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# TOTAL PROGENY IN THE NEAR-CRITICAL MULTI-TYPE GALTON-WATSON PROCESSES WITH IMMIGRATION

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In this paper, we study Galton-Watson branching processes with immigration. These processes are an extension of the classical Galton-Watson model, incorporating an additional mechanism where new individuals, called immigrants, enter the population independently of the reproduction dynamics of existing individuals. We focus on the multi-type case, where individuals are classified into several distinct types, and the reproduction law depends on the type. A crucial role in the study of multi-type Galton-Watson processes is played by the matrix $M$, which represents the expected number of descendants of different particle types, and its largest positive eigenvalue, $\rho$. Sequences of branching processes with primitive matrices $M$ and eigenvalues $\rho$ converging to 1 are referred to as near-critical.

Our focus is on the random vector $Y_n$, representing the total number of particles across all generations up to generation $n$, commonly called the total progeny, in near-critical multi-type Galton-Watson processes with immigration. Assuming the double limit $n(\rho-1)$ exists as $n \to \infty$ and $\rho \to 1$, we establish the limiting distribution of the properly normalized vector $Y_n$. This result is derived under standard conditions imposed on the probability generating functions of the offspring and immigration component.

**1. Introduction.** The study of branching processes dates back to the 19th century when Francis Galton and Henry Watson posed the problem of the extinction of aristocratic families. This question had been investigated independently earlier by the French mathematician Irenee-Jules Bienayme. Today, they are used in biology, physics, epidemiology, computer science, genetics, and finance to model branching phenomena and population dynamics. Objects, typically called particles or individuals, represent entities within the system that contribute to the branching structure, whether through reproduction, transformation, or propagation of influence.

One of many generalizations of Galton-Watson processes is allowing a number of distinguishable particles with different probabilistic behavior. Such processes are called multi-type Galton-Watson processes. Those processes can represent genetic types in animal populations, mutant types in bacterial populations, electrons, photons, etc. Another generalization allows particles to enter the process from outside the system, forming what are known as processes with immigration. We refer the reader to the classic papers [1, 6] for fundamental results in Galton-Watson theory. Pakes [11] studied processes describing the total number of particles (called total progeny) living in the critical process. Pakes [12] also studied the total progeny in critical processes with immigration in the one-dimensional case.

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Fahady has studied the near-critical Galton-Watson processes [4] in the one-dimensional case and Quine [13] in the multidimensional case. Inhomogeneous multi-type nearly critical Galton-Watson branching processes with immigration were studied in [14]. Nagaev and Karpenko [7, 8] studied the total descendants in the near-critical one-dimensional processes without immigration. The functional limit theorem for the total progeny in the one-dimensional near-critical processes with immigration was proved in [9] (see also the references there). In addition, large and moderate deviations for the total progeny in the one-dimensional near-critical Galton-Watson processes were investigated in [5] and [10].

In this paper, we derive limiting distributions for the total progeny in multi-type near-critical processes with immigration, a result that, to the best of our knowledge, has not appeared in the literature before.

**2. Main results.** Let $X$ be the set of all $d$-tuples $i = (i_1, \dots, i_d)'$ with non-negative integer entries and let $C = \{s \in X : s_k \le 1 \text{ for all } k \in \{1, \dots, d\}\}$ be the $d$-dimensional unit 'cube'. Set $\mathbf{1} = (1, 1, \dots, 1)'$ and $\mathbf{0} = (0, 0, \dots, 0)'$. Here and subsequently, the prime $'$ denotes the transpose. Given $s, w \in \mathbb{R}^d$, we write $sw$, $s/w$ and $s^w$ instead of $(s_1w_1, \dots, s_dw_d)'$, $(s_1/w_1, \dots, s_d/w_d)'$ and $s_1^{w_1} s_2^{w_2} \cdots s_d^{w_d}$. Let $vw'$ and $v'w$ denote the dot and tensor product of vectors $v$ and $w$ respectively. Then $v'wu$ is a product of a vector $u$ and the scalar $v'w$. For any square matrix $G$ of order $d$ and vector $v$, $v'G$ stands for the standard product of the row $v'$ and the matrix $G$, while $vG$ denotes a new matrix in which the $i$-th row is the corresponding row of $G$ multiplied by $v_i$. We also write $v \le w$ or $v < w$, if $v_k \le w_k$ or $v_k < w_k$ for all $k \in \{1, \dots, d\}$.

