-type integral equations. Their studying gives us the opportunity to obtain the theorems of existence, uniqueness, and continuous dependence of both classical solutions and different classes of generalized solutions to boundary value problems. Some of results presented here can be obtained by other methods. However, many of the problems considered in this paper are sufficiently comprehensive in the method of characteristics. Such a circumstance justifies the choice of this method as the main one.

In addition, we restrict ourselves to studying only piecewise smooth solutions. Although by means of the results mentioned above, one could significantly extend the solutions class to the problems under consideration by standard methods.

The review is restricted to linear problems, since in this case the theory is largely complete. Cases of other boundary-value problems with detailed literature review, in particular, for nonlinear hyperbolic equations and systems have been considered in [16,26,32,63].

2. Auxiliary statements. Let l_1 and l_2 be smooth curves located into the half-plane $t \geqslant 0$ of the plane xOt, $x = a_1(t)$ and $x = a_2(t)$ its equations respectively, and $a_1(0) = a_1$, $a_2(0) = a_2$, $a_1(t) < a_2(t)$, $t \in [0, T]$, $T > 0$. Denote $G_\tau = \{(x,t) \in \mathbb{R}^2 \colon a_1(t) < x < a_2(t),\ 0 < t \leqslant \tau\}$, where $\tau \in (0, T]$, and $G = G_T$. Let us consider the matrix differential equation

$$\sum_{i=0}^{n} A_i\Big(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)u = f(x,t) \tag{1}$$

of order $n \geqslant 1$ on G, where A_i is a linear homogeneous differential operator

$$A_i\Big(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)u \equiv \sum_{j=0}^{i} A_{ij}(x,t)\frac{\partial^i u}{\partial x^j \partial t^{i-j}}$$

of order i. The coefficients A_{ij} are square matrices of order $m \geqslant 1$ with elements $a_{ij}^{pq} \in C^1(\overline{G})$; u and f are columns of height m with components $u_1, \ldots, u_m$ and $f_1, \ldots, f_m$, respectively. It will be further assumed that $A_{n0}(x,t) \equiv E_m$, where E_m is the unit matrix of order m.

Equation (1) is strictly hyperbolic in G. That is, the *eigenvalue of matrix bundle*

$$A_n(x,t,1,\lambda) = \lambda^n + \sum_{j=1}^{n} A_{nj}\lambda^{n-j},$$

namely the roots λ of the *characteristic equation*

$$\det A_n(x,t,1,\lambda) \equiv |A_n(x,t,1,\lambda)| = 0,$$

are real and different for any $(x,t) \in \overline{G}$.

Let us denote these roots by $-\lambda_1(x,t), \dots, -\lambda_{mn}(x,t)$. Suppose a non-zero m-dimensional vector $h_i = h_i(x,t)$ satisfies $A_n(x,t,1,\lambda_i)h_i = 0$, then it said to be the *eigenvector of the matrix bundle* A_n corresponding to the eigenvalue λ_i for $i = 1, \dots, mn$.

All eigenvectors $h_1, \dots, h_{mn}$ and eigenvalues $\lambda_1, \dots, \lambda_{mn}$ can be partitioned into n non-intersecting groups of m elements such that the vectors of every group form a base in $\mathbb{R}^m$. Denote by $H_i = H_i(x,t)$ the square matrix of order m such that $h_{(i-1)m+1}, \dots, h_{im}$ are its columns and $\Lambda_i = \operatorname{diag}(\lambda_{(i-1)m+1}, \dots, \lambda_{im})$ the diagonal matrix of order m. Then $\det H_i(x,t) \neq 0$ and $\det(\Lambda_i - \Lambda_j) \neq 0$ for any $(x,t) \in \overline{G}$, where $i \neq j$.

Consider the following block matrices

$$A = A(x,t) = \begin{pmatrix} A_{n1}(x,t) & A_{n2}(x,t) & \cdots & A_{nn-1}(x,t) & A_{nn}(x,t) \\ -E_m & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -E_m & 0 \end{pmatrix}$$

and

$$P = P(x,t) = \begin{pmatrix} H_1\Lambda_1^{n-1} & \cdots & H_n\Lambda_n^{n-1} \\ \vdots & \ddots & \vdots \\ H_1\Lambda_1 & \cdots & H_n\Lambda_n \\ H_1 & \cdots & H_n \end{pmatrix} = (P_{ij})_{i,j=1}^n,$$

where $P_{ij} = H_j\Lambda_j^{n-i}$. Because $\det(A + \lambda E_{mn}) = \det A_n(x,t,1,\lambda)$, the values $-\lambda_i(x,t)$ are eigenvalues of the matrix A. Therefore, this matrix is similar to a diagonal one. It is easy to see that

$$\det P(x,t) = \pm \prod_{i=1}^{n} \det H_i(x,t) \prod_{1 \le j < i \le n} \det\big(\Lambda_i(x,t) - \Lambda_j(x,t)\big) \neq 0$$

for any $(x,t) \in \overline{G}$ and $P^{-1}AP = \operatorname{diag}(\Lambda_1, \dots, \Lambda_n)$.

Suppose $P^{-1} = (P^{ij})_{i,j=1}^n$, where $P^{ij} = P^{ij}(x,t)$ are square matrices of order m. Let us introduce the linear homogeneous matrix operators

$$M_i u \equiv M_i\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u \equiv \sum_{j=1}^{n} P^{ij}(x,t)\frac{\partial^{n-1}u}{\partial t^{n-j}\partial x^{j-1}}, \qquad i = 1, \dots, n. \tag{2}$$

It is easy to see that its form a base in the space of linear homogeneous matrix operators of order $n-1$, in particular

$$\frac{\partial^{n-1}u}{\partial t^{n-i}\partial x^{i-1}} \equiv \sum_{j=1}^{n} P_{ij}M_j\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u, \qquad i = 1, \dots, n. \tag{3}$$

From the above we immediately obtain the decomposition

$$A_n\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u \equiv \sum_{i=1}^{n} P_{1i}\Big(E_m\frac{\partial}{\partial t} + \Lambda_i\frac{\partial}{\partial x}\Big)M_i u + \bar{A}_{n-1}\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u. \tag{4}$$

Here $\bar{A}_{n-1}$ is a linear homogeneous differential operator of order $n-1$, defined in an obvious way, with coefficients depending on the operator A_n coefficients and the first derivatives of the matrics P block elements P_{ij}.

3. A priori estimates for solutions to general nonlocal mixed problems for systems with smooth coefficients. It is well known about the important role of a priori estimates for solutions in boundary-value theory of hyperbolic-type differential equations [12,41,68]. In the case of one spatial variable, the a priori estimates for solutions to general mixed problems of first-order hyperbolic systems by different methods have been obtained in [70–73,78]. The same for one high-order equation have been done in [10,44,60,61]. In this section, we mainly use the ideas of [60,78] to obtain a priori estimates for solutions to general mixed problems for system (1).

Suppose system (1) is given on G; $A_{nj}(x,t) \in C^1(\overline{G})$, $j = 1,\dots,n$; $A_{ij}(x,t)$, $f(x,t) \in C(\overline{G})$, $i = 0,\dots,n-1$, $j = 0,\dots,i$, and $a_1(t)$, $a_2(t) \in C^1[0,T]$.

Assume that the differences $w_i^1(t) \equiv \lambda_i(a_1(t),t) - a_1'(t)$ and $w_i^2(t) \equiv \lambda_i(a_2(t),t) - a_2'(t)$ have no zeros on $[0,T]$. Let us denote by I_1^+ (I_1^-) the set of indices i such that $w_i^1(t) > 0$ ($w_i^1(t) < 0$), and by I_2^+ (I_2^-) the set of indices i such that $w_i^2(t) > 0$ ($w_i^2(t) < 0$).

Suppose $F(x,t)$ is an arbitrary function defined on G and M is a boundary point of $\overline{G}$. Then a value of F at M means the limit value of $F(x,t)$ as (x,t) approaches M by arbitrarily way from G.

For (1), we impose the initial conditions

$$\frac{\partial^i u(x,0)}{\partial t^i} = g_i(x), \quad i = 0,1,\dots,n-1, \quad x \in [a_1,a_2] \tag{5}$$

and the boundary ones

$$\sum_{i=1}^{2}\sum_{j=0}^{n-1} B_{ijk}\Big(t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)u\Big|_{x=a_i(t)} = h_k(t), \quad k = 1,\dots,N, \quad t \in [0,T]. \tag{6}$$

Here $B_{ijk}(t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})$ is a linear homogeneous differential operator of order j such that its coefficients are continuous functions of the variable t; $g_i(x)$ and $h_k(t)$ are given functions; $N = \operatorname{card} I_1^+ + \operatorname{card} I_2^-$.

Suppose $g_i(x) \in C^{n-i-1}[a_1,a_2]$, $i = 0,1,\dots,n-1$, $h_k(t) \in C[0,T]$, $k = 1,\dots,N$. Equations (6) show that the operator B_{ijk} must be a string of length m

$$B_{ijk}\Big(t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big) = \Big(B_{ijk}^1\Big(t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big), \dots, B_{ijk}^m\Big(t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)\Big)$$

with elements being scalar homogeneous differential operators

$$B_{ijk}^p\Big(t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big) = \sum_{q=0}^{j} B_{ijk}^{pq}(t)\frac{\partial^j}{\partial x^q \partial t^{j-q}}, \quad p = 1,\dots,m,$$

of order j.

By $P_{ij}^{rs}(x,t)$, $r,s = 1,\dots,m$, denote the elements of the matrix $P_{ij}(x,t)$. Put

$$\alpha_{k,(j-1)m+s}^1(t) = \sum_{r=1}^{m}\sum_{q=0}^{n-1} B_{1,n-1,k}^{rq}(t)P_{q+1,j}^{rs}(a_1(t),t), \quad k = 1,\dots,N, \quad (j-1)m+s \in I_1^+;$$

$$\alpha^2_{k,(j-1)m+s}(t) = \sum_{r=1}^{m}\sum_{q=0}^{n-1} B^{rq}_{2,n-1,k}(t) P^{rs}_{q+1,j}(a_2(t),t), \quad k=1,\dots,N, \quad (j-1)m+s \in I_2^-.$$

Let ν and μ be the cardinalities of the sets I_1^+ and I_2^-, respectively. Suppose $\alpha_1(t)$ is the square matrix of order N such that its first ν columns are formed from $\alpha^1_{k,(j-1)m+s}(t)$, and the rest μ ones are formed from $\alpha^2_{k,(j-1)m+s}(t)$. The index k corresponds to the line number. In every column the index k runs through the values $1,\dots,N$ in the same order. We assume that the condition

$$|\alpha_1(t)| \equiv \det \alpha_1(t) \neq 0, \quad \text{for all } t \in [0,T] \tag{7}$$

holds. It is an analogue of the well-known Lopatinsky condition for the case of problem (1), (5), (6). In what follows, we will call (7) the solvability condition of the problem. If $\nu+\mu=0$, then boundary conditions (6) are missing and the condition (7) is not required. In this case, we have the Cauchy problem for system (1) with initial conditions (5).

Let us denote

$$\begin{gathered}\|u\|^2_{r,G} = \iint\limits_G \sum_{i=1}^{m}\sum_{j=0}^{r}\sum_{k=0}^{j}\Big(\frac{\partial^j u_i}{\partial t^{j-k}\partial x^k}\Big)^2 dxdt; \quad \|f\|^2 = \iint\limits_G \sum_{i=1}^{m} f_i^2 dxdt; \\ \|g\|^2 = \int_{a_1}^{a_2}\sum_{i=0}^{n-1}\sum_{j=1}^{m}\Big(\frac{d^{n-i-1}g_{ij}}{dx^{n-i-1}}\Big)^2 dx; \quad \|h\|^2 = \int_0^T \sum_{k=0}^{N} h_k^2 dt,\end{gathered} \tag{8}$$

where $g_{i1},\dots,g_{im}$ are the elements of the column g_i.

Theorem 1. *Under all the above conditions, for any $C^{n-1}(G)$-solution $u=u(x,t)$ to problem* (1), (5), (6) *the estimate*

$$\|u\|^2_{n-1,G} \leqslant C(\|f\|^2 + \|g\|^2 + \|h\|^2) \tag{9}$$

holds. A constant $C>0$ depends only on the coefficients of system (1), *on the ones of boundary conditions* (6), *and on the domain G measure.*

Proof. Put

$$V_i(x,t) \equiv M_i\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u, \quad i=1,\dots,n. \tag{10}$$

Let $v_{(i-1)m+1},\dots,v_{im}$ be components of the vector V_i. Then, by (3), (4), system (1) can be written in one of n equivalent forms

$$\frac{\partial V_i}{\partial t} + \Lambda_i \frac{\partial V_i}{\partial x} + \sum_{j=1}^{n} a_{ij}V_j = R_i\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u + P^{i1}f, \quad i=1,\dots,n, \tag{11}$$

where $a_{ij}(x,t)$ are known square matrices of order m; R_j are linear combinations of matrix coefficients at derivatives up to order $n-2$ inclusive; the elements of this matrices are continuous on $\overline{G}$.

Initial conditions (5) are reduced to

$$V_i(x,0) = \varphi_i(x) \equiv \sum_{j=1}^{n} P^{ij}(x,0)\frac{d^{j-1}g_{n-j}(x)}{dx^{j-1}}, \qquad i=1,\dots,n. \tag{12}$$

Due to (2), (3), (10), boundary conditions (6) can be written as

$$\sum_{i=1}^{2}\sum_{j=1}^{n}\sum_{q=0}^{n-1}\sum_{r,s=1}^{m} B_{i,n-1,k}^{rq}(t)P_{q+1,j}^{rs}(a_i(t),t)v_{(j-1)m+s}(a_i(t),t) = \\ = h_k(t) - \sum_{i=1}^{2}\sum_{j=0}^{n-2} B_{ijk}\Big(t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u\Big|_{x=a_i(t)}, \quad k=1,\dots,N.$$

Suppose that (7) holds, then the above equalities can be written equivalently as

$$\begin{gathered} v_i(a_1(t),t) = \sum_{j\in I_1^-}\sigma_{ij}^1(t)v_j(a_1(t),t) + \sum_{j\in I_2^+}\sigma_{ij}^2(t)v_j(a_2(t),t) + \\ + \sum_{k=1}^{N}\sigma_{ik}(t)\Big\{h_k(t) - \sum_{r=1}^{2}\sum_{j=0}^{n-2}B_{ijk}\Big(t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u\Big|_{x=a_r(t)}\Big\}, \quad i\in I_1^+, \\ v_i(a_2(t),t) = \sum_{j\in I_1^-}\rho_{ij}^1(t)v_j(a_1(t),t) + \sum_{j\in I_2^+}\rho_{ij}^2(t)v_j(a_2(t),t) + \\ + \sum_{k=1}^{N}\rho_{ik}(t)\Big\{h_k(t) - \sum_{r=1}^{2}\sum_{j=0}^{n-2}B_{ijk}\Big(t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u\Big|_{x=a_r(t)}\Big\}, \quad i\in I_2^-, \end{gathered} \tag{13}$$

where $\sigma_{ij}^1(t)$, $\sigma_{ik}^2(t)$, $\sigma_{ij}(t)$ $\rho_{ij}^1(t)$, $\rho_{ij}^2(t)$, $\rho_{ij}(t)$ are known continuous functions on $[0,T]$.

Let $\gamma > 0$ be some constant and $Q_1(x,t),\dots,Q_{nm}(x,t)$ positive continuously differentiable functions on $\overline{G}$. We denote $V_i^* = \operatorname{diag}(v_{(i-1)m+1},\dots,v_{im})$, $Q_i^* = (Q_{(i-1)m+1},\dots,Q_{im})$. Multiply left-hand side of (11) by $2e^{-\gamma t}Q_i^*(x,t)V_i^*(x,t)$. Then sum up obtained equalities over i from 1 to n and integrate the result over subset $G_\tau \subset G$, where $0 < \tau \le T$. After simple transformations, we get

$$\iint\limits_{G_\tau} e^{-\gamma t}F(v)\,dxdt + \int\limits_0^\tau e^{-\gamma t}S(v)\,dt = N_1 + N_2 + N_3 - N_4, \tag{14}$$

where

$$N_1 = \iint\limits_{G_\tau} 2e^{-\gamma t}\sum_{i=1}^{n} Q_i^*(x,t)V_i^*(x,t)R_i\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u\,dxdt,$$

$$N_2 = \iint\limits_{G_\tau} 2e^{-\gamma t}\sum_{i=1}^{n} Q_i(x,t)V_i^*(x,t)P^{i1}(x,t)f(x,t)\,dxdt,$$

$$N_3 = \int\limits_{a_1}^{a_2}\sum_{i=1}^{nm} Q_i(x,0)v_i^2(x,0)dx, \qquad N_4 = \int\limits_{a_1(\tau)}^{a_2(\tau)} e^{-\gamma\tau}\sum_{i=1}^{nm} Q_i(x,\tau)v_i^2(x,\tau)dx,$$

$$F(v) = \gamma\sum_{i=1}^{nm} Q_i v_i^2 + 2\sum_{i,j=1}^{n} Q_i^*V_i^*a_{ij}V_j - \sum_{i=1}^{n}\Big(\frac{\partial Q_i^*}{\partial t}V_i^*V_i + \frac{\partial Q_i^*}{\partial x}V_i^*\Lambda_iV_i + \\ + Q_i^*V_i^*\frac{\partial \Lambda_i}{\partial x}V_i\Big) \equiv \gamma\sum_{i=1}^{nm} Q_i v_i^2 + \sum_{i,j=1}^{nm} b_{ij}v_iv_j;$$

$$S(v)=\sum_{i=1}^{nm}\bigl(Q_i(a_2(t),t)w_i^2(t)v_i^2(a_2(t),t)-Q_i(a_1(t),t)w_i^1(t)v_i^2(a_1(t),t)\bigr)\equiv S_2(v)-S_1(v).$$

Obviously, $b_{ij}(x,t)$ are continuous functions on $\overline{G}$.

Given a set of positive continuous functions $Q_i(x,t)$ on $\overline{G}$, we choose a number γ so large that the quadratic form F is positive-definite one

$$F(v)\geqslant C_1\sum_{i=1}^{nm}v_i^2(x,t),\quad C_1>0.$$

Let us write $S_2(v)=S_2^+(v)+S_2^-(v)$, where addends of the sum S_2 are indexed by $i\in I_2^+$, and ones of S_2^- does so by $i\in I_2^-$. Analogously, $S_1(v)=S_1^+(v)+S_1^-(v)$, where S_1^+ and S_1^- are corresponded to the indices $i\in I_1^+$ and $i\in I_1^-$, respectively.

Since $w_i^2(t)>0$, $i\in I_2^+$ and $w_i^1(t)<0$, $i\in I_1^-$, for $t\in[0,T]$, then for arbitrary strictly positive $Q_i(a_1(t),t)$, $i\in I_1^-$, and $Q_i(a_2(t),t)$, $i\in I_2^+$, the difference $S_2^+(v)-S_1^-(v)$ is a positive-definite quadratic form of $v_i(a_1(t),t)$, $i\in I_1^-$, and $v_i(a_2(t),t)$, $i\in I_2^+$. In the sums S_1^+ and S_2^-, let us replace $v_i(a_1(t),t)$, $i\in I_1^+$, and $v_i(a_2(t),t)$, $i\in I_2^-$ by their expressions (13).

Let M, m_1, m_2 be positive integers. Consider as $Q_i(x,t)$, $i=1,\dots,mn$, arbitrary smooth functions on $\overline{G}$ satisfying the conditions

$$\begin{aligned}\bigl|Q_i(a_1(t),t)w_i^1(t)\bigr|&\geqslant M,\quad i\in I_1^-;\\ \bigl|Q_i(a_2(t),t)w_i^2(t)\bigr|&\geqslant M,\quad i\in I_2^+;\\ m_1\leqslant\bigl|Q_i(a_1(t),t)w_i^1(t)\bigr|&\leqslant m_2,\quad i\in I_1^+;\\ m_1\leqslant\bigl|Q_i(a_2(t),t)w_i^2(t)\bigr|&\leqslant m_2,\quad i\in I_2^-.\end{aligned}$$

We take M so large, and m_1, m_2 so small that the following inequality

$$S(v)\geqslant C_2\Bigl(\sum_{i\in I_1^-}v_i^2(a_1(t),t)+\sum_{i\in I_2^+}v_i^2(a_2(t),t)\Bigr)-\\ -C_3\Bigl(\sum_{q=1}^{m}\sum_{r=1}^{2}\sum_{j=0}^{n-2}\sum_{s=0}^{j}\Bigl(\frac{\partial^j u_q}{\partial t^{j-s}\partial x^s}\Bigr)^2\Big|_{x=a_r(t)}+\|h\|^2\Bigr)$$

holds, where C_2, C_3 are some positive constants.

It is readily seen that

$$N_4\geqslant m_1e^{-\gamma\tau}\int_{a_1(\tau)}^{a_2(\tau)}\sum_{i=1}^{nm}v_i^2(x,\tau)dx,\quad |N_1+N_2+N_3|\leqslant C_4\bigl(\|u\|^2_{n-1,G_\tau}+\|f\|^2+\|g\|^2\bigr),\ C_4>0,$$

in (14). Also, it is easy to deduce that

$$\int_0^\tau\sum_{q=1}^{m}\sum_{r=1}^{2}\sum_{j=0}^{n-2}\sum_{s=0}^{j}\Bigl(\frac{\partial^j u_q}{\partial t^{j-s}\partial x^s}\Bigr)^2\Big|_{x=a_r(t)}dt\leqslant C_5\bigl(\|u\|^2_{n-1,G_\tau}+\|g\|^2\bigr),\quad C_5>0.$$

Then, from what has been said and (14), we immediately obtain

$$\int_{a_1(\tau)}^{a_2(\tau)}\sum_{i=1}^{nm}v_i^2(x,\tau)\,dx\leqslant C_6\bigl(\|u\|^2_{n-1,G_\tau}+\|f\|^2+\|g\|^2+\|h\|^2\bigr),\quad C_6>0,$$

and as a result

$$\int\limits_{a_1(\tau)}^{a_2(\tau)} \sum_{q=1}^{m}\sum_{j=0}^{n-1}\sum_{k=0}^{j}\Big(\frac{\partial^j u_q(x,\tau)}{\partial t^{j-k}\partial x^k}\Big)^2\,dx \leqslant C_7\Bigg(\int\limits_{a_1(\tau)}^{a_2(\tau)}\sum_{i=1}^{nm} v_i^2(x,\tau)\,dx + \|u\|^2_{n-1,G_\tau} + \|g\|^2\Bigg) \leqslant$$

$$\leqslant C_8\big(\|u\|^2_{n-1,G_\tau} + \|f\|^2 + \|g\|^2 + \|h\|^2\big),$$

where a constant $C_8 > 0$ does not depend on u. Using Gronwall's lemma, we get (9). □

4. The case of a system with piecewise smooth coefficients. The aim of this section is to establish an a priori estimate, analogous to (9), for the solution to a general mixed problem for a hyperbolic system with discontinuous coefficients. Many questions arising by consideration of such problems are described in [4].

Let $R \geqslant 1$ be a given positive integer. Let l_r, $r \in \{1,\dots,R+1\}$, be a curve defined by the equation $x = a_r(t)$, $t \in [0,T]$. Suppose $a_r(t) \in C^1[0,T]$. We will assume that $a_r(t) < a_{r+1}(t)$, $r = 1,\dots,R$, for all $t \in [0,T]$. Denote

$$G^r = \big\{(x,t) \in \mathbb{R}^2\colon 0 < t \leqslant T,\ a_r(t) < x < a_{r+1}(t)\big\},\quad G = \bigcup_{r=1}^{R} G^r.$$

On every G^r, let us consider a strictly hyperbolic system

$$\sum_{i=0}^{n} A_i^r\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u^r = f^r(x,t),\quad r = 1,\dots,R \tag{15}$$

of the form (1), where A_i^r is the linear homogeneous matrix differential operator

$$A_i^r\Big(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u^r = \sum_{i=0}^{i} A_{ij}^r(x,t)\frac{\partial^i u^r}{\partial x^j\partial t^{i-j}}$$

of order i. Its coefficients A_{ij}^r are square matrices of order m; u^r and f^r are columns of height m with components $u_1^r,\dots,u_m^r$ and $f_1^r,\dots,f_m^r$, respectively. Put $A_{n0}^r(x,t) \equiv E_m$, $r = 1,\dots,R$, just like before. Suppose $A_{nj}^r \in C^1(\overline{G^r})$, $j = 1,\dots,n$; $A_{ij}^r(x,t)$, $f^r(x,t) \in C(\overline{G^r})$, $i = 0,\dots,n-1$, $j = 0,1,\dots,i$, $r = 1,\dots,R$.

For every $r \in \{1,\dots,R\}$, denote by $-\lambda_1^r(x,t),\dots,-\lambda_{nm}^r(x,t)$ the λ-roots of the characteristic equation $\det A_n^r(x,t,1,\lambda) = 0$. Suppose the differences $w_{ir}^1 \equiv \lambda_i(a_r(t),t) - a_r'(t)$ and $w_{ir}^2 \equiv \lambda_i(a_{r+1}(t),t) - a_{r+1}'(t)$ do not have zeros on $[0,T]$. Let I_{1r}^+ (resp. I_{1r}^-) denote the set of those indices i for which $w_{ir}^1(t) > 0$ (resp. $w_{ir}^1(t) < 0$). Likewise, I_{2r}^+ (resp. I_{2r}^-) does so for $w_i^{2r}(t) > 0$ (resp. $w_i^{2r}(t) < 0$).

For (15), we consider the initial conditions

$$\frac{\partial^i u^r(x,0)}{\partial t^i} = g_i^r(x),\qquad i = 0,1,\dots,n-1,\quad r = 1,\dots,R,\quad x \in [a_r(0),a_{r+1}(0)], \tag{16}$$

together with following boundary conditions on l_1 and l_{R+1} and conjugation conditions on $l_2,\dots,l_R$

$$\sum_{r=1}^{R}\sum_{i=r}^{r+1}\sum_{j=0}^{n-1} B_{ijk}^r\Big(t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)u^r\Big|_{x=a_i(t)} = h_k(t),\quad k = 1,\dots,N_1,\quad t \in [0,T]. \tag{17}$$

Here $N_1 = \sum\limits_{r=1}^{R} (\operatorname{card} I_{1r}^{+} + \operatorname{card} I_{2r}^{-})$; B_{ijk}^{r} are the operators

$$B_{ijk}^{r} = \big(B_{ijk}^{r1}, \dots, B_{ijk}^{rm}\big), \qquad B_{ijk}^{rp} \equiv \sum_{q=0}^{j} B_{ijk}^{rpq}(t) \frac{\partial^{j}}{\partial x^{q} \partial t^{j-q}}.$$

Their structure, for every r, is the same as for the operators B_{ijk} in (6).

Suppose $g_i^r \in C^{n-i-1}[a_r(0), a_{r+1}(0)]$, and the coefficients and free terms of (17) are continuous on $[0,T]$. Evidently, at $R = 1$, the formulated problem is the one from the previous section.

For every fixed $r = 1, \dots, R$, we introduce the matrices $A^r(x,t)$ and $P^r(x,t)$ as well as A and P in Section 2. Let P_{ij}^{r}, $i,j = 1,\dots,n$, be the block elements of the matrix P^r, and P_{ij}^{rps}, $p,s = 1,\dots,m$, the elements of P_{ij}^{r}. Put

$$\alpha_{k,(j-1)m+s}^{r1}(t) = \sum_{p=1}^{m} \sum_{q=0}^{n-1} B_{r,n-1,k}^{rpq}(t) P_{q+1,j}^{rps}(a_r(t),t),, \qquad (j-1)m+s \in I_{1r}^{+},$$
$$\alpha_{k,(j-1)m+s}^{r2}(t) = \sum_{p=1}^{m} \sum_{q=0}^{n-1} B_{r+1,n-1,k}^{rpq}(t) P_{q+1,j}^{rps}(a_{r+1}(t),t), \qquad (j-1)m+s \in I_{2r}^{-}$$

for all $k = 1, \dots, N_1$ and $r = 1, \dots, R$.

Let $\alpha_2(t)$ be a square matrix of order N_1 such that $\alpha_{kp}^{rq}(t)$, introduced above, are its elements. Here $p = (j-1)m + s \in I_{1r}^{+}$ for $q = 1$ but $p(j-1)m + s \in I_{2r}^{-}$ for $q = 2$ and $j = 1, \dots, n$, $s = 1, \dots, m$. For every fixed triple p, r, q, the elements $\alpha_{kp}^{rq}(t)$, $k = 1, \dots, N_1$, present a column of the matrix.

Suppose the condition

$$\det \alpha_2(t) \neq 0, \quad \text{for all } t \in [0,T], \tag{18}$$

similar to that of Lopatinsky is satisfied. It is also called the solvability conditions of problem (15)–(17).

We denote by $\|u^r\|_{n,G^r}^2$, $\|f^r\|_{G^r}^2$ the corresponding norms defined by (8) for $G = G^r$.

Theorem 2. *Under the assumptions of the current section, for any $C^{n-1}(G)$-solution of problem (15)–(17) the estimate*

$$\sum_{r=1}^{R} \|u^r\|_{n-1,G^r}^2 \leqslant C \sum_{r=1}^{R} \big(\|f^r\|_{G^r}^2 + \|g^r\|^2\big) + C\,\|h\|^2 \tag{19}$$

holds. Here a constant $C > 0$ does not depend on the solution.

Proof. The theorem can be proved in the same scheme as can Theorem 1, so let us restrict ourselves only to general remarks.

On every G^r, we consider the operators M_i^r and the vector functions V_i^r with components $v_{(i-1)m+1}^r, \dots, v_{im}^r$, $i = 1, \dots, n$, analogous to ones in (2) and (10). For functions V_i^r we obtain a first-order system (11), assuming that all coefficients in (11) are marked by the upper index r. Respectively, initial conditions (16) induce the ones for V_i^r. From (17) due to (18), we can expressed $v_i^r(a_r(t),t)$, $i \in I_{1r}^{+}$, $r = 1, \dots, R$ and $v_i^r(a_{r+1}(t),t)$, $i \in I_{2r}^{-}$, $r = 2, \dots, R+1$, in terms of $h_k(t)$, $k = 1, \dots, N_1$, the values of u^r and of their derivatives up to order $n-2$ inclusive on the lines l_r, $r = 1, \dots, R+1$, and the values of the rest function v_i^r on the ones.

Thus, the equalities of form (13) are obtained. Suppose γ is sufficiently large and the same for all G^r, and the functions $Q_1(x,t), \dots, Q_n(x,t)$ are chosen, on each component G^r separately, as in previous section. They are piecewise smooth ones on G. Thus we have an equality of form (14). Finally, acting as in Section 3 with obvious changes, we get an inequality of type (19). □

Remark 1. Suppose the initial data of the problems considered in Sections 2–4 are smoother; then one can get estimates for the norms $\|\cdot\|_{N,G}$ of solutions, where $N > n-1$.

Remark 2. By slightly changing the proofs of Theorems 1 and 2, one can obtain, under the appropriate conditions for the initial data, a priori estimates for L_p-norms of solutions and of their derivatives up to order of not less than $n-1$.

5. Correct solvability of a nonlocal problem with nonseparated and integral initial data. Various methods have been used to prove the correct solvability of mixed problems for two-dimensional hyperbolic equations and systems with smooth coefficients. Depending on the proof technique, there are three basic ones ([16]). The first is the approximation of smooth functions by analytic ones; the second is the using of the finite difference method (both covering the multidimensionality of spatial variables); and the third is the characteristics method (specific in the case of one spatial variable). Various problems for a second order equation have been studied in [14, 27, 34–37, 39, 47]. Mixed problems for first-order systems have been considered in [1, 8, 18, 20, 21, 28, 33, 40–43, 56, 62, 69–75, 77, 79]. Cases of an single equation or high order system of equations have been investigated in [2, 7, 10, 22, 24, 44, 45, 47, 49, 51, 60, 61, 70, 71, 76]. In [54, 57, 58] the simplest mixed problems for equations and systems with boundary conditions involving integrals, over the spatial variable, of the desired solution have been studied. Various models of problems with nonlocal conditions, such as nonseparated and integral ones, have also been studied in [26, 32, 63].

This section deals with conditions for correct solvability of a general problem, with integral constraints, for system (1) defined in Section 2 on G.

We find a solution of (1) on G satisfying the initial conditions (5) and constraints

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{p=1}^{m}\left(\sum_{q=1}^{2} B_{qik}^{pj}(t)\frac{\partial^i u_p}{\partial t^{i-j}\partial x^j}\bigg|_{x=a_q(t)} + \right.$$
$$\left. + \int_{a_1(t)}^{a_2(t)} C_{ik}^{pj}(\xi,t)\frac{\partial^i u_p(\xi,t)}{\partial t^{i-j}\partial \xi^j}\,d\xi\right) = h_k(t), \qquad k = 1,\dots,N_0, \quad t\in[0,T]; \tag{20}$$
$$\sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{p=1}^{m}\int_{a_1(t)}^{a_2(t)} C_{ik}^{pj}(\xi,t)\frac{\partial^i u_p(\xi,t)}{\partial t^{i-j}\partial \xi^j}\,d\xi = h_k(t), \qquad k = N_0+1,\dots,N,$$

where B_{qik}^{pj}, C_{ik}^{pj}, h_k are known functions, $0 \leqslant N_0 \leqslant N$. In extreme cases $N_0 = 0$ and $N_0 = N$ the corresponding conditions are absent.

Suppose initial conditions (5) and condition (20) are agreed at $(a_1, 0)$ and $(a_2, 0)$. Namely

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{p=1}^{m}\left(\sum_{q=1}^{2} B_{qik}^{pj}(0) g_{i-j,p}^{(j)}(a_q) + \int_{a_1}^{a_2} C_{ik}^{pj}(\xi, 0) g_{i-j,p}^{(j)}(\xi)\, d\xi\right) = h_k(0), \quad k = 1, \ldots, N_0.$$
$$\sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{p=1}^{m}\int_{a_1(t)}^{a_2(t)} C_{ik}^{pj}(\xi, 0) g_{i-j,p}^{(j)}(\xi) d\xi = h_k(0), \quad k = N_0 + 1, \ldots, N. \tag{21}$$

Let us construct matrices $\beta_1(t)$, $\beta_2(t)$, $\beta_3(t)$, $\beta_4(t)$ as follows. The matrices $\beta_1(t)$ and $\beta_2(t)$, likewise to ones in Section 4, consist of elements $\alpha_{k,(j-1)m+s}^1(t)$ and $\alpha_{k,(j-1)m+s}^2(t)$, respectively, where $k = 1, \ldots, N_0$. The matrices $\beta_3(t)$ and $\beta_4(t)$ are composed in the same way, but they consist of elements

$$\alpha_{k,(j-1)m+s}^3(t) = \sum_{r=1}^{m}\sum_{q=0}^{n-1} C_{n-1,k}^{rq}(a_1(t), t) P_{q+1,j}^{rs}(a_1(t), t) w_{(j-1)m+s}^1(a_1(t), t),$$

where $k = N_0 + 1, \ldots, N$, $(j-1)m + s \in I_1^+$, and

$$\alpha_{k,(j-1)m+s}^4(t) = -\sum_{r=1}^{m}\sum_{q=0}^{n-1} C_{n-1,k}^{rq}(a_2(t), t) P_{q+1,j}^{rs}(a_2(t), t) w_{(j-1)m+s}^2(a_2(t), t),$$

where $k = N_0 + 1, \ldots, N$, $(j-1)m + s \in I_2^-$.

Note for the pairs of matrices $\beta_1(t)$, $\beta_2(t)$ and $\beta_3(t)$, $\beta_4(t)$, the index k runs in the same order over the sets $1, \ldots, N_0$ and $N_0 + 1, \ldots, N$, respectively. The same being true for the pairs $\beta_1(t)$, $\beta_2(t)$ and $\beta_3(t)$, $\beta_4(t)$, but for index $(j-1)m + s$ running over I_1^+ and I_2^-.

Therefore the square matrix $\beta(t) = \begin{pmatrix} \beta_1(t) & \beta_2(t) \\ \beta_3(t) & \beta_4(t) \end{pmatrix}$ has order N.

Let the solvability condition

$$\det \beta(t) \neq 0 \quad \text{for all } t \in [0, T] \tag{22}$$

be satisfied. It is analogous, for given problem, to the Lopatinsky condition.

Theorem 3. *Suppose the following conditions hold:*

1) *in* (1), *the coefficients of the operator A_n are continuously differentiable; the coefficients rest of A_i, $i < n$, and the free term f are continuous on $\overline{G}$;*
2) *the initial functions g_i are continuously differentiable $n - i - 1$ times on $[a_1, a_2]$;*
3) *for $k = 1, \ldots, N_0$, the coefficients and free terms in* (20) *are continuous on $\overline{G}$ and $[0, T]$ respectively;*
4) *for $k = N_0 + 1, \ldots, N$, the coefficients $C_{n-1,k}^{pj}(x, t)$ are continuously differentiable in x and t, and the coefficients $C_{ik}^{pj}(x, t)$, $i < n - 1$, are continuous and continuously differentiable in t on $\overline{G}$;*
5) *the functions $a_1(t)$, $a_2(t)$, $h_k(t)$, $k = N_0 + 1, \ldots, N$, are continuously differentiable on $[0, T]$;*

6) *agreement conditions* (21) *and solvability conditions* (22) *are satisfied.*

Then there exists a unique solution of class $C^{n-1}(\overline{G})$ *to problem* (1), (5), (20) *continuously depending on the right parts of* (1) *and* (20), *and initial functions* (5) *in the sense of metrics* (8).

The meaning of this solution will be specified below.

Proof. The proof is based on the method of characteristics and is to reduce the problem to an equivalent system of Volterra integral equations. This system is solved by the iteration method.

Introducing functions (10), let us write equation (1) in form (11), namely as

$$\frac{\partial v_i}{\partial t} + \lambda_i(x,t)\frac{\partial v_i}{\partial x} = F_i(x,t,u,v), \quad i = 1,\dots,nm. \tag{23}$$

Here F_i is a linear function with continuous coefficients on $\overline{G}$; the ones are depended on v_i, $i = 1,\dots,nm$, on u_j, $j = 1,\dots,m$, and on all derivatives of the functions u_j up to order $n-2$ inclusive; v is a vector with components $v_1,\dots,v_{nm}$. Taking into account (12), initial conditions (5) can be written as

$$v_i(x,0) = \psi_i(x), \quad i = 1,\dots,nm, \quad x \in [a_1,a_2]. \tag{24}$$

Let us consider the auxiliary functions

$$\begin{aligned} v_i(a_1(t),t) &= \nu_i^1(t), \quad i \in I_1^+, \\ v_i(a_2(t),t) &= \nu_i^2(t), \quad i \in I_2^-. \end{aligned} \tag{25}$$

Suppose $x = \varphi_i(t,\xi,\tau)$ is a solution of the characteristic equation $x'_t = \lambda_i(x,t)$ satisfying the initial condition $x(\tau) = \xi$, where $(\xi,\tau) \in \overline{G}$. Let $L_i(\xi,\tau)$ be the corresponding characteristic passing through (ξ,τ), let it be extended backwards in time to the intersection with the boundary of G, and $t_i(\xi,\tau)$ the smallest value of t for its points. Evidently, $0 \leqslant t_i(\xi,\tau) \leqslant \tau$. If $t_i(\xi,\tau) > 0$, then $\varphi_i(t_i(\xi,\tau),\xi,\tau)$ is equal to $a_1(t_i(\xi,\tau))$ or to $a_2(t_i(\xi,\tau))$.

Accordingly, the domain G is split into three parts

$$G_i^0 := \{t_i(\xi,\tau) \equiv 0\}, \quad \begin{aligned} G_i^1 &:= \{t_i(\xi,\tau) > 0, \varphi_i(t_i(\xi,\tau),\xi,\tau) \equiv a_1(t_i(\xi,\tau))\}, \\ G_i^2 &:= \{t_i(\xi,\tau) > 0, \varphi_i(t_i(\xi,\tau),\xi,\tau) \equiv a_2(t_i(\xi,\tau))\}. \end{aligned}$$

Any of the sets G_i^1 or G_i^2 may be empty.

Integrating (23) along characteristics [1, 16, 46], and taking into account (24) and (25), we obtain in the domain $\overline{G}$ the system of integro-differential relationships

$$v_i(x,t) = w_i(x,t) + \int\limits_{t_i(x,t)}^{t} F_i\big(\varphi_i(\tau,x,t),\tau,u\big(\varphi_i(\tau,x,t),\tau\big),v\big(\varphi_i(\tau,x,t),\tau\big)\big)\,d\tau, \tag{26}$$

where

$$w_i(x,t) = \begin{cases} \psi_i(\varphi_i(0,x,t)), & \text{if } (x,t) \in G_i^0, \\ \nu_i^1(t_i(x,t)), & \text{if } (x,t) \in G_i^1, \\ \nu_i^2(t_i(x,t)), & \text{if } (x,t) \in G_i^2 \end{cases} \qquad \text{for} \quad i = 1,\dots,nm.$$

For the purpose of the function $v_i(x,t)$ to be continuous when passing from G_i^0 to G_i^1 and to G_i^2, the fulfillment of the agreement conditions

$$\nu_i^1(0) = \psi_i(a_1), \quad i \in I_1^+; \qquad \nu_i^2(0) = \psi_i(a_2), \quad i \in I_2^- \tag{27}$$

is required. Conditions (20) can be written as

$$\begin{gathered}\sum_{j=1}^{n}\sum_{q=0}^{n-1}\sum_{r,s=1}^{m}\Bigg(\sum_{i=1}^{2} B_{i,n-1,k}^{rq}(t)P_{q+1,j}^{rs}\big(a_i(t),t\big)\nu_{(j-1)m+s}^{i}(t) + \\ + \int\limits_{a_1(t)}^{a_2(t)} C_{n-1,k}^{rq}(\xi,t)P_{q+1,j}^{rs}(\xi,t)v_{(j-1)m+s}(\xi,t)\,d\xi\Bigg) = H_k^1(t,u), \quad k=1,\dots,N_0; \\ \sum_{j=1}^{n}\sum_{q=0}^{n-1}\sum_{r,s=1}^{m}\int\limits_{a_1(t)}^{a_2(t)} C_{n-1,k}^{rq}(\xi,t)P_{q+1,j}^{rs}(\xi,t)v_{(j-1)m+s}(\xi,t)\,d\xi = H_k^2(t,u), \\ k=N_0+1,\dots,N,\end{gathered} \tag{28}$$

where H_k^1 and H_k^2 coincide with the free members $h_k(t)$ of (20) minus all the left side addends containing derivatives of u_p up to order $n-2$ inclusive.

Functions (25) must be such that for functions v_i determined from (26), conditions (28) are satisfied. For this to be the case, in (28) we replace v_i with right-hand sides of (26). In the one-fold integrals of the left-hand sides of the obtained equations, we do the change of variable ξ by $\tau = t_{(j-1)m+s}(\xi,t)$. Then we differentiate equalities corresponding to the indices $k=N_0+1,\dots,N$ with respect to t. As a result, from the expressions $\partial H_k^2(t,u)/\partial t$, $(n-1)$-th derivatives of the functions u_p are appeared. We replace them with v_i according to (3). Then we do so with regard to v_i and (26).

Finally, equalities (28) take the form

$$\begin{gathered}\sum_{(j-1)m+s\in I_1^+} \alpha_{k,(j-1)m+s}^{q}\nu_{(j-1)m+s}^{q}(t) + \sum_{(j-1)m+s\in I_2^-} \alpha_{k,(j-1)m+s}^{q}\nu_{(j-1)m+s}^{q}(t) = \\ = \int\limits_0^t \sum_{(j-1)m+s\in I_1^+} R_{k,(j-1)m+s}^{q1}(t,\tau)\nu_{(j-1)m+s}^{1}(\tau)\,d\tau + \\ + \int\limits_0^t \sum_{(j-1)m+s\in I_2^-} R_{k,(j-1)m+s}^{q2}(t,\tau)\nu_{(j-1)m+s}^{2}(\tau)\,d\tau + H_k^{q+2}(t,u,v),\end{gathered}$$

where $q=1$ for $k=1,\dots,N_0$, and $q=2$ for $k=N_0+1,\dots,N$.

Here $R_{kp}^{q1}(t,\tau)$, $R_{kp}^{q2}(t,\tau)$ are known continuous functions; $H_k^3(t,u,v)$ and $H_k^4(t,u,v)$ are expressions containing v_i, u_p and derivatives of u_p up to order $n-2$ inclusive.

Taking into account (22), these equalities can be solved with respect to ν_i^1 and ν_i^2. Let us write the result in the matrix form

$$\nu(t) = \int\limits_0^t R(t,\tau)\nu(\tau)\,d\tau + H(t,u,v).$$

Here $\nu(\tau)$ is a column of height N with the components ν_i^1 and ν_i^2, $R(t,\tau)$ is a known continuous kernel, $H(t,u,v)$ is a vector with components of the form $H_k^{q+2}(t,u,v)$.

Let $S(t,\tau)$ be resolvent of the kernel $R(t,\tau)$. Therefore the last equality can be written as

$$\nu(t) = H(t,u,v) + \int_0^t S(t,\tau)H(\tau,u,v)\,d\tau. \tag{29}$$

Suppose conditions (21) are held; then it is easy to show that the functions ν_i^1 and ν_i^2, defined by (29), satisfy condition (27) for arbitrary u, v.

Substituting ν_i^1 and ν_i^2 mentioned above in (26), we get

$$v_i(x,t) = W_i(x,t) + U_{i1}(x,t,u) + W_{i1}(x,t,u,v), \quad i = 1,\ldots,mn, \tag{30}$$

where W_i are known continuous functions; U_{i1} is a linear combination, with continuous coefficients, of the values of the functions u_p and their derivatives up to order $n-2$ on the lines l_1 and l_2; W_{i1} are Volterra type addends, with continuous kernels, containing v_i and all derivatives of u_p up to order $n-2$.

Let $L\colon x = a(t)$ be an arbitrary smooth curve connecting a point $(x,t) \in \overline{G}$ and the segment $[a_1,a_2]$ of the axis Ox, and completely lying in $\overline{G}$. Denote by $D_t f$ the full derivative of f along L with respect to t.

Replacing v_i in the right-hand sides of (30) with (10) and using (3), we rewrite (30) as

$$\frac{\partial^{n-1}u}{\partial t^{n-i-1}\partial x^i} = T_i^{n-1}(x,t) + R_i^{n-1}(x,t,u) + S_i^{n-1}(x,t,u), \tag{31}$$

where T_i^{n-1}, R_i^{n-1}, S_i^{n-1} are columns of height m with an obvious structure of their components.

It follows from (3) and (10) that

$$D_t\frac{\partial^{n-2}u}{\partial t^{n-i-2}\partial x^i} = \sum_{j=1}^{n}(P_{i+1,j} + a'(t)P_{i+2,j})V_j, \quad i = 0,1,\ldots,n-2.$$

Let us replace V_i in the right-hand sides with (10), then replace the derivatives of u of order $n-1$ with (31). Integrating the resulting equality along L with respect to t, we get

$$\frac{\partial^{n-2}u}{\partial t^{n-i-2}\partial x^i} = T_i^{n-2}(x,t) + R_i^{n-2}(x,t,u), \tag{32}$$

where T_i^{n-2} are known continuous functions, R_i^{n-2} are Volterra type addends.

After that, we consistently form expressions

$$\frac{\partial^{n-3}u}{\partial t^{n-i-3}\partial x^i}, \quad i = 0,1,\ldots,n-3, \quad \frac{\partial^{n-4}u}{\partial t^{n-i-4}\partial x^i}, \quad i = 0,1,\ldots,n-4, \text{ etc.}$$

Integrating the ones along L with respect to t, we get relationships, similar to (32), for derivatives of order $n-3$, $n-4,\ldots,$ 0. Finally, we substitute these derivatives up to order $n-2$ inclusive into (31), namely in R_i^{n-1}.

Then for u and its all derivatives up to order $n-1$ inclusive, we obtain a system of $(n+1)n/2$ vector Volterra-type integral equations of the form

$$\frac{\partial^{n-i}u}{\partial t^{n-i-j}\partial x^j} = T_j^{n-i}(x,t) + R_j^{n-i}(x,t,u), \qquad i = 1,\ldots,n, \;\; j = 0,1,\ldots,n-i, \tag{33}$$

where T_j^{n-i} are known functions, R_j^{n-i} are Volterra operators of u and of its all derivatives up to order $n-1$ inclusive.

Under the conditions of Theorem 3, it is easy to show that there exists a unique continuous solution of system (33) on $\overline{G}$, obtaining by the iteration method. We call it the generalized solution to problem (1), (5), (20) of class $C^{n-1}(\overline{G})$. The continuous dependence of the solution on the free member f and on the initial functions h_k also follows easily from (33). □

Remark 3. Increasing the smoothness of the initial data, one can similarly proof the existence and uniqueness of a classical solution to the problem.

Remark 4. If l_1 and l_2 (or one of them) are characteristics of (1), then the number of conditions (20) is decreased by two (or one) units.

6. Correct solvability of a nonlocal problem with piecewise smooth initial data. A number of papers are devoted to the problem of the correct solvability of mixed problems for hyperbolic equations and systems with discontinuous coefficients on a line. In particular, in [46] a common mixed problem for a first-order system with discontinuous data has been investigated by the method of characteristics. Mixed problems with discontinuous data for a second-order equation have been studied by various methods in [2, 9, 19, 48, 66]. The case of one arbitrary order equation has been considered in [48, 50]. Using the contour integral method, in [55, 64] a broad class of problems of special kind for higher order systems has been researched.

This section deals with the general case of a nonlocal mixed problem for a hyperbolic arbitrary order system with discontinuous data.

Under the conditions of Section 4, let us consider system (15) with initial conditions (16) and the conditions

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{p=1}^{m}\sum_{r=1}^{R}\left(\sum_{q=r}^{r+1} B_{qik}^{rpj}(t)\frac{\partial^i u_p^r}{\partial t^{i-j}\partial x^j}\bigg|_{x=a_q(t)} + \right.$$

$$\left. + \int\limits_{a_r(t)}^{a_{r+1}(t)} C_{ik}^{rpj}(\xi,t)\frac{\partial^i u_p^r(\xi,t)}{\partial t^{i-j}\partial \xi^j}\,d\xi\right) = h_k(t), \quad k=1,\dots,K; \tag{34}$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{p=1}^{m}\sum_{r=1}^{R}\int\limits_{a_r(t)}^{a_{r+1}(t)} C_{ik}^{rpj}(\xi,t)\frac{\partial^i u_p^r(\xi,t)}{\partial t^{i-j}\partial \xi^j}\,d\xi = h_k(t), \qquad k=K+1,\dots,N_1,$$

where B_{qik}^{rpj}, C_{ik}^{rpj}, $h_k(t)$ are given functions, $0 \leqslant K \leqslant N_1$. This conditions are stated instead of boundary ones on l_1 and l_{R+1}, and conjugation conditions on $l_2,\dots,l_R$.

It is assumed that the following agreement conditions of (16) and (34) at the points $(a_r(0),0)$, $r=1,\dots,R+1$, are satisfied

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{p=1}^{m}\sum_{r=1}^{R}\left(\sum_{q=r}^{r+1} B_{qik}^{rpj}(0)\frac{d^j g_{i-j,p}^r(x)}{dx^j}\bigg|_{x=a_q(0)} + \right.$$

$$\left. + \int\limits_{a_r(0)}^{a_{r+1}(0)} C_{ik}^{rpj}(\xi,0)\frac{d^j g_{i-j,p}^r(\xi)}{dx^j}\,d\xi\right) = h_k(0), \quad k=1,\dots,K. \tag{35}$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{p=1}^{m}\sum_{r=1}^{R}\int\limits_{a_r(0)}^{a_{r+1}(0)} C_{ik}^{rpj}(\xi,0)\frac{d^j g_{i-j,p}^r(\xi)}{dx^j}\,d\xi = h_k(0), \quad k=K+1,\dots,N_1.$$

Let $\alpha_{k,(j-1)m+s)}^{rp}(t)$ be the functions introduced in Section 5, where $k=1,\dots,K$. For fixed r, p, j, s, of these quantities we make up a column $\gamma_{k,(j-1)m+s)}^{rp}(t)$ of height K. Then for fixed $r=1,\dots,m$ and $p=1,2$, of these columns we do the matrices $\gamma^{r1}(t)$ and $\gamma^{r2}(t)$ with $(j-1)m+s\in I_{1q}^{+}$ and $(j-1)m+s\in I_{2q}^{-}$, respectively.

Let us consider the functions

$$\alpha_{k,(j-1)m+s}^{r3}(t)=\sum_{p=1}^{m}\sum_{q=0}^{n-1}C_{n-1,k}^{rpq}(a_r(t),t)P_{q+1,j}^{rps}(a_r(t),t)w_{(j-1)m+s}^{1}(a_r(t),t),$$
$$k=K+1,\dots,N_1,\qquad (j-1)m+s\in I_{1q}^{+},\qquad r=1,\dots,R;$$
$$\alpha_{k,(j-1)m+s}^{r4}(t)=-\sum_{p=1}^{m}\sum_{q=0}^{n-1}C_{n-1,k}^{rpq}(a_{r+1}(t),t)P_{q+1,j}^{rps}(a_{r+1}(t),t)w_{(j-1)m+s}^{2}(a_{r+1}(t),t),$$
$$k=K+1,\dots,N_1,\qquad (j-1)m+s\in I_{2q}^{-},\qquad r=1,\dots,R.$$

By the same way as $\gamma^{r1}(t)$ and $\gamma^{r2}(t)$, of these quantities we make up the matrices $\gamma^{r3}(t)$ and $\gamma^{r4}(t)$ of height N_1-K.

It is assumed that

$$\det\gamma(t)=\det\begin{pmatrix}\gamma^{11}(t) & \cdots & \gamma^{R1}(t) & \gamma^{12}(t) & \cdots & \gamma^{R2}(t)\\ \gamma^{13}(t) & \cdots & \gamma^{R3}(t) & \gamma^{14}(t) & \cdots & \gamma^{R4}(t)\end{pmatrix}\neq 0,\quad t\in[0,T]. \tag{36}$$

For the case under consideration, this condition is analogous to the Lopatinsky one. We will call it the solvability condition to problem (15), (16), (34).

Theorem 4. *Suppose the following conditions hold:*

1) *the coefficients of the operators A_n^r, $r=1,\dots,R$, are of class $C^1(\overline{G^r})$, and the coefficients the rest of operators and free members f^r of system* (15) *are of class $C(\overline{G^r})$;*
2) *the initial functions g_i^r are of class $C^{n-i-1}[a_r(0),a_{r+1}(0)]$, $i=0,1,\dots,n-1$, $r=1,\dots,R$;*
3) *for $k=1,\dots,K$, the coefficients C_{ik}^{rpj} are of class $C(\overline{G})$, and B_{qik}^{rpj} and $h_k(t)$ are of class $C[0,T]$; for $k=K+1,\dots,N$, the functions $C_{ik}^{rpj}(\xi,t)$ and $h_k(t)$ are of class $C^1(\overline{G^r})$ and $C^1[0,T]$, respectively;*
4) *agreement conditions* (35) *and solvability conditions* (36) *are satisfied.*

Then there exists a unique C^{n-1}-solution to problem (15), (16), (34) *on every G_r. It is continuously depended on $f^r(x,t)$, $h_k(t)$ and $g_i^r(t)$.*

Proof. The proof is similar in spirit to the one of Theorem 3, so let us restrict ourselves to some general instructions.

On every G^r, system (15) can be written as (23). Taking into account (36), the equality (34) can be solved with respect to $v_i^r(a_r(t),t)$, $i\in I_{1r}^{+}$ and $v_i^r(a_{r+1}(t),t)$, $i\in I_{2r}^{-}$, where $r=1,\dots,R$.

Then, for every $r = 1, \dots, R$, we repeat the reasoning of Section 5 and use the considerations of [46] to transform the obtained system of integro-differential equations into a system of Volterra-type integral equations. As a result, we come to a system of $(n+1)nR/2$ Volterra-type equations of the second kind with respect to $u_i^r(x,t)$ and their derivatives up to order $n-1$ inclusive. It is equivalent to the original problem. Solving this system by the method of successive approximations, we obtain the desired result. □

Remark 5. Let boundaries of $\overline{G}$ be unknown, that is, we are dealing with a hyperbolic Stefan problem [59]. In this case conditions for unknown boundaries should be added, for all $t \in [0,T]$, to the initial and boundary conditions of problem (1), (5), (20). For example,

$$\begin{gathered} a_l''(t) = \Phi_l\big(t, a_1(t), a_2(t), a_1'(t), a_2'(t), u(a_1(t),t), u(a_2(t),t))\big), \\ a_l(0) = a_l^0, \quad a_l'(0) = a_l^1, \quad l = 1,\, 2, \\ \max_{i\in I_l^-} \lambda_i\big(a_l(t),t)\big) < a_l'(t) < \max_{i\in I_l^+} \lambda_i\big(a_l(t),t)\big). \end{gathered} \tag{37}$$

Nonlinear problems (1), (5), (20), (37), and various problems with unknown boundaries for hyperbolic equations and systems have been investigated in [3,5,6,31,32].

Remark 6. For the reasonig of this section, similar to that of Sections 4 and 5, it is important that no pair of lines l_r have points in common. The intersection of some of these lines is much more difficult [5,6,29–32,52,53]. In $\overline{G}$, the lines of initial conditions can be degenerated to a point (the Darboux problems). Such problems have been investigated in [52,53]; and the case of the Darboux problem with unknown boundaries of $\overline{G}$ have been considered in [6,32].

7. About requirement of additional conditions. In each of the problems discussed above, it was assumed that some conditions, called the solvability conditions, are satisfied (conditions (7), (18), (22), (36)). And if the smoothness and agreement conditions of data are completely natural, then the occurrences of the solvability conditions may seem artificial at first glance. Let us show by examples that these conditions are significant.

By G denote the rectangle $\{(x,t) \in \mathbb{R}^2 \colon 0 < x < 1, \;\; 0 < t < 0,5\}$. Let us consider the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{38}$$

on G. We are seeking for its solution that the initial conditions

$$u(x,0) = 0, \quad u_t'(x,0) = 0, \quad 0 \leqslant x \leqslant 1 \tag{39}$$

and the conditions

$$\frac{\partial u}{\partial t}\Big|_{x=0} - \frac{\partial u}{\partial x}\Big|_{x=1} = h_1(t), \qquad \frac{\partial u}{\partial x}\Big|_{x=0} + \frac{\partial u}{\partial t}\Big|_{x=1} = h_2(t), \tag{40}$$

are satisfied. Here h_1 i h_2 are given continuous functions on $[0,1/2]$ such that $h_1(0) = h_2(0) = 0$.

All the assumptions of Theorem 1 hold, except for condition (7), since it is easy to verify that $\det \alpha_1(t) \equiv 0$ in this case.

It can be easily checked that any continuously differentiable solution of (38) on G satisfying initial conditions (39) is given by

$$u(x,t) = \begin{cases} f(t-x), & 0 \leqslant x \leqslant t, \\ 0, & t \leqslant x \leqslant 1-t, \quad t \in [0,1/2], \\ g(t+x-1), & 1-t \leqslant x \leqslant 1, \end{cases} \tag{41}$$

where $f(t)$ and $g(t)$ are arbitrary continuously differentiable functions on $[0,1/2]$ that satisfy the conditions $f(0)=f'(0)=g(0)=g'(0)=0$. By (40), we come to

$$f'(t)-g'(t)\equiv h_1(t), \quad -f'(t)+g'(t)\equiv h_2(t).$$

Therefore, problem (38)–(40) is solvable if and only if $h_2(t)=-h_1(t)$. If this condition is fulfilled, then

$$g(t)=f(t)-\int\limits_0^t h_1(\tau)\,d\tau, \tag{42}$$

and the function f remains arbitrary. Thus, the problem above is either unsolvable or has an infinite set of solutions. Hence, the estimate (9) is impossible.

Now let us seek the solution of (38) on G satisfying initial conditions (39) and the conditions

$$\begin{gathered}\frac{\partial u}{\partial t}\Big|_{x=0}-\frac{\partial u}{\partial x}\Big|_{x=1}=h_1(t),\\ \int\limits_0^1(1-2\xi)\frac{\partial u(\xi,t)}{\partial t}\,d\xi=h_2(t),\end{gathered} \tag{43}$$

where $h_1(t)$ and $h_2(t)$ are given continuous functions on $[0,1/2]$ such that $h_1(0)=h_2(0)=0$, and, in addition, the function h_2 is continuously differentiable.

We see that all the conditions of Theorem 3 are satisfied except for condition (22), since in the case under consideration $\det\beta(t)\equiv 0$.

By requiring that (41) again satisfy conditions (43), we conclude that problem (38), (39), (43) is solvable if and only if h_1 and h_2 are connected by

$$h_2(t)=\int\limits_0^t\big(1-2(t-\tau)\big)h_1(\tau)\,d\tau.$$

If this condition is satisfied, then the problem has an infinite set of solutions of the form (41), where f is arbitrary function, and g has the form (42).

It is easy to build similar examples for cases of failure of condition (19) or (36).

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