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COMPOSITION OF ENTIRE FUNCTION AND ANALYTIC FUNCTIONS IN THE UNIT BALL WITH A VANISHED GRADIENT

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The composition $H(z) = f(\Phi(z))$ is studied, where f is an entire function of a single complex variable and Φ is an analytic function in the n-dimensional unit ball $\mathbb{B}^n$, $n \geq 2$, with a vanished gradient. We found conditions by the function Φ providing boundedness of the $\mathbf{L}$-index in joint variables for the function H, if the function f has bounded l-index for some positive continuous function l and $\mathbf{L}(w) = l(\Phi(w))(\max\{1, |\Phi'_{w_1}(w)|\}, \ldots, \max\{1, |\Phi'_{w_n}(w)|\})\colon \mathbb{B}^n \to \mathbb{R}^n_+$.

Such a constructed vector-valued function $\mathbf{L}\colon \mathbb{B}^n \to \mathbb{R}^n_+$ allows us to consider a function Φ with a nonempty zero set for its gradient. The obtained results complement earlier published results with $\operatorname{grad}\Phi(w) = \nabla\Phi(z) = (\frac{\partial\Phi(w)}{\partial w_1}, \ldots, \frac{\partial\Phi(w)}{\partial w_n}) \neq \mathbf{0}$. We prove the following statement (Theorem 3): Suppose that the function $l\colon \mathbb{C} \to \mathbb{R}_+$ satisfies some condition for the regularity of behavior. Let $g\colon \mathbb{C} \to \mathbb{C}$ be an entire function of bounded l-index $N(g, l)$, and $\Phi\colon \mathbb{B}^n \to \mathbb{C}$ be an analytic function such that the vector-valued function $\mathbf{L}(w)\colon \mathbb{B}^n \to \mathbb{R}^n_+$ defined above also satisfies some conditions for regularity of behavior. If there exists $C_2 \geq 1$ such that for all $w \in \mathbb{B}^n$ and for all $J \in \mathbb{Z}^n_+ \setminus \{\mathbf{0}\}$ with $\|J\| \leq N(g, l) + 1$, one has

$$|\Phi^{(J)}(w)| \leq C_2(l(\Phi(w)))^{1/(N(g,l)+1)}(|\nabla|\Phi(w))^J,$$

then the analytic function $H(w) = g(\Phi(w))\colon \mathbb{B}^n \to \mathbb{C}$ has bounded $\mathbf{L}$-index in joint variables.

1. Introduction. We examine some compositions of entire and analytic functions with usage of the notion of $\mathbf{L}$-index in joint variables. Our investigation is a generalization of the results of paper [1] to the case where the first-order partial derivatives of the inner analytic function can vanish in the unit ball, i.e., at least one of them (or each of them) has nonempty zero set (the case of non-vanished gradiend in the unit ball was studied in [3]).

Let us recall some standard notations from [1–6]. Let $\mathbb{R}^n$ and $\mathbb{C}^n$ be n-dimensional real and complex vector spaces, respectively, $n \in \mathbb{N}$. Denote $\mathbb{R}_+ = (0, +\infty)$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$, $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$, $\mathbf{1}_j = (0, \ldots, 0, \underbrace{1}_{j\text{-th place}}, 0, \ldots, 0)$,

$$\mathbb{B}^n = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon |z| = \sqrt{|z_1|^2 + \ldots + |z_n|^2} < 1 \right\}.$$

For $K = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$, let us write $\|K\| = k_1 + \cdots + k_n$, $K! = k_1! \cdot \ldots \cdot k_n!$. For $A = (a_1, \ldots, a_n) \in \mathbb{C}^n$, $B = (b_1, \ldots, b_n) \in \mathbb{C}^n$, we will use formal notations without violation

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of the existence of these expressions $A \pm B = (a_1 \pm b_1, \dots, a_n \pm b_n)$, $AB = (a_1 b_1, \dots, a_n b_n)$, $A/B = (a_1/b_1, \dots, a_n/b_n)$, and if A, $B \in \mathbb{R}^n$, then $A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}$, $\mathbf{max}\{A; B\} = (\max\{a_1; b_1\}, \max\{a_2; b_2\}, \dots, \max\{a_n; b_n\})$, and the notation $A < B$ means that $a_j < b_j$ for all $j \in \{1, \dots, n\}$. Similarly, the relation $A \le B$ is defined.