The *multi-type Galton-Watson process* is a *homogeneous vector Markov process* $\{T_n\}_{n \in \mathbb{Z}_+}$, where $T_n \in X$ and $T_0 = \mathbf{1}$. Those vectors denote generations of particles, starting from the parent particle, which forms generation $T_0$, descendants of this particle form generation $T_1$, and so on. The $i$-th entry $T_{n,i}$ of the vector $T_n$ is treated as the number of particles of type $i$ in the $n$-th generation.

Let $f_{1,m}$ be the probability generating function (p.g.f.) of $T_1$, provided the process started with a particle of type $m$, i.e., $T_0 = e_m$. Then

$$f_{1,m}(s) = \sum_{i \in X} p_m(i) s^i, \qquad s \in C,$$

where $p_m(i) = P(T_1 = i | T_0 = e_m)$. By $f_{n,m}$ we denote the p.g.f. of $T_n$, provided the process started with a particle of type $m$. Write

$$f = (f_{1,1}, \dots, f_{1,d})', \qquad f_n = (f_{n,1}, \dots, f_{n,d})'.$$

It is well known [6, p.36], that $f_n(s) = f(f_{n-1}(s))$, i.e., $f_{n,m}(s) = f_{1,m}(f_{n-1}(s))$. We also introduce the matrix $M(s, f) = \|M_{kl}(s)\|_{k,l=1}^d$ of the first moments $M_{kl}(s) = \frac{\partial f_{1,k}}{\partial s_l}(s)$ of $T_1$.

Let $\{H_i\}_{i=1}^\infty \subset X$ be i.i.d. vectors with the p.g.f. $g(s) = \sum_{i \in X} p(i) s^i$.

Let $W_l^{n,j}$ be random vectors with p.g.f. $f_{1,j}$. The *Galton-Watson process with immigration* is defined by the sequence

$$Z_0 = H_0, \quad Z_{n+1} = \sum_{j=1}^{d} \sum_{l=1}^{Z_{n,j}} W_l^{n,j} + H_{n+1}, \qquad n = 0, 1, \dots, \tag{1}$$

where vectors $W_l^{n,j}$ belonging to the same generation are independent, and the vectors $H$'s and $W$'s are also assumed to be independent. The random process is uniquely defined by the vector $f$ and function $g$.

The random vector $Y_n = \sum_{j=0}^n Z_j$ is called the *total progeny* of a Galton-Watson process with immigration.

For a given natural number $U$ and positive constants $a$, $b$, $c$, we introduce the class $\mathcal{K} = \mathcal{K}(a, b, c, U)$ of probability generating functions $f$ such that

$$(A)\ [M^U(\mathbf{1}, f)]_{kl} \geq a\ (\forall k, l \in \{1, \dots, d\});$$

$$(B) \sum_{k,l,m=1}^{d} \frac{\partial f_{1,k}^2}{\partial s_l \partial s_m}(\mathbf{1}) \geq b; \qquad (C) \sum_{k,l,m,j=1}^{d} \frac{\partial f_{1,k}^3}{\partial s_l \partial s_m \partial s_j}(\mathbf{1}) \leq c.$$

The first two conditions ensure that the process is positively regular and not singular [6, p. 38]. Given the Perron-Frobenius theorem, the process is also irreducible and the matrix $M(\mathbf{1}, f)$ has the *Perron root* $\rho(f)$, that is a *positive eigenvalue* and *any other eigenvalue* (possibly complex) *in absolute value is strictly smaller than* $\rho(f)$.

We call the Galton-Watson process *subcritical* if $\rho(f) < 1$, *critical* if $\rho(f) = 1$, and *supercritical* if $\rho(f) > 1$.

Given $f \in \mathcal{K}$, let $\mu(f)$ be the vector of extinction probabilities

$$\mu_j(f) = P(\cup_{n \geq 1}\{Z_n = \mathbf{0}\} | Z_0 = e_j), \quad j \in \{1, \dots, d\}.$$

This vector is the smallest root of the equation $f(s) = s$ [6, Theorem 7.1.]. Let $\rho_\mu(f)$ be the Perron root of $M(\mu, f)$. If $\rho(f) \leq 1$, then $\mu(f) = \mathbf{1}$, and therefore $\rho_\mu(f) = \rho(f)$. In the case $\rho(f) > 1$, we have $\mu(f) < \mathbf{1}$, and , in general, the values $\rho_\mu(f)$ and $\rho(f)$ are different.

We introduce the vector of immigration $\lambda(b) = \nabla g(s)|_{s=\mathbf{1}}$ as the gradient of p.g.f. $b$ for $H_i$ at the point $\mathbf{1}$. Let $\mathcal{J} = \mathcal{J}(d_1, d_2)$ be the class of immigration p.g.f.'s $g$, for which the following conditions

$$(A0) \quad g(\mathbf{1}) = 1; \qquad (B0) \quad \sum_{k=1}^{d} \lambda_k(g) \geq d_1; \qquad (C0) \quad \sum_{k,l=1}^{d} \frac{\partial^2 g}{\partial s_k \partial s_l}(\mathbf{1}) \leq d_2$$

hold. Also, for each $f \in \mathcal{K}$, we set

$$q(f) = \frac{1}{2} \sum_{k,l,m=1}^{d} \frac{\partial^2 f_{1,k}}{\partial s_l \partial s_m}(\mathbf{1})\, u_l u_m v_k,$$

where $v = v(f)$ and $u = u(f)$ are the left and the right eigenvectors of $M(\mathbf{1}, f)$ that correspond the Perron root and subject to the normalized conditions $u'\mathbf{1} = 1$ and $u'v = 1$.

Before formulating the main result, we introduce some notation. For $x, y > 0$, we write

$$z_\pm(x, y, f) = \frac{1}{2}\sqrt{1 + \frac{4q(f)y}{x^{\mp 1} - 1 \pm \ln x}}, \qquad w_\pm(x, y, f) = \frac{2z_\pm(x, y, f) \mp 1}{2z_\pm(x, y, f) \pm 1}.$$

In this paper, we study the nearly critical case of the Galton-Watson process. So, we choose a sequence $\{f^{(n)}\}_{n=1}^\infty \subset \mathcal{K}(a, b, c, U)$ of p.g.f.'s such that $f^{(n)} \to f^*$ in $\mathbb{R}^d$ and $\rho(n) = \rho(f^{(n)})$ is close to 1 for large $n$. We assume that the asymptotic formula $\rho(n) = 1 \mp \frac{\ln r}{n} + o\left(\frac{1}{n}\right)$ holds as $n \to \infty$, where $r$ is some fixed number taken from $(0, 1]$. Define

$$\ell_n(r) = \frac{n^2}{2} \quad \text{if } r = 1, \quad \ell_n(r) = \frac{\rho_n^n - 1 + n(1 - \rho_n)}{(1 - \rho_n)^2} \quad \text{if } r < 1 \text{ and } \rho(n) < 1,$$

$$\ell_n(r) = \frac{\rho_{\mu,n}^{-n} - 1 - n(1 - \rho_{\mu,n})}{(1 - \rho_{\mu,n})^2} \quad \text{if } r < 1 \text{ and } \rho(n) > 1,$$

where $\rho_{\mu,n} = \rho_\mu(f^{(n)})$. Let $m_n = \ell_n(r)v(f^{(n)})$. We set $\gamma(f, g) = \lambda'(g)u(f)/q(f)$ and introduce the function

$$\Psi_\pm(\tau, r, f, g) = r^{\pm\frac{1}{2}\gamma(f,g)} \left(\frac{r^{z_\pm(\tau,r,f)} + w_\pm(\tau, r, f) r^{-z_\pm(\tau,r,f)}}{1 + w_\pm(\tau, r, f)}\right)^{-\gamma(f,g)}.$$

Let us choose positive numbers $\tau_1, \dots, \tau_d$ such that $\tau = \sum_{k=1}^{d} \tau_k$ and set

$$s_n = \left(e^{\frac{-\tau_1}{m_{n,1}}}, \dots, e^{\frac{-\tau_d}{m_{n,d}}}\right).$$

**Theorem 1.** *For $f^{(n)} \in \mathcal{K}$ and $g \in \mathcal{J}$, we consider the sequence of the Galton-Watson processes with immigration $Z_n^{(n)}$ given by (1). Let $Y_n$ be the total progeny of the $n$-th random process for the $n$-th generation. If $f^{(n)} \to f^*$ and $\rho(f^*) = 1$, then there exists a limit, as $n \to \infty$, of the probadility generating function $E\left(s_n^{Y_n}\right)$ of the total progeny $Y_n$.*

*Suppose the Perron roots for $f^{(n)}$ admit the asymptotics $\rho(n) = 1 + o\left(\frac{1}{n}\right)$ as $n \to \infty$, then*

$$E\left(s_n^{Y_n}\right) \to \left(\cosh\sqrt{2\tau q(f^*)}\right)^{-\gamma(f^*,g)}, \quad n \to \infty.$$

*If the Perron roots approach 1 more slowly, namely $\rho(n) = 1 \mp \frac{\ln r}{n} + o\left(\frac{1}{n}\right)$ as $n \to \infty$ for some fixed $r \in (0,1)$, then*

$$E\left(s_n^{Y_n}\right) \to \Psi_{\pm}(\tau, r, f^*, g), \quad n \to \infty.$$

**3. Preliminary results.** The proof of Theorem 1 is divided into several lemmas.

By $\mathcal{K}_\rho$ we denote subset of functions $f$ from $\mathcal{K}$ for which the Perron root of $M(\mathbf{1}, f)$ equals to $\rho$. We define $\sigma(n,s) = \sigma(n,s,\rho(f)) = \frac{1}{n} + (1-\rho(f))^2 + |\mathbf{1} - s|$. By $\alpha(n,s) = \alpha(n,s,\rho(f))$ we denote such infinitesimal that $\sup\{\alpha(n,s,\rho(f)) : f \in \mathcal{K}_\rho\} \to 0$, $\sigma(n,s) \to 0$. Unnecessary arguments in $\sigma(n,s)$ and $\alpha(n,s)$ will be dropped. For simplicity, we will assume throughout the paper that all coordinates $\mathbf{1} - s$ converge to $\mathbf{0}$ at the same rate as $\sigma(n,s)$ goes to 0, meaning that there exist positive constants $c_1$ and $c_2$ such that $c_1 \leq \frac{1-s_k}{1-s_m} \leq c_2, \sigma(n,s) \to 0, k, m \in \{1,2,\ldots,d\}$. We also assume that rate of convergence of components $1 - s_k$ of the vector $\mathbf{1} - s$ are at least of the order $O(\min\{\frac{1}{n^2}, \frac{1-\rho(f)}{n}\})$ as $\sigma(n,s) \to 0$. Our choice of normalizing sequences in Theorem 1 guarantees that those assumptions are satisfied.

For matrices $M(s,f)$, let $\rho_s(f)$ be the Perron root; $u_s(f), v_s(f)$ be the right and the left eigenvectors corresponding to $\rho_s(f)$, for which the normalisation conditions $u_s'(f)v_s(f) = u_s'(f)\mathbf{1} = 1$ are also satisfied. By $\widehat{\rho}_s(f)$, we denote the second largest in magnitude eigenvector of $M(s,f)$. If $s$ is sufficiently close to $\mathbf{1}$, then the Perron root exists, as guaranteed by the Perron-Frobenius theorem. To simplify the notation, we will omit the argument $f$ in the defined above functions. Thus, we will write $M(s)$, $\rho_s$, $q$, $w_\pm(\tau,r)$, and similar expressions instead of $M(s,f)$, $\rho_s(f)$, $q(f)$, $w_\pm(\tau,r,f)$, etc., unless it becomes necessary to explicitly specify the dependence on $f$.

**Lemma 1.** *For a fixed $s_0$, $\mathbf{0} < s_0 \leq \mathbf{1}$, there exist constants $0 < \eta = \eta(s_0, a, U), \widehat{d} = \widehat{d}(s_0, c, U, d_2) < +\infty$, such that for all $f \in \mathcal{K}, g \in J$, $k, l, m \in \{1,2,\ldots,d\}$ and $s \in C_{s_0} : \{s : \mathbf{0} < s_0 \leq s \leq \mathbf{1}\}$ the following inequalities hold:*

*i) $\rho_s(f) \geq \eta$; ii) $\frