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ISOMORPHISMS OF ALGEBRAS OF SYMMETRIC FUNCTIONS ON SPACES ℓ_p

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The work is devoted to the study of algebras of entire symmetric functions on some Banach spaces of sequences. A function on a vector space is called symmetric with respect to some fixed group G of operators acting on this space, or G-symmetric, if it is invariant under the action of elements of the group G on its argument. For different vector spaces there exist some natural groups of symmetries. In the case of vector spaces of sequences the most natural are groups of operators permuting coordinates of sequences. Such groups of operators are generated by some groups of bijections on the set $\mathbb{N}$ of positive integers. The most commonly used for this purpose is the group $\mathcal{S}$ of all bijections on $\mathbb{N}$. We consider entire functions and polynomials that are symmetric with respect to the group of operators, generated by $\mathcal{S}$, on the complex Banach space $\ell_p(\mathbb{C}^n)$ of all absolutely summable in a power $p \in [1, +\infty)$ sequences of n-dimensional complex vectors. We construct some natural isomorphism between the space $\ell_p(\mathbb{C}^n)$ and its partial case – the classical Banach space ℓ_p. Also we construct the group of operators on ℓ_p that is consistent with the isomorphism and the above-mentioned group of operators on $\ell_p(\mathbb{C}^n)$. This group is generated by the subgroup of $\mathcal{S}$, elements of which permute elements of $\mathbb{N}$ "by blocks". We obtain the isomorphism between Fréchet algebras of complex-valued entire functions of bounded type on ℓ_p and $\ell_p(\mathbb{C}^n)$ that are symmetric with respect to the above-mentioned respective groups of operators. The respective subalgebras of continuous symmetric polynomials on these spaces are also isomorphic.

Introduction. The most general notion of symmetry of a function on a vector space is introduced in [1]. A function on a vector space is called symmetric with respect to some fixed group G of operators acting on this space, or G-symmetric, if it is invariant under the action of elements of the group G on its argument. In the case of Banach spaces with some symmetric structure it is natural to consider groups of operators that preserve this structure. For example, if a Banach space has a symmetric Schauder basis, then commonly used group is the group of operators permuting elements of this basis. One of such spaces is the Banach space ℓ_p of all absolutely summable in a power $p \in [1, +\infty)$ sequences of real or complex numbers. In [6, 7, 12] there were studied algebras of continuous polynomials that are symmetric with respect to the group of all permutations of coordinates of elements of ℓ_p. One of the key properties of such algebras is that they have countable algebraic bases (see definition below). Consequently, algebras of entire symmetric functions of bounded type

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on ℓ_p belong to the class of the so-called countably generated algebras (see [13, 14]). This property is significantly used in the description of the spectra of such algebras [4, 5].

In [9], it is constructed a countable algebraic basis of the algebra of symmetric continuous polynomials (see Theorem 3 below) on the complex Banach space $\ell_p(\mathbb{C}^n)$ of all absolutely summable in a power $p \in [1, +\infty)$ sequences of n-dimensional complex vectors. Consequently, the Fréchet algebra of entire symmetric functions of bounded type on this space is countably generated. We extend these results to functions on ℓ_p that satisfy weaker symmetry requirements than the classical ones. We construct some natural isomorphism between spaces ℓ_p and $\ell_p(\mathbb{C}^n)$. Although ℓ_p and $\ell_p(\mathbb{C}^n)$ are isomorphic, the properties of symmetric functions with respect to classical groups of symmetry on these spaces are different. For example, each algebraic basis of the algebra of all continuous symmetric polynomials on ℓ_p is the sequence of polynomials that cannot contain more than one m-homogeneous element for each integer $m \geq p$. Meanwhile, in the case $n \geq 2$ the algebraic basis of the algebra of all continuous symmetric polynomials on $\ell_p(\mathbb{C}^n)$ contains more than one m-homogeneous element for each integer $m \geq p$. We construct some special group of operators G_{Θ_n,ℓ_p} on ℓ_p, consistent with the classical group of symmetry on $\ell_p(\mathbb{C}^n)$ and with the above-mentioned isomorphism. This group is a proper subgroup of the classical group of all operators of permutation of coordinates of elements of ℓ_p. Elements of the group G_{Θ_n,ℓ_p} permute coordinates of elements of ℓ_p "by blocks" of the length n. We show that the Fréchet algebra of entire functions of bounded type on ℓ_p that are symmetric with respect to this group is isomorphic to the above-mentioned Fréchet algebra of entire symmetric functions of bounded type on $\ell_p(\mathbb{C}^n)$. The respective subalgebras of continuous symmetric polynomials on these spaces are also isomorphic. We construct an algebraic basis of the algebra of continuous G_{Θ_n,ℓ_p}-symmetric polynomials on ℓ_p as the image (with respect to the isomorphism of algebras) of the above-mentioned algebraic basis of the algebra of continuous symmetric polynomials on $\ell_p(\mathbb{C}^n)$.

Note that if we consider the sequence of groups G_{Θ_n,ℓ_p}, where instead of n we substitute powers of some integer $a \geq 2$, then each element of the sequence contains the next element as a proper subgroup. Consequently, the conditions of the symmetry with respect to the next element are weaker than the conditions of the symmetry with respect to the current one. Therefore such sequences of groups can be used in investigations of the so-called weakly symmetric functions on ℓ_p (see [3, 18]). Firstly, the notion of weak symmetry was defined in [18], Weak symmetric functions are applied to the approximation of non-symmetric functions by symmetric functions.

1. Preliminaries. Let $\mathbb{N}$ be the set of all positive integers. Let $\mathbb{Z}_+$ be the set of all nonnegative integers.

Symmetric mappings. Let A, B be arbitrary nonempty sets. Let S be an arbitrary fixed set of mappings that act from A to itself. A mapping $f\colon A \to B$ is called *S-symmetric* if $f(s(a)) = f(a)$ for every $a \in A$ and $s \in S$.

The algebra $H_b(X)$. Let X be a complex Banach space. Let $H_b(X)$ be the Fréchet algebra of all entire functions $f\colon X \to \mathbb{C}$, which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets.

Let
$$\|f\|_r = \sup_{\|x\| \leq r} |f(x)|$$
for $f \in H_b(X)$ and $r > 0$. The topology of $H_b(X)$ can be generated by an arbitrary set of norms $\{\|\cdot\|_r \colon r \in \Gamma\}$, where Γ is any unbounded subset of $(0, +\infty)$.

The algebras $H_{b,S}(X)$ and $\mathcal{P}_S(X)$. Let X be a complex Banach space. Let S be a set of operators on X. Let $H_{b,S}(X)$ be the subalgebra of all S-symmetric elements of $H_b(X)$. By [16, Lemma 3], $H_{b,S}(X)$ is closed in $H_b(X)$. So, $H_{b,S}(X)$ is a Fréchet algebra. Let $\mathcal{P}_S(X)$ be the subalgebra of $H_{b,S}(X)$ consisting of all S-symmetric continuous polynomials on X.

Algebraic basis. Let A be a unital commutative algebra over the field $\mathbb{C}$. For every polynomial $Q\colon \mathbb{C}^n \to \mathbb{C}$ of the form

$$Q(z_1,\ldots,z_n) = \sum_{(k_1,\ldots,k_n)\in\Omega} \alpha_{(k_1,\ldots,k_n)} z_1^{k_1}\cdots z_n^{k_n},$$

where $\alpha_{(k_1,\ldots,k_n)} \in \mathbb{C}$ and Ω is some nonempty finite subset of $\mathbb{Z}_+^n$, let us define the mapping $Q_A\colon A^n \to A$ by

$$Q_A(a_1,\ldots,a_n) = \sum_{(k_1,\ldots,k_n)\in\Omega} \alpha_{(k_1,\ldots,k_n)} a_1^{k_1}\cdots a_n^{k_n}, \tag{1}$$

where $a_1,\ldots,a_n \in A$ (we consider the zeroth power a_j^0 of an element a_j to be the unit element of A).

Let $a, a_1,\ldots,a_n \in A$. The element a is called an *algebraic combination* of $a_1,\ldots,a_n$ if there exists a polynomial $Q\colon \mathbb{C}^n \to \mathbb{C}$ such that $a = Q_A(a_1,\ldots,a_n)$.

A nonempty set $B \subset A$ is called an *algebraic basis* of A if every element of A can be uniquely represented as an algebraic combination of some elements of B.

Isomorphisms of Fréchet algebras of entire symmetric functions. We will use the following result, the item a) of which is proven in [19, Theorem 2] and the items b) and c) are proven in [17, Theorem 4]).

Theorem 1. *Let X and Y be complex Banach spaces. Let S_1 and S_2 be semigroups of operators on X and Y resp. Let $\iota_{X,Y}\colon X \to Y$ be an isomorphism such that*

1) for every $x \in X$ and $s_1 \in S_1$, there exists $s_2 \in S_2$ such that $\iota_{X,Y}(s_1(x)) = s_2(\iota_{X,Y}(x))$;

2) for every $y \in Y$ and $s_2 \in S_2$, there exists $s_1 \in S_1$ such that $\iota_{X,Y}^{-1}(s_2(y)) = s_1(\iota_{X,Y}^{-1}(y))$.

Then

a) the mapping

$$I\colon f \in H_{b,S_2}(Y) \mapsto f \circ \iota_{X,Y} \in H_{b,S_1}(X) \tag{2}$$

is an isomorphism, i.e., I is a continuous linear multiplicative bijection;

b) the restriction of I to $\mathcal{P}_{S_2}(Y)$ is an isomorphism between algebras $\mathcal{P}_{S_2}(Y)$ and $\mathcal{P}_{S_1}(X)$;

c) if $\mathcal{P}_{S_2}(Y)$ has some algebraic basis B, then $I(B)$ is an algebraic basis in $\mathcal{P}_{S_1}(X)$.

The space $\ell_p(\mathbb{C}^n)$. Let $n \in \mathbb{N}$ and $p \in [1,+\infty)$. Let us denote $\ell_p(\mathbb{C}^n)$ the vector space of all sequences $x = (x_1, x_2, \ldots)$, where $x_j = \big(x_j^{(1)},\ldots,x_j^{(n)}\big) \in \mathbb{C}^n$ for $j \in \mathbb{N}$, such that the series $\sum_{j=1}^\infty \sum_{s=1}^n |x_j^{(s)}|^p$ is convergent. The space $\ell_p(\mathbb{C}^n)$ with norm

$$\|x\|_{\ell_p(\mathbb{C}^n)} = \left(\sum_{j=1}^\infty \sum_{s=1}^n |x_j^{(s)}|^p\right)^{1/p}$$

is a Banach space. If $n = 1$, then as usual we will write $\ell_p(\mathbb{C}) = \ell_p$.

2. The main result.

2.1. Isometrical isomorphism between ℓ_p and $\ell_p(\mathbb{C}^n)$. Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. Let us construct an isomorphism between ℓ_p and $\ell_p(\mathbb{C}^n)$. Let $\iota\colon \ell_p \to \ell_p(\mathbb{C}^n)$ be defined in the following way. For $y = (y_1, y_2, \ldots) \in \ell_p$, let

$$\iota(y) = x, \tag{3}$$

where

$$x = \big((x_1^{(1)}, \ldots, x_1^{(n)}), (x_2^{(1)}, \ldots, x_2^{(n)}), \ldots\big) \tag{4}$$

is such that

$$x_j^{(s)} = y_{(j-1)n+s} \tag{5}$$

for every $j \in \mathbb{N}$ and $s \in \{1, \ldots, n\}$. Let us show that the mapping ι is well defined.

Lemma 1. *For every $y \in \ell_p$, the element $\iota(y)$, defined by* (3)*, belongs to $\ell_p(\mathbb{C}^n)$ and $\|\iota(y)\|_{\ell_p(\mathbb{C}^n)} = \|y\|_{\ell_p}$.*

Proof. Let $y = (y_1, y_2, \ldots) \in \ell_p$. Then the series $\sum_{m=1}^{\infty} |y_m|^p$ is convergent. Let $x = \iota(y)$. Then, taking into account (4) and (5),

$$\sum_{j=1}^{\infty} \sum_{s=1}^{n} |x_j^{(s)}|^p = \sum_{j=1}^{\infty} \sum_{s=1}^{n} |y_{(j-1)n+s}|^p = \sum_{m=1}^{\infty} |y_m|^p. \tag{6}$$

By (6), since the series $\sum_{m=1}^{\infty} |y_m|^p$ is convergent, it follows that the series $\sum_{j=1}^{\infty} \sum_{s=1}^{n} |x_j^{(s)}|^p$ is convergent too. Therefore $x \in \ell_p(\mathbb{C}^n)$. Also, by (6), $\|x\|_{\ell_p(\mathbb{C}^n)} = \|y\|_{\ell_p}$. □

Let us define the mapping $\varkappa_n\colon \mathbb{N} \to \mathbb{N} \times \{1, \ldots, n\}$ in the following way. Let $m \in \mathbb{N}$. Then there exist unique $j \in \mathbb{N}$ and $s \in \{1, \ldots, n\}$ such that $m = (j-1)n + s$. We set

$$\varkappa_n(m) = (j, s). \tag{7}$$

It can be checked that the mapping $\varkappa_n$ is a bijection. Evidently,

$$\varkappa_n^{-1}((j, s)) = (j-1)n + s \tag{8}$$

for every $(j, s) \in \mathbb{N} \times \{1, \ldots, n\}$.

Let us establish some properties of the mapping ι.

Lemma 2. *The mapping ι, defined by* (3)*, is surjective.*

Proof. Let us show that ι is surjective. Let $x = \big((x_1^{(1)}, \ldots, x_1^{(n)}), (x_2^{(1)}, \ldots, x_2^{(n)}), \ldots\big) \in \ell_p(\mathbb{C}^n)$. Let us construct $y = (y_1, y_2, \ldots) \in \ell_p$ such that $\iota(y) = x$. For $m \in \mathbb{N}$, we set

$$y_m = x_j^{(s)}, \tag{9}$$

where $(j, s) = \varkappa_n(m)$ and the mapping $\varkappa_n$ is defined by (7). Let us show that $y \in \ell_p$. By (9),

$$\sum_{m=1}^{\infty} |y_m|^p = \sum_{(j,s) \in \varkappa_n(\mathbb{N})} |x_j^{(s)}|^p.$$

Since $\varkappa_n$ is surjective, $\varkappa_n(\mathbb{N}) = \mathbb{N} \times \{1, \ldots, n\}$. Consequently,

$$\sum_{(j,s)\in\varkappa_n(\mathbb{N})} |x_j^{(s)}|^p = \sum_{j=1}^{\infty}\sum_{s=1}^{n} |x_j^{(s)}|^p.$$

Therefore

$$\sum_{m=1}^{\infty} |y_m|^p = \sum_{j=1}^{\infty}\sum_{s=1}^{n} |x_j^{(s)}|^p. \tag{10}$$

Since $x \in \ell_p(\mathbb{C}^n)$, the series in the right-hand side of the equality (10) is convergent. Consequently, the series in the left-hand side of the equality (10) is convergent too. Therefore $y \in \ell_p$.

Let us show that $\iota(y) = x$. Let $z = \iota(y)$. Let us show that $z = x$. Let

$$z = \big(\big(z_1^{(1)}, \ldots, z_1^{(n)}\big), \big(z_2^{(1)}, \ldots, z_2^{(n)}\big), \ldots\big).$$

Let $j_0 \in \mathbb{N}$ and $s_0 \in \{1, \ldots, n\}$. By (5),

$$z_{j_0}^{(s_0)} = y_{(j_0-1)n+s_0}. \tag{11}$$

Let $m_0 = (j_0 - 1)n + s_0$. Then

$$y_{(j_0-1)n+s_0} = y_{m_0}. \tag{12}$$

Since $m_0 = (j_0 - 1)n + s_0$, by the definition of the mapping $\varkappa_n$, it follows that $(j_0, s_0) = \varkappa_n(m_0)$. Consequently, by (9),

$$y_{m_0} = x_{j_0}^{(s_0)}. \tag{13}$$

By (11),(12) and (13), $z_{j_0}^{(s_0)} = x_{j_0}^{(s_0)}$. Thus, $z = x$. $\square$

Theorem 2. *Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. The mapping ι, defined by (3), is an isometrical isomorphism between ℓ_p and $\ell_p(\mathbb{C}^n)$.*

Proof. It can be checked that ι is linear. By Lemma 1, ι is isometrical. Therefore, since every linear isometrical mapping is injective, it follows that ι is injective. By Lemma 2, ι is surjective. So, ι is bijective. Thus, ι is an isometrical isomorphism. $\square$

2.2. Algebras of symmetric functions on ℓ_p and $\ell_p(\mathbb{C}^n)$. Let B be some set of bijections, acting from $\mathbb{N}$ to itself. Let X be an arbitrary vector space of sequences of scalars or vectors such that $(x_{b(1)}, x_{b(2)}, \ldots) \in X$ for every $b \in B$ and $(x_1, x_2, \ldots) \in X$. For $b \in B$, let the operator $g_{b,X}\colon X \to X$ be defined by

$$g_{b,X}((x_1, x_2, \ldots)) = (x_{b(1)}, x_{b(2)}, \ldots), \tag{14}$$

where $(x_1, x_2, \ldots) \in X$. Let

$$G_{B,X} = \{g_{b,X}\colon b \in B\}, \tag{15}$$

where operators $g_{b,X}$ are defined by (14). It can be checked that if B is a group with respect to the operation of composition, then $G_{B,X}$ is also a group.

Let $\mathcal{S}$ be the group of all bijections, acting from $\mathbb{N}$ to itself. Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. Let us consider $G_{\mathcal{S},\ell_p(\mathbb{C}^n)}$-symmetric functions on $\ell_p(\mathbb{C}^n)$, where the group $G_{\mathcal{S},\ell_p(\mathbb{C}^n)}$ is defined by (15). Usually such functions are called symmetric (see [2, 9, 15]) or block-symmetric (see [8, 10, 11]).

Let us define the so-called power sum symmetric polynomials on $\ell_p(\mathbb{C}^n)$. For a multi-index $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, let $\langle k \rangle = k_1 + \ldots + k_n$. For every $k \in \mathbb{Z}_+^n$ such that $\langle k \rangle \geq \lceil p \rceil$, where $\lceil p \rceil$ is the ceiling of p, let us define the mapping $H_k \colon \ell_p(\mathbb{C}^n) \to \mathbb{C}$ by

$$H_k(x) = \sum_{j=1}^{\infty} \prod_{\substack{s=1 \\ k_s>0}}^{n} \big(x_j^{(s)}\big)^{k_s}, \tag{16}$$

where $x = \big(\big(x_1^{(1)}, \ldots, x_1^{(n)}\big), \big(x_2^{(1)}, \ldots, x_2^{(n)}\big), \ldots\big) \in \ell_p(\mathbb{C}^n)$. Note that H_k is a $G_{\mathcal{S},\ell_p(\mathbb{C}^n)}$-symmetric $\langle k \rangle$-homogeneous polynomial. By [9, Proposition 2], the polynomial H_k is continuous. We will use the following result from [9]:

Theorem 3 ([9], Corollary 15)**.** *Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. The set of polynomials*

$$\big\{H_k \colon k \in \mathbb{Z}_+^n \text{ such that } \langle k \rangle \geq \lceil p \rceil\big\} \tag{17}$$

is an algebraic basis of the algebra $\mathcal{P}_{G_{\mathcal{S},\ell_p(\mathbb{C}^n)}}(\ell_p(\mathbb{C}^n))$ of all $G_{\mathcal{S},\ell_p(\mathbb{C}^n)}$-symmetric continuous complex-valued polynomials on $\ell_p(\mathbb{C}^n)$.

Note that the analogical result for the real case is established in [15].

Let $n \in \mathbb{N}$. Let us construct the group of operators on ℓ_p, whose elements permute coordinates of elements of ℓ_p by blocks of the length n.

Recall that $\mathcal{S}$ is the group of all bijections, acting from $\mathbb{N}$ to itself. Let $\mathrm{id}_{\{1,\ldots,n\}}$ be the identical mapping on $\{1, \ldots, n\}$. For $\sigma \in \mathcal{S}$, let $\sigma \times \mathrm{id}_{\{1,\ldots,n\}}$ be the mapping, acting from $\mathbb{N} \times \{1, \ldots, n\}$ to itself, defined by

$$\big(\sigma \times \mathrm{id}_{\{1,\ldots,n\}}\big)((j, s)) = (\sigma(j), s), \tag{18}$$

where $(j, s) \in \mathbb{N} \times \{1, \ldots, n\}$. Since both σ and $\mathrm{id}_{\{1,\ldots,n\}}$ are bijections, it follows that $\sigma \times \mathrm{id}_{\{1,\ldots,n\}}$ is a bijection too.

For $\sigma \in \mathcal{S}$, let $\theta_{\sigma,n} \colon \mathbb{N} \to \mathbb{N}$ be defined by

$$\theta_{\sigma,n} = \varkappa_n^{-1} \circ \big(\sigma \times \mathrm{id}_{\{1,\ldots,n\}}\big) \circ \varkappa_n, \tag{19}$$

where $\varkappa_n$ and $\sigma \times \mathrm{id}_{\{1,\ldots,n\}}$ are defined by (7) and (18) resp. Since mappings $\varkappa_n^{-1}$, $\sigma \times \mathrm{id}_{\{1,\ldots,n\}}$ and $\varkappa_n$ are bijections, it follows that $\theta_{\sigma,n}$ is a bijection too.

Let

$$\Theta_n = \{\theta_{\sigma,n} \colon \sigma \in \mathcal{S}\}, \tag{20}$$

where $\theta_{\sigma,n}$ is defined by (19). Since, for every $\sigma \in \mathcal{S}$, the mapping $\theta_{\sigma,n}$ is a bijection, acting from $\mathbb{N}$ to itself, it follows that Θ_n is a subset of $\mathcal{S}$.

Let us establish some properties of mappings, defined by (18) and (19).

Lemma 3. *For every $n \in \mathbb{N}$ and $\sigma_1, \sigma_2 \in \mathcal{S}$,*

$$\big(\sigma_1 \times \mathrm{id}_{\{1,\ldots,n\}}\big) \circ \big(\sigma_2 \times \mathrm{id}_{\{1,\ldots,n\}}\big) = (\sigma_1 \circ \sigma_2) \times \mathrm{id}_{\{1,\ldots,n\}}. \tag{21}$$

Proof. For $(j,s) \in \mathbb{N} \times \{1,\ldots,n\}$, by (18),

$$\big((\sigma_1 \times \mathrm{id}_{\{1,\ldots,n\}}) \circ (\sigma_2 \times \mathrm{id}_{\{1,\ldots,n\}})\big)((j,s)) = (\sigma_1 \times \mathrm{id}_{\{1,\ldots,n\}})\big((\sigma_2(j),s)\big) = \\ = \big(\sigma_1(\sigma_2(j)),s\big) = \big((\sigma_1 \circ \sigma_2)(j),s\big) = \big((\sigma_1 \circ \sigma_2) \times \mathrm{id}_{\{1,\ldots,n\}}\big)((j,s)).$$

So, the equality (21) holds. □

Lemma 4. *For every $n \in \mathbb{N}$ and $\sigma_1, \sigma_2 \in \mathcal{S}$,*

$$\theta_{\sigma_1,n} \circ \theta_{\sigma_2,n} = \theta_{\sigma_1 \circ \sigma_2,n}.$$

Proof. Taking into account (19) and Lemma 3,

$$\theta_{\sigma_1,n} \circ \theta_{\sigma_2,n} = \varkappa_n^{-1} \circ (\sigma_1 \times \mathrm{id}_{\{1,\ldots,n\}}) \circ \varkappa_n \circ \varkappa_n^{-1} \circ (\sigma_2 \times \mathrm{id}_{\{1,\ldots,n\}}) \circ \varkappa_n = \\ = \varkappa_n^{-1} \circ (\sigma_1 \times \mathrm{id}_{\{1,\ldots,n\}}) \circ (\sigma_2 \times \mathrm{id}_{\{1,\ldots,n\}}) \circ \varkappa_n = \\ = \varkappa_n^{-1} \circ \big((\sigma_1 \circ \sigma_2) \times \mathrm{id}_{\{1,\ldots,n\}}\big) \circ \varkappa_n = \theta_{\sigma_1 \circ \sigma_2,n}.$$

□

Proposition 1. *For every $n \in \mathbb{N}$, the set Θ_n, defined by* (20), *is a subgroup of the group $\mathcal{S}$.*

Proof. As it was mentioned above, Θ_n is a subset of $\mathcal{S}$. Let us show that Θ_n is a subgroup of $\mathcal{S}$.

Let $a_1, a_2 \in \Theta_n$. Let us show that $a_1 \circ a_2 \in \Theta_n$. By (20), there exist $\sigma_1, \sigma_2 \in \mathcal{S}$ such that $a_1 = \theta_{\sigma_1,n}$ and $a_2 = \theta_{\sigma_2,n}$. By Lemma 4, $a_1 \circ a_2 = \theta_{\sigma_1 \circ \sigma_2,n}$. Since $\sigma_1, \sigma_2 \in \mathcal{S}$ and $\mathcal{S}$ is a group, it follows that $\sigma_1 \circ \sigma_2 \in \mathcal{S}$. Consequently, $\theta_{\sigma_1 \circ \sigma_2,n} \in \Theta_n$, that is, $a_1 \circ a_2 \in \Theta_n$.

Let $a \in \Theta_n$. Let us show that the inverse element of a belongs to Θ_n. By (20), there exists $\sigma \in \mathcal{S}$ such that $a = \theta_{\sigma,n}$. Since $\sigma \in \mathcal{S}$ and $\mathcal{S}$ is a group, it follows that $\sigma^{-1} \in \mathcal{S}$. Consequently, $\theta_{\sigma^{-1},n} \in \Theta_n$. By Lemma 4,

$$a \circ \theta_{\sigma^{-1},n} = \theta_{\sigma,n} \circ \theta_{\sigma^{-1},n} = \theta_{\sigma \circ \sigma^{-1},n} = \theta_{e,n}$$

and

$$\theta_{\sigma^{-1},n} \circ a = \theta_{\sigma^{-1},n} \circ \theta_{\sigma,n} = \theta_{\sigma^{-1} \circ \sigma,n} = \theta_{e,n},$$

where e is the identity element of the group $\mathcal{S}$. Note that $e \times \mathrm{id}_{\{1,\ldots,n\}}$ is the identical mapping on $\mathbb{N} \times \{1,\ldots,n\}$. Consequently, taking into account (19), $\theta_{e,n} = e$. So, $\theta_{\sigma^{-1},n}$ is the inverse element of a.

Thus, Θ_n is a subgroup of $\mathcal{S}$. □

Let us show how elements of the group Θ_n act on elements of $\mathbb{N}$.

Lemma 5. *Let $n \in \mathbb{N}$, $\sigma \in \mathcal{S}$ and $m \in \mathbb{N}$. Let $(j,s) = \varkappa_n(m)$. Then*

$$\theta_{\sigma,n}(m) = (\sigma(j) - 1)n + s,$$

where $\theta_{\sigma,n}$ is defined by (19).

Proof. By (19), (18) and (8), taking into account that $(j,s) = \varkappa_n(m)$,

$$\theta_{\sigma,n}(m) = \big(\varkappa_n^{-1} \circ (\sigma \times \mathrm{id}_{\{1,\ldots,n\}}) \circ \varkappa_n\big)(m) = \\ = \big(\varkappa_n^{-1} \circ (\sigma \times \mathrm{id}_{\{1,\ldots,n\}})\big)((j,s)) = \varkappa_n^{-1}((\sigma(j),s)) = (\sigma(j) - 1)n + s.$$

□

Let us show that, in fact, elements of the group Θ_n permute subsets $\{1,\ldots,n\},\{n+1,\ldots,2n\},\ldots$ of $\mathbb{N}$.

Corollary 1. *Let $n\in\mathbb{N}$, $\sigma\in\mathcal{S}$ and $k\in\mathbb{N}$. Then*

$$\theta_{\sigma,n}\left(\{(k-1)n+1,\ldots,(k-1)n+n\}\right)=\{(\sigma(k)-1)n+1,\ldots,(\sigma(k)-1)n+n\}\,, \tag{22}$$

where $\theta_{\sigma,n}$ is defined by (19).

Proof. By (7),

$$\varkappa_n\big((k-1)n+s\big)=(k,s)$$

for every $s\in\{1,\ldots,n\}$. Consequently, by Lemma 5,

$$\theta_{\sigma,n}((k-1)n+s)=(\sigma(k)-1)n+s$$

for every $s\in\{1,\ldots,n\}$. Therefore the equality (22) holds. □

Let $n\in\mathbb{N}$ and $p\in[1,+\infty)$. Let G_{Θ_n,ℓ_p} be the group of operators on ℓ_p, defined by (15), where Θ_n is defined by (20). Let us prove some properties of elements of groups $G_{\mathcal{S},\ell_p(\mathbb{C}^n)}$ and G_{Θ_n,ℓ_p}.

Lemma 6. *For every $\sigma\in\mathcal{S}$ and $y\in\ell_p$,*

$$\iota\big(g_{\theta_{\sigma,n},\ell_p}(y)\big)=g_{\sigma,\ell_p(\mathbb{C}^n)}\big(\iota(y)\big), \tag{23}$$

where ι is defined by (3), *$g_{\theta_{\sigma,n},\ell_p}$ and $g_{\sigma,\ell_p(\mathbb{C}^n)}$ are defined by* (14), *and $\theta_{\sigma,n}$ is defined by* (19).

Proof. Let $\sigma\in\mathcal{S}$ and $y=(y_1,y_2,\ldots)\in\ell_p$. Let us show that the equality (23) holds. Let sequences $x=(x_1,x_2,\ldots),z=(z_1,z_2,\ldots)\in\ell_p(\mathbb{C}^n)$, where $x_m=\big(x_m^{(1)},\ldots,x_m^{(n)}\big),z_m=\big(z_m^{(1)},\ldots,z_m^{(n)}\big)\in\mathbb{C}^n$ for $m\in\mathbb{N}$, be defined by

$$x=\iota\big(g_{\theta_{\sigma,n},\ell_p}(y)\big) \tag{24}$$

and

$$z=g_{\sigma,\ell_p(\mathbb{C}^n)}\big(\iota(y)\big). \tag{25}$$

Let us show that $x=z$.

Let $(j,s)\in\mathbb{N}\times\{1,\ldots,n\}$. Let us show that $x_j^{(s)}=z_j^{(s)}$.

Let us find the value of $x_j^{(s)}$. Let $v=(v_1,v_2,\ldots)\in\ell_p$ be defined by

$$v=g_{\theta_{\sigma,n},\ell_p}(y). \tag{26}$$

By (24) and (26),

$$x=\iota(v).$$

Therefore, by (3),

$$x_j^{(s)}=v_{(j-1)n+s}. \tag{27}$$

By (14) and (26),

$$v_{(j-1)n+s}=y_{\theta_{\sigma,n}((j-1)n+s)}. \tag{28}$$

So, by (27) and (28),

$$x_j^{(s)}=y_{\theta_{\sigma,n}((j-1)n+s)}. \tag{29}$$

By Lemma 5, taking into account that, by (7), $\varkappa_n((j-1)n+s) = (j,s)$,

$$\theta_{\sigma,n}((j-1)n+s) = (\sigma(j)-1)n+s. \tag{30}$$

By (29) and (30),

$$x_j^{(s)} = y_{(\sigma(j)-1)n+s}. \tag{31}$$

Let us find the value of $z_j^{(s)}$. Let $w = (w_1, w_2, \ldots) \in \ell_p(\mathbb{C}^n)$, where $w_m = \big(w_m^{(1)}, \ldots, w_m^{(n)}\big) \in \mathbb{C}^n$ for $m \in \mathbb{N}$, be defined by

$$w = \iota(y). \tag{32}$$

By (25) and (32),

$$z = g_{\sigma,\ell_p(\mathbb{C}^n)}\big(w\big). \tag{33}$$

By (14) and (33), $z_j = w_{\sigma(j)}$ and, consequently,

$$z_j^{(s)} = w_{\sigma(j)}^{(s)}. \tag{34}$$

By (3) and (32),

$$w_{\sigma(j)}^{(s)} = y_{(\sigma(j)-1)n+s}. \tag{35}$$

By (34) and (35),

$$z_j^{(s)} = y_{(\sigma(j)-1)n+s}. \tag{36}$$

By (31) and (36), $x_j^{(s)} = z_j^{(s)}$. Thus, $x = z$. So, the equality (23) holds. □

Corollary 2. *For every $\sigma \in \mathcal{S}$ and $x \in \ell_p(\mathbb{C}^n)$,*

$$\iota^{-1}\big(g_{\sigma,\ell_p(\mathbb{C}^n)}(x)\big) = g_{\theta_{\sigma,n},\ell_p}\big(\iota^{-1}(x)\big), \tag{37}$$

where ι is defined by (3), $g_{\theta_{\sigma,n},\ell_p}$ and $g_{\sigma,\ell_p(\mathbb{C}^n)}$ are defined by (14), and $\theta_{\sigma,n}$ is defined by (19).

Proof. Let $\sigma \in \mathcal{S}$ and $x \in \ell_p(\mathbb{C}^n)$. Let us show that the equality (37) holds. Substituting $y = \iota^{-1}(x)$ into Lemma 6, we obtain $\iota\big(g_{\theta_{\sigma,n},\ell_p}\big(\iota^{-1}(x)\big)\big) = g_{\sigma,\ell_p(\mathbb{C}^n)}(x)$. Consequently,

$$\iota^{-1}\big(\iota\big(g_{\theta_{\sigma,n},\ell_p}\big(\iota^{-1}(x)\big)\big)\big) = \iota^{-1}\big(g_{\sigma,\ell_p(\mathbb{C}^n)}(x)\big),$$

that is, $g_{\theta_{\sigma,n},\ell_p}\big(\iota^{-1}(x)\big) = \iota^{-1}\big(g_{\sigma,\ell_p(\mathbb{C}^n)}(x)\big)$. So, the equality (37) holds. □

Let us apply Theorem 1 to algebras of entire symmetric functions on ℓ_p and $\ell_p(\mathbb{C}^n)$.

Theorem 4. *Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. Let ι be the isometrical isomorphism between ℓ_p and $\ell_p(\mathbb{C}^n)$, defined by (3). Then*

a) the mapping

$$I\colon f \in H_{b,G_{\mathcal{S},\ell_p(\mathbb{C}^n)}}(\ell_p(\mathbb{C}^n)) \mapsto f \circ \iota \in H_{b,G_{\Theta_n,\ell_p}}(\ell_p) \tag{38}$$

is an isomorphism, where $H_{b,G_{\mathcal{S},\ell_p(\mathbb{C}^n)}}(\ell_p(\mathbb{C}^n))$ is the Fréchet algebra of entire $G_{\mathcal{S},\ell_p(\mathbb{C}^n)}$-symmetric functions of bounded type on $\ell_p(\mathbb{C}^n)$ and $H_{b,G_{\Theta_n,\ell_p}}(\ell_p)$ is the Fréchet algebra of entire G_{Θ_n,ℓ_p}-symmetric functions of bounded type on ℓ_p;

b) the restriction of the isomorphism I to $\mathcal{P}_{G_{\mathcal{S},\ell_p(\mathbb{C}^n)}}(\ell_p(\mathbb{C}^n))$ is an isomorphism between algebras of polynomials $\mathcal{P}_{G_{\mathcal{S},\ell_p(\mathbb{C}^n)}}(\ell_p(\mathbb{C}^n))$ and $\mathcal{P}_{G_{\Theta_n,\ell_p}}(\ell_p)$;

c) the set of polynomials

$$\big\{I(H_k)\colon k \in \mathbb{Z}_+^n \text{ such that } \langle k\rangle \geq \lceil p\rceil\big\} \tag{39}$$

is an algebraic basis of the algebra $\mathcal{P}_{G_{\Theta_n,\ell_p}}(\ell_p)$, where polynomials H_k are defined by (16).

Proof. Let us substitute $\ell_p, \ell_p(\mathbb{C}^n), G_{\Theta_n,\ell_p}, G_{\mathcal{S},\ell_p(\mathbb{C}^n)}, \iota$ instead of $X, Y, S_1, S_2, \iota_{X,Y}$ respectively into Theorem 1. Let us check the conditions of Theorem 1.

Note that, by Theorem 2, ι is an isometrical isomorphism between ℓ_p and $\ell_p(\mathbb{C}^n)$.

Let us show that the condition 1) of Theorem 1 holds. Let $x \in \ell_p$ and $s_1 \in G_{\Theta_n,\ell_p}$. By (15) and (20), there exists $\sigma \in \mathcal{S}$ such that $s_1 = g_{\theta_{\sigma,n},\ell_p}$, where $\theta_{\sigma,n}$ is defined by (19). Let $s_2 = g_{\sigma,\ell_p(\mathbb{C}^n)}$, where $g_{\sigma,\ell_p(\mathbb{C}^n)}$ is defined by (14). Note that $s_2 \in G_{\mathcal{S},\ell_p(\mathbb{C}^n)}$. By Lemma 6, where we substitute x instead of y, $\iota\big(g_{\theta_{\sigma,n},\ell_p}(x)\big) = g_{\sigma,\ell_p(\mathbb{C}^n)}\big(\iota(x)\big)$, that is, $\iota(s_1(x)) = s_2(\iota(x))$. So, the condition 1) of Theorem 1 holds.

Let us show that the condition 2) of Theorem 1 holds. Let $y \in \ell_p(\mathbb{C}^n)$ and $s_2 \in G_{\mathcal{S},\ell_p(\mathbb{C}^n)}$. By (15), there exists $\sigma \in \mathcal{S}$ such that $s_2 = g_{\sigma,\ell_p(\mathbb{C}^n)}$. Let $s_1 = g_{\theta_{\sigma,n},\ell_p}$, where $g_{\theta_{\sigma,n},\ell_p}$ is defined by (14) and $\theta_{\sigma,n}$ is defined by (19). Note that $s_1 \in G_{\Theta_n,\ell_p}$. By Corollary 2, where we substitute y instead of x, $\iota^{-1}\big(g_{\sigma,\ell_p(\mathbb{C}^n)}(y)\big) = g_{\theta_{\sigma,n},\ell_p}\big(\iota^{-1}(y)\big)$, that is, $\iota^{-1}(s_2(y)) = s_1(\iota^{-1}(y))$. So, the condition 2) of Theorem 1 holds.

Thus, the conditions of Theorem 1 hold. Therefore, by the item a) of Theorem 1, the mapping I, defined by (38), is an isomorphism. By the item b) of Theorem 1, the restriction of the isomorphism I to $\mathcal{P}_{G_{\mathcal{S},\ell_p(\mathbb{C}^n)}}(\ell_p(\mathbb{C}^n))$ is an isomorphism between algebras of polynomials $\mathcal{P}_{G_{\mathcal{S},\ell_p(\mathbb{C}^n)}}(\ell_p(\mathbb{C}^n))$ and $\mathcal{P}_{G_{\Theta_n,\ell_p}}(\ell_p)$. By Theorem 3, the set (17) is an algebraic basis of the algebra $\mathcal{P}_{G_{\mathcal{S},\ell_p(\mathbb{C}^n)}}(\ell_p(\mathbb{C}^n))$. Consequently, by the item c) of Theorem 1 the set (39) is an algebraic basis of the algebra $\mathcal{P}_{G_{\Theta_n,\ell_p}}(\ell_p)$. □

Let us represent elements of the algebraic basis (39) in the explicit form.

Lemma 7. *Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. For every $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ such that $\langle k \rangle \geq \lceil p \rceil$ and for every $y = (y_1, y_2, \ldots) \in \ell_p$,*

$$I(H_k)(y) = \sum_{j=1}^{\infty} \prod_{\substack{s=1 \\ k_s>0}}^{n} (y_{(j-1)n+s})^{k_s} \tag{40}$$

where H_k is defined by (16) *and I is defined by* (38).

Proof. By (38), $I(H_k) = H_k \circ \iota$, where ι is defined by (3). Therefore

$$I(H_k)(y) = H_k(\iota(y)). \tag{41}$$

Let $x = \big((x_1^{(1)}, \ldots, x_1^{(n)}), (x_2^{(1)}, \ldots, x_2^{(n)}), \ldots\big) \in \ell_p(\mathbb{C}^n)$ be defined by

$$x = \iota(y). \tag{42}$$

By (3), $x_j^{(s)} = y_{(j-1)n+s}$ for every $(j, s) \in \mathbb{N} \times \{1, \ldots, n\}$. Therefore, taking into account (16),

$$H_k(x) = \sum_{j=1}^{\infty} \prod_{\substack{s=1 \\ k_s>0}}^{n} \big(x_j^{(s)}\big)^{k_s} = \sum_{j=1}^{\infty} \prod_{\substack{s=1 \\ k_s>0}}^{n} (y_{(j-1)n+s})^{k_s}. \tag{43}$$

By (41), (42) and (43), the equality (40) holds. □

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REFERENCES

1. R. Aron, P. Galindo, D. Pinasco, I. Zalduendo, *Group-symmetric holomorphic functions on a Banach space*, Bull. Lond. Math. Soc., **48** (2016), №5, 779–796. doi:10.1112/blms/bdw043
2. A. Bandura, V. Kravtsiv, T. Vasylyshyn, *Algebraic basis of the algebra of all symmetric continuous polynomials on the Cartesian product of ℓ_p-spaces*, Axioms, **11** (2022), №2, art. no.41. doi:10.3390/axioms11020041
3. I.V. Burtnyak, Yu.Yu. Chopyuk, S.I. Vasylyshyn, T.V. Vasylyshyn, *Algebras of weakly symmetric functions on spaces of Lebesgue measurable functions*, Carpathian Math. Publ., **15** (2023), №2, 411–419. doi:10.15330/cmp.15.2.411-419
4. I. Chernega, P. Galindo, A. Zagorodnyuk, *Some algebras of symmetric analytic functions and their spectra*, Proc. Edinb. Math. Soc., **55** (2012), №1, 125–142. doi:10.1017/S0013091509001655
5. I. Chernega, P. Galindo, A. Zagorodnyuk, *On the spectrum of the algebra of bounded-type symmetric analytic functions on ℓ_1*, Math. Nachr., (2024). doi:10.1002/mana.202300415
6. M. González, R. Gonzalo, J.A. Jaramillo, *Symmetric polynomials on rearrangement invariant function spaces*, J. Lond. Math. Soc., **59** (1999), №2, 681–697. doi:10.1112/S0024610799007164
7. O.V. Handera-Kalynovska, V.V. Kravtsiv, *The Waring-Girard formulas for symmetric polynomials on spaces ℓ_p*, Carpathian Math. Publ., **16** (2024), №2, 407–413. doi:10.15330/cmp.16.2.407-413
8. V. Kravtsiv, *The analogue of Newton's formula for block-symmetric polynomials*, International Journal of Mathematical Analysis, **10** (2016), №5–8, 323–327. doi:10.12988/ijma.2016.617
9. V. Kravtsiv, T. Vasylyshyn, A. Zagorodnyuk, *On algebraic basis of the algebra of symmetric polynomials on $\ell_p(\mathbb{C}^n)$*, J. Funct. Spaces, **2017** (2017), Article ID 4947925, 8 p. doi:10.1155/2017/4947925
10. V.V. Kravtsiv, A.V. Zagorodnyuk, *Multiplicative convolution on the algebra of block-symmetric analytic functions*, J. Math. Sci. (N.Y.), **246** (2020), №2, 245–255. doi:10.1007/s10958-020-04734-z
11. V.V. Kravtsiv, A.V. Zagorodnyuk, *Spectra of algebras of block-symmetric analytic functions of bounded type*, Mat. Stud., **58** (2022), №1, 69–81. doi:10.30970/ms.58.1.69-81
12. A.S. Nemirovskii, S.M. Semenov, *On polynomial approximation of functions on Hilbert space*, Mat. USSR Sbornik, **21** (1973), №2, 255–277. doi:10.1070/SM1973v021n02ABEH002016
13. Z. Novosad, S. Vasylyshyn, A. Zagorodnyuk, *Countably generated algebras of analytic functions on Banach spaces*, Axioms, **12** (2023), №8, Article number 798. doi:10.3390/axioms12080798
14. S.I. Vasylyshyn, *Spectra of algebras of analytic functions, generated by sequences of polynomials on Banach spaces, and operations on spectra*, Carpathian Math. Publ., **15** (2023), №1, 104–119. doi:10.15330/cmp.15.1.104-119
15. T. Vasylyshyn, *Symmetric functions on spaces $\ell_p(\mathbb{R}^n)$ and $\ell_p(\mathbb{C}^n)$*, Carpathian Math. Publ., **12** (2020), №1, 5–16. doi:10.15330/cmp.12.1.5-16
16. T. Vasylyshyn, *Algebras of symmetric analytic functions on Cartesian powers of Lebesgue integrable in a power $p \in [1, +\infty)$ functions*, Carpathian Math. Publ., **13** (2021), №2, 340–351. doi:10.15330/cmp.13.2.340-351
17. T. Vasylyshyn, *Algebras of symmetric and block-symmetric functions on spaces of Lebesgue measurable functions*, Carpathian Math. Publ., **16** (2024), №1, 174–189. doi:10.15330/cmp.16.1.174-189
18. T. Vasylyshyn, V. Zahorodniuk, *Weakly symmetric functions on spaces of Lebesgue integrable functions*, Carpatian Math. Publ., **14** (2022), №2, 437–441. doi:10.15330/cmp.14.2.437-441
19. T. Vasylyshyn, V. Zahorodniuk, *On isomorphisms of algebras of entire symmetric functions on Banach spaces*, J. Math. Anal. Appl., **529** (2024), №2, Article number 127370. doi:10.1016/j.jmaa.2023.127370

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