We denote the K-th order partial derivative of the entire function $F(z) = F(z_1, \dots, z_n)$ by

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1 + \dots + k_n} F}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}, \text{ where } K = (k_1, \dots, k_n) \in \mathbb{Z}_+^n.$$

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z) \colon \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function such that

$$(\forall z \in \mathbb{B}^n)\colon \ l_j(z) > \beta/(1 - |z|), \ j \in \{1, \dots, n\}, \tag{1}$$

where $\beta > \sqrt{n}$ is a some constant.

An analytic function F in the unit ball is called *a function of bounded* $\mathbf{L}$*-index in joint variables* ([3]) if there exists a number $m \in \mathbb{Z}_+$ such that for all $z \in \mathbb{B}^n$ and $J = (j_1, j_2, \dots, j_n) \in \mathbb{Z}_+^n$, one has

$$\frac{|F^{(J)}(z)|}{J! \mathbf{L}^J(z)} \le \max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} \colon \ K \in \mathbb{Z}_+^n, \ \|K\| \le m \right\}. \tag{2}$$

The least integer m for which inequality (2) holds is called the $\mathbf{L}$-index in joint variables of the function F and is denoted by $N(F, \mathbf{L}, \mathbb{B}^n)$. If we will consider entire functions instead analytic functions in the unit ball and $l_j(z_j) \equiv 1, \ j \in \{1, 2, \dots, n\}$, then the entire function F is called a function of bounded index in joint variables or function of bounded index [8]. If $n = 1$ then we obtain a notion $N(g, l)$ of the l-index for entire function g of one complex variable with a continuous function $l \colon \mathbb{C} \to \mathbb{R}_+$. This class of functions was actively studied by M. Sheremeta and his scholars ([11, 13, 17]) as an extension of the index of an entire function ([9]) to a wider class of entire functions whose growth exceed the growth of exponential-type functions. He put the j-th power of the function l in the denominator of the Taylor coefficient from (2).

Many papers are devoted to a composition of two holomorphic functions belonging to different classes and various definitions of the index. Nowadays, the most exhaustively investigated cases are case of the bounded l-index for entire functions of single variable ([10, 11]) and analytic in a disc functions ([12, 13]) and the case of the bounded L-index in a direction for multivariate entire functions ([14]) and analytic functions in a unit ball ([15]). The case of the bounded index in joint variables is more difficult (see [2, 3, 14]). Here, we will try to implement the mentioned approach for the notion of bounded $\mathbf{L}$-index in joint variables and obtained Theorem 3. The first theorem can be applied to consider a nonlinear partial differential equation. For example, we have a composite PDE, make changes to the variables, and transform the equation to a simpler form. If the simpler equation has analytic solutions of bounded index in some sense (see [16, 17]), then we can apply Theorem 3 to learn the properties of the analytic solutions of the composite PDE.

2. Auxiliary propositions. To prove the main theorem, we need an auxiliary proposition. For $R \in \mathbb{R}_+^n$, $j \in \{1, \dots, n\}$ and $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, we define

$$D^n \left[z^0, R/\mathbf{L}(z^0) \right] = \{ z \in \mathbb{C}^n \colon |z_j - z_j^0| \le r_j / l_j(z^0), j \in \{1, \dots, n\} \}$$
$$\lambda_{1,j}(z_0, R) = \inf \left\{ l_j(z)/l_j(z^0) \colon z \in D^n \left[z^0, R/\mathbf{L}(z^0) \right] \right\}, \ \ \lambda_{1,j}(R) = \inf_{z^0 \in \mathbb{C}^n} \lambda_{1,j}(z_0, R),$$

$$\lambda_{2,j}(z_0,R)=\sup\left\{l_j(z)/l_j(z^0)\colon z\in D^n\left[z^0,R/\mathbf{L}(z^0)\right]\right\},\quad \lambda_{2,j}(R)=\sup_{z^0\in\mathbb{C}^n}\lambda_{2,j}(z_0,R).$$

By $Q(\mathbb{B}^n)$ we denote a class of functions $\mathbf{L}(z)$ for which every $R\in\mathbb{R}^n_+$ with $|R|\le\beta$ satisfies the condition $0<\lambda_{1,j}(R)\le\lambda_{2,j}(R)<+\infty$. If $n=1$, then $Q\equiv Q^1$.

Theorem 1 ([4])**.** *Let $\mathbf{L}\in Q(\mathbb{B}^n)$. An analytic function $F\colon\mathbb{B}^n\to\mathbb{C}$ has bounded $\mathbf{L}$-index in joint variables if and only if there exist $p\in\mathbb{Z}_+$ and $C_1\in\mathbb{R}_+$ such that for each $z\in\mathbb{B}^n$*

$$\max\left\{\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)}\colon \|J\|=p+1\right\}\le C_1\cdot\max\left\{\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)}\colon \|K\|\le p\right\}. \tag{3}$$

Theorem 1 was fisrtly deduced by W. K. Hayman ([18]) for single-variate entire functions having bounded index ($n=1$, $\mathbf{L}(z)\equiv 1$). M. M. Sheremeta ([13]) proved it for analytic functions of one variable with finite bounded l-index. Note that Hayman's Theorem is very convenient for investigating the properties of entire solutions of differential equations ([13, 19]).

For an analytic function $\Phi\colon\mathbb{B}^n\to\mathbb{C}$ and $z\in\mathbb{B}^n$ we put

$$\nabla\Phi(z)=\left(\frac{\partial\Phi(z)}{\partial z_1},\ldots,\frac{\partial\Phi(z)}{\partial z_n}\right),\quad |\nabla|\Phi(z)=\left(\left|\frac{\partial\Phi(z)}{\partial z_1}\right|,\ldots,\left|\frac{\partial\Phi(z)}{\partial z_n}\right|\right). \tag{4}$$

Denote by $Q^m_{\mathbf{1}}$ the class of positive continuous functions $L\colon\mathbb{C}^m\to\mathbb{R}_+:=(0,+\infty)$ such that for every $\eta>0$

$$\sup_{z\in\mathbb{C}^m}\sup_{t_1,t_2\in\mathbb{C}}\left\{\frac{L(z+t_1\mathbf{1})}{L(z+t_2\mathbf{1})}\colon |t_1-t_2|\le\frac{\eta}{\min\{L(z+t_1\mathbf{1}),L(z+t_2\mathbf{1})\}}\right\}<+\infty,$$

where $\mathbf{1}=(1,\ldots,1)\in\mathbb{C}^m$.

We recall known results on the composition of analytic functions in the unit ball having bounded $\mathbf{L}$-index in joint variables.

Theorem 2 ([3])**.** *Let $L\in Q^m_{\mathbf{1}}$, $G\colon\mathbb{C}^m\to\mathbb{C}$ be an entire function of bounded L-index in the direction $\mathbf{1}=(1,\ldots,1)\in\mathbb{C}^m$, $\Phi\colon\mathbb{B}^n\to\mathbb{C}$ be analytic function such that $\frac{\partial\Phi(z)}{\partial z_k}\ne 0$ for every $k\in\{1,\ldots,n\}$ and $|\Phi^{(J)}(z)|\le C|\nabla|\Phi(z)^J$, $C\equiv\mathrm{const}>0$, for all $z\in\mathbb{C}^n$, $J\in\mathbb{Z}^n_+$, $\|J\|\le p$, where $p=N(G,\mathbf{L})$.Then $H(z)=G(\Phi(z),\ldots,\Phi(z))$ has bounded $\mathbf{L}$-index in joint variables, where $\mathbf{L}(z)=(L(\Phi(z),\ldots,\Phi(z))|\frac{\partial\Phi(z)}{\partial z_1}|,\ldots,L(\Phi(z),\ldots,\Phi(z))|\frac{\partial\Phi(z)}{\partial z_n}|)\in Q(\mathbb{B}^n)$.*

There exist simple examples of functions which do not satisfy the conditions in Theorem 2. At first, they were presented for entire functions in [1].

3. Results on composition of entire and analytic functions. Removing the condition: $\frac{\partial\Phi(z)}{\partial z_k}\ne 0$ for every $k\in\{1,\ldots,n\}$, in Theorem 2 and slightly increasing the function $\mathbf{L}$, we deduce a new result.

Theorem 3. *Let $l\in Q$ such that $l(z)\ge 1$ for all $z\in\mathbb{C}$, $g\colon\mathbb{C}\to\mathbb{C}$ be an entire function of bounded l-index $N(g,l)$, and $\Phi\colon\mathbb{B}^n\to\mathbb{C}$ be an analytic function, $n\ge 2$, such that*

$$\mathbf{L}\in Q(\mathbb{B}^n),\quad \mathbf{L}(w)=\max\{\mathbf{1},|\nabla|\Phi(w)\}l(\Phi(w)). \tag{5}$$

If there exists $C_2\ge 1$ such that for all $w\in\mathbb{B}^n$ and for all $J\in\mathbb{Z}^n_+\setminus\{\mathbf{0}\}$ with $\|J\|\le N(g,l)+1$, one has

$$|\Phi^{(J)}(w)|\le C_2(l(\Phi(w)))^{1/(N(g,l)+1)}(|\nabla|\Phi(w))^J, \tag{6}$$

then the analytic function $H(w)=g(\Phi(w))\colon\mathbb{B}^n\to\mathbb{C}$ has bounded $\mathbf{L}$-index in joint variables.

Proof. The following formula was proved in [2]:

$$H^{(K)}(w) = g^{(\|K\|)}(\Phi(w))(\nabla\Phi(w))^K + \sum_{j=1}^{\|K\|-1} g^{(j)}(\Phi(w))Q_{j,K}(w), \tag{7}$$

where

$$Q_{j,K}(w) = \sum_{\substack{\mathbf{1}_1 p_{\mathbf{1}_1}+\ldots+\mathbf{1}_n p_{\mathbf{1}_n}+\ldots+K p_K=K \\ 0\le p_{\mathbf{1}_1}+\ldots+p_{\mathbf{1}_n}\le j-1}} c_{j,K,p_{\mathbf{1}_1},\ldots,p_K}(\Phi^{(\mathbf{1}_1)}(w))^{p_{\mathbf{1}_1}}\ldots(\Phi^{(\mathbf{1}_n)}(w))^{p_{\mathbf{1}_n}}\ldots(\Phi^{(K)}(w))^{p_K},$$

and $c_{j,K,p_{\mathbf{1}_1},\ldots,p_K} \in \mathbb{Z}_+$ are some coefficients, $K \in \mathbb{Z}_+^n$. Also, it was deduced [2] that

$$g^{(k)}(\Phi(w)) = \frac{H^{(k\mathbf{1}_i)}(w)}{(\Phi^{(\mathbf{1}_i)}(w))^k} + \frac{1}{(\Phi^{(\mathbf{1}_i)}w))^{2k}}\sum_{j=1}^{k-1} H^{(j\mathbf{1}_i)}(w)(\Phi^{(\mathbf{1}_i)}(w))^j \widetilde{Q}_{j,k}(w), \tag{8}$$

where

$$\widetilde{Q}_{j,k}(w) = \sum_{m_1+\ldots+km_k=2(k-j)} b_{j,k,m_1,\ldots,m_k}(\Phi^{(\mathbf{1}_i)}(w))^{m_1}\ldots(\Phi^{(k\mathbf{1}_i)}(w))^{m_k}, \tag{9}$$

and $b_{j,k,m_1,\ldots,m_k} \in \mathbb{Z}$ are some coefficients, $i \in \{1,\ldots,n\}$, $k \in \mathbb{Z}_+$.

We suppose that the hypothesis of the theorem is satisfied. It means that the entire function $g\colon \mathbb{C} \to \mathbb{C}$ is of bounded l-index, and the analytic function $\Phi\colon \mathbb{B}^n \to \mathbb{C}$ obeys (6). Denote $\widetilde{\mathbf{L}}(w) = |\nabla|\Phi(w)l(\Phi(w))$. Replacing K in (7) by J and dividing it by $(\widetilde{\mathbf{L}}(w))^J$, for any $J \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$ we have

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \le \frac{|g^{(\|J\|)}(\Phi(w))|}{(\widetilde{\mathbf{L}}(w))^J}\cdot|\nabla|\Phi(w))^J + \sum_{k=1}^{\|J\|-1}\frac{|g^{(k)}(\Phi(w))|}{(\widetilde{\mathbf{L}}(w))^J}\cdot|Q_{k,J}(w)|.$$

Substituting $\widetilde{\mathbf{L}}(w) = |\nabla|\Phi(w)l(\Phi(w))$ in the last estimate, we deduce

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \le \frac{|g^{(\|J\|)}(\Phi(w))|}{(|\nabla|\Phi(w))^J(l(\Phi(w)))^{\|J\|}}\cdot|\nabla|\Phi(w))^J + \sum_{k=1}^{\|J\|-1}\frac{|g^{(k)}(\Phi(w))|}{(|\nabla|\Phi(w))^J(l(\Phi(w)))^{\|J\|}}\cdot|Q_{k,J}(w)| \le$$
$$\le \frac{|g^{(\|J\|)}(\Phi(w))|}{(l(\Phi(w)))^{\|J\|}} + \sum_{k=1}^{\|J\|-1}\frac{|g^{(k)}(\Phi(w))|}{(|\nabla|\Phi(w))^J(l(\Phi(w)))^{\|J\|}}\cdot|Q_{k,J}(w)|. \tag{10}$$

By Theorem 1, inequality (3) is valid for $F = g$, $p = N(g,l)$:

$$\frac{|g^{(N(g,l)+1)}(z)|}{(l(z))^{N(g,l)+1}} \le C_1\cdot\max\left\{\frac{|g^{(k)}(z)|}{(l(z))^k}\colon 0\le k\le N(g,l)\right\}.$$

Applying this inequality with $z = \Phi(w)$ to (10), we obtain for $\|J\| = N(g,l)+1$:

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \le \max\left\{\frac{|g^{(k)}(\Phi(w))|}{(l(\Phi(w)))^k}\colon\ 0\le k\le N(g,l)\right\}\times$$
$$\times \max_{\|J\|=N(g,l)+1}\Big(C_1 + \sum_{k=1}^{\|J\|-1}\frac{|Q_{k,J}(w)|}{(|\nabla|\Phi(w))^J(l(\Phi(w)))^{\|J\|-k}}\Big). \tag{11}$$

In view of (6), it is possible to deduce the upper estimate of $|Q_{k,J}(w)|$

$$|Q_{k,J}(w)| \le$$
$$\le \sum_{\substack{\mathbf{1}_1 p_{\mathbf{1}_1}+\ldots+\mathbf{1}_n p_{\mathbf{1}_n}+\ldots+J p_J=J\\ 0\le p_{\mathbf{1}_1}+\ldots+p_{\mathbf{1}_n}\le k-1}} |c_{k,J,p_{\mathbf{1}_1},\ldots,p_J}||\Phi^{(\mathbf{1}_1)}(w)|^{p_{\mathbf{1}_1}}\ldots|\Phi^{(\mathbf{1}_n)}(w)|^{p_{\mathbf{1}_n}}\ldots|\Phi^{(J)}(w)|^{p_J} \le$$
$$\le \sum_{\substack{\mathbf{1}_1 p_{\mathbf{1}_1}+\ldots+\mathbf{1}_n p_{\mathbf{1}_n}+\ldots+J p_J=J\\ 0\le p_{\mathbf{1}_1}+\ldots+p_{\mathbf{1}_n}\le k-1}} |c_{k,J,p_{\mathbf{1}_1},\ldots,p_J}|C_2^{\|J\|}(l(\Phi(w)))^{\|J\|/(N(g,l)+1)}(|\nabla|\Phi(w))^J \le$$
$$\le \hat{c}_{k,J}(l(\Phi(w)))^{\|J\|/(N(g,l)+1)}(|\nabla|\Phi(w))^J, \tag{12}$$

where

$$\hat{c}_{k,J} = C_2^{\|J\|}\sum_{\substack{\mathbf{1}_1 p_{\mathbf{1}_1}+\ldots+\mathbf{1}_n p_{\mathbf{1}_n}+\ldots+J p_J=J\\ 0\le p_{\mathbf{1}_1}+\ldots+p_{\mathbf{1}_n}\le k-1}} |c_{k,J,p_{\mathbf{1}_1},\ldots,p_J}|.$$

We substitute estimate (12) in (11) and use $\|J\| = N(g,l)+1$, that is, $(l(\Phi(w)))^{\|J\|/(N(g,l)+1)} = l(\Phi(w))$. Therefore, the following inequality is valid:

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \le \max\Big\{\frac{|g^{(k)}(\Phi(w))|}{(l(\Phi(w)))^k}\colon\ 0\le k\le N(g,l)\Big\}\times$$
$$\times \max_{\|J\|=N(g,l)+1}\left(C_1+\sum_{k=1}^{\|J\|-1}\frac{\hat{c}_{k,J}(l(\Phi(w)))^{\|J\|/(N(g,l)+1)}(|\nabla|\Phi(w))^J}{(|\nabla|\Phi(w))^J(l(\Phi(w)))^{\|J\|-k}}\right)=$$
$$=\max\Big\{\frac{|g^{(k)}(\Phi(w))|}{(l(\Phi(w)))^k}\colon\ 0\le k\le N(g,l)\Big\}\max_{\|J\|=N(g,l)+1}\left(C_1+\sum_{k=1}^{\|J\|-1}\frac{\hat{c}_{k,J}}{(l(\Phi(w)))^{\|J\|-1-k}}\right).$$

Since $l(\Phi(w)) \ge 1$, one has $(l(\Phi(w)))^{\|J\|-1-k} \ge 1$ for $k \le \|J\|-1$. Thus, for $\|J\| = N(g,l)+1$

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \le C_3\max\Big\{\frac{|g^{(k)}(\Phi(w))|}{(l(\Phi(w)))^k}\colon\ 0\le k\le N(g,l)\Big\}, \tag{13}$$

where $C_3 = \max_{\|J\|=N(g,l)+1}(C_1+\sum_{k=1}^{N(g,l)}\hat{c}_{k,J})$. Dividing equality (8) by $l^k(\Phi(w))$ and estimating by the modulus, we deduce for each $i\in\{1,\ldots,n\}$

$$\frac{|g^{(k)}(\Phi(w))|}{l^k(\Phi(w))} \le$$
$$\le \frac{|H^{(k\mathbf{1}_i)}(w)|}{l^k(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^k}+\frac{1}{|\Phi^{(\mathbf{1}_i)}(w)|^{2k}l^k(\Phi(w))}\sum_{j=1}^{k-1}|H^{(j\mathbf{1}_i)}(w)||\Phi^{(\mathbf{1}_i)}(w)|^j|\widetilde{Q}_{j,k}(w)| =$$
$$= \frac{|H^{(k\mathbf{1}_i)}(w)|}{l^k(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^k}+\sum_{j=1}^{k-1}\frac{\widetilde{Q}_{j,k}(w)|}{|\Phi^{(\mathbf{1}_i)}(w)|^{2k-2j}l^{k-j}(\Phi(w))}\frac{|H^{(j\mathbf{1}_i)}(w)|}{|\Phi^{(\mathbf{1}_i)}(w)|^j l^j(\Phi(w))}.$$

Introducing the maximum of the fraction $\frac{|H^{(j\mathbf{1}_i)}(w)|}{|\Phi^{(\mathbf{1}_i)}(w)|^j l^j(\Phi(w))}$ over $j\in\{1,\ldots,k\}$, we can increase the previous estimate

$$\frac{|g^{(k)}(\Phi(w))|}{l^k(\Phi(w))} \le \max_{1\le j\le k}\left\{\frac{|H^{(j\mathbf{1}_i)}(w)|}{l^j(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^j}\right\}\Big(1+\sum_{j=1}^{k-1}\frac{|\widetilde{Q}_{j,k}(w)|}{l^{k-j}(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^{2(k-j)}}\Big).$$

Substituting the expression from (9) instead of $\widetilde{Q}_{j,k}(w)$, we obtain

$$\frac{|g^{(k)}(\Phi(w))|}{l^k(\Phi(w))} \le \max\left\{\frac{|H^{(j\mathbf{1}_i)}(w)|}{l^j(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^j} : 1 \le j \le k\right\} \times$$
$$\times\left(1 + \sum_{j=1}^{k-1} \sum_{m_1+2m_2+\ldots+km_k=2(k-j)} |b_{j,k,m_1,\ldots,m_k}| \frac{|\Phi^{(\mathbf{1}_i)}(w)|^{m_1}\cdot\ldots\cdot|\Phi^{(k\mathbf{1}_i)}(w)|^{m_k}}{l^{k-j}(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^{2(k-j)}}\right). \tag{14}$$

Estimating (6) and $l(z) \ge 1$ gives us

$$|\Phi^{(s\mathbf{1}_i)}(w)| \le C_2 l^{s/2}(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^s,$$

because $s/2 \ge 1/(N(g,l)+1)$ for $s \in \{1, 2, \ldots, N(g,l)+1\}$. Substituting the right-hand side of this inequality in (14), we deduce

$$\frac{|g^{(k)}(\Phi(w))|}{l^k(\Phi(w))} \le \max\left\{\frac{|H^{(j\mathbf{1}_i)}(w)|}{l^j(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^j} : 1 \le j \le k\right\}\left(1 + \sum_{j=1}^{k-1} \sum_{m_1+2m_2+\ldots+km_k=2(k-j)}\right.$$
$$\left.|b_{j,k,m_1,\ldots,m_k}|(C_2)^{m_1+m_2+\ldots+m_k}\frac{(l(\Phi(w)))^{(m_1+2m_2+\ldots+km_k)/2}|\Phi^{(\mathbf{1}_i)}(w)|^{m_1+2m_2+\ldots+km_k}}{l^{k-j}(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^{2(k-j)}}\right).$$

Since $m_1 + 2m_2 + \ldots + km_k = 2(k-j)$, we obtain $(l(\Phi(w)))^{k-j}|\Phi^{(\mathbf{1}_i)}(w)|^{2(k-j)}$ in the nominator under the sum. The expression matches with the denominator and reduces with it. Thus, it yields

$$\frac{|g^{(k)}(\Phi(w))|}{l^k(\Phi(w))} \le C_4 \max\left\{\frac{|H^{(j\mathbf{1}_i)}(w)|}{l^j(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^j} : 1 \le j \le k\right\}, \tag{15}$$

where

$$C_4 = 1 + \sum_{j=1}^{k-1} \sum_{m_1+2m_2+\ldots+km_k=2(k-j)} |b_{j,k,m_1,\ldots,m_k}|(C_2)^{m_1+m_2+\ldots+m_k}.$$

Then, from inequality (13) and (15), we obtain for each $i \in \{1, \ldots, n\}$ and $\|J\| = N(g,l)+1$

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \le C_5 \max\left\{\frac{|H^{(j\mathbf{1}_i)}(w)|}{l^j(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^j} : 0 \le j \le N(g,l)\right\}, \tag{16}$$

where $C_5 = C_3C_4$. Estimate (16) is established by the assumption that every component of the gradient $\nabla\Phi$ does not vanish, i.e., $\Phi^{(\mathbf{1}_i)}(w) \ne 0$. Our proof is significant for equality (8), providing an estimate of the k-th order derivative of the function g by smaller-order partial derivatives of the function H in the variable w_i. Similarly to [2], by the method of mathematical induction an analog of (8) can be proved for the mixed partial derivative $J \in \mathbb{Z}^n_+$:

$$g^{(\|J\|)}(w) = \frac{H^{(J)}(w)}{(\nabla\Phi(w))^J} + \frac{1}{(\nabla\Phi(w))^{2J}} \sum_{\substack{0<\|K\|\le\|J\|-1,\\ K\le J}} H^{(K)}(w)(\nabla\Phi(w))^K Q^*(w; J, K),$$

where $Q^*(w;J,K)$ is constructed by analogy to $Q_{j,K}(w)$, $\widetilde{Q}_{j,k}(w)$. Then, repeating considerations from (10) to (16) as in [2], we deduce for $\|J\|=N(g,l)+1$

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J}\le C\max\left\{\frac{|H^{(K)}(w)|}{(l(\Phi(w)))^{\|K\|}|\nabla\Phi(w)|^K}\colon 0<\|K\|\le N(g,l), K\le J\right\}, \tag{17}$$

where $C>1$ is a constant.

We should like to point out that $\mathbf{L}(w)=l(\Phi(w))\mathbf{max}\{\mathbf{1},|\nabla|\Phi(w)\}$. We introduce the function $\mathbf{L}$ to inequality (17) for $\|J\|=N(g,l)+1$ in the following form:

$$\frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J}\cdot\frac{(\mathbf{L}(w))^J}{(\widetilde{\mathbf{L}}(w))^J}\le C\max\left\{\frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K}\cdot\frac{(\mathbf{L}(w))^K}{(\widetilde{\mathbf{L}}(w))^K}\colon 0<\|K\|\le N(g,l), K\le J\right\}.$$

Inverting the fraction $\frac{(\mathbf{L}(w))^J}{(\widetilde{\mathbf{L}}(w))^J}$, one has

$$\frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J}\le C\frac{(\widetilde{\mathbf{L}}(w))^J}{(\mathbf{L}(w))^J}\max\left\{\frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K}\cdot\frac{(\mathbf{L}(w))^K}{(\widetilde{\mathbf{L}}(w))^K}\colon 0<\|K\|\le N(g,l), K\le J\right\}.$$

Applying $\max\limits_{a,b\in\mathcal{A}\subset\mathbb{R}_+}\{a\cdot b\}\le\max\limits_{a\in\mathcal{A}\subset\mathbb{R}_+}\{a\}\cdot\max\limits_{b\in\mathcal{A}\subset\mathbb{R}_+}\{b\}$ with a finite set $\mathcal{A}$ to the right-hand side of the inequality, we establish

$$\frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J}\le C\frac{(\widetilde{\mathbf{L}}(w))^J}{(\mathbf{L}(w))^J}\max\left\{\frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K}\colon 0<\|K\|\le N(g,l), K\le J\right\}\times$$
$$\times\max\left\{\frac{(\mathbf{L}(w))^K}{(\widetilde{\mathbf{L}}(w))^K}\colon 0<\|K\|\le N(g,l), K\le J\right\}.$$

Since $\max\{a\colon a\in\mathcal{A}\subset\mathbb{R}_+\}=\frac{1}{\min\{1/a\colon a\in\mathcal{A}\subset\mathbb{R}_+\}}$, the last estimate can be rewritten as

$$\frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J}\le\frac{C(\widetilde{\mathbf{L}}(w)/\mathbf{L}(w))^J\max\left\{\frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K}\colon 0<\|K\|\le N(g,l), K\le J\right\}}{\min\{(\widetilde{\mathbf{L}}(w)/\mathbf{L}(w))^K\colon 0<\|K\|\le N(g,l), K\le J\}}. \tag{18}$$

Let $T_0=T(w)=\widetilde{\mathbf{L}}(w)/\mathbf{L}(w)=\frac{|\nabla|\Phi(w)}{\mathbf{max}\{\mathbf{1},|\nabla|\Phi(w)\}}\in\mathbb{R}^n_+$ and $K_0\le J$, $0<\|K_0\|\le N(g,l)$ ($K_0\in\mathbb{Z}^n_+$) be such that $(T_0)^{K_0}=\min\{T_0^K\colon\ 0<\|K\|\le N(g,l), K\le J\}$. One should observe that $T_0\in(0,1]^n$ and $\|J-K_0\|\ge N(g,l)+1-N(g,l)=1$, and $J-K_0\ge\mathbf{1}_s$ for some $s\in\{1,\dots,n\}$. Hence, $\frac{T_0^J}{T_0^{K_0}}=T_0^{J-K_0}\le T_0^{\mathbf{1}_s}\le 1$.

Therefore,

$$\frac{(\widetilde{\mathbf{L}}(w)/\mathbf{L}(w))^J}{\min\{(\widetilde{\mathbf{L}}(w)/\mathbf{L}(w))^K\colon\ 0<\|K\|\le N(g,l), K\le J\}}=T_0^{J-K_0}\le T_0^{\mathbf{1}_s}\le 1.$$

Thus, from inequality (18), we obtain

$$\frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J}\le C\max\left\{\frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K}\colon 0<\|K\|\le N(g,l), K\le J\right\}. \tag{19}$$

Let $J^* \in \mathbb{Z}_+$ be such that $\max_{\|J\|=N(g,l)+1} \frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J} = \frac{|H^{(J^*)}(w)|}{(\mathbf{L}(w))^{J^*}}$ and $\|J^*\| = N(g,l)+1$. Then, we deduce from (19)

$$\max\left\{\frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J} \colon \|J\| = N(g,l)+1\right\} = \frac{|H^{(J^*)}(w)|}{(\mathbf{L}(w))^{J^*}} \leq$$
$$\leq C\max\left\{\frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K} \colon 0 < \|K\| \leq N(g,l), K \leq J^*\right\} \leq$$
$$\leq C\max\left\{\frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K} \colon 0 < \|K\| \leq N(g,l)\right\} \tag{20}$$

for all w such that $\Phi^{(\mathbf{1}_i)}(w) \neq 0$.

If $\Phi^{(\mathbf{1}_i)}(w) = 0$ for some i, then for any $J \in \mathbb{Z}^n_+$ with $\|J\| \leq N(g,l)+1$ and $J_i \neq 0$ inequality (6) implies $\Phi^{(J)}(w) = 0$. In view of (7), it means $H^{(J)}(w) = 0$. Thus, inequality (20) also holds for the points w belonging to zero for at least one component of the gradient $\nabla\Phi$.

Therefore, by Theorem 1, we conclude that the function H belongs to the class of functions with finite $\mathbf{L}$-index in joint variables. □

Remark 1. On our opinion, it would be interesting to establish an analogue of Theorem 3 for the composition $H(w) = g(\Phi(w)) \colon \mathbb{B}^n \to \mathbb{C}^m$, where $g \colon \mathbb{C} \to \mathbb{C}^m$ is an entire vector-valued function and $\Phi \colon \mathbb{B}^n \to \mathbb{C}$ is an analytic function in the unit ball $\mathbb{B}^n$, or even for the composition of the most general form $H(w) = G(\Phi(w)) \colon \mathbb{B}^n \to \mathbb{C}^m$, where $G \colon \mathbb{C}^k \to \mathbb{C}^m$ is an entire vector-valued function of several complex variables and $\Phi \colon \mathbb{B}^n \to \mathbb{C}^k$ is an analytic vector-valued function in the unit ball $\mathbb{B}^n$. There are few papers on the classes of vector-valued functions of a single variable ([20–22]), and of several variables ([19, 23]). Similar problem also can be considered for functions analytic in the unit polydisc (see auxiliary results in [7] for this class of analytic functions with bounded L-index in direction).

Remark 2. We can not deduce analog of Theorem 3 for a more general composition of entire and analytic functions in the unit ball such as $f(\varphi_1(w_1), \varphi_2(w_2))$ because under usage of methods from this assertion we obtain that $|\varphi_1'(w_1)/\varphi_2'(w_2)|$ or similar expression must be bounded in w_1 and w_2.

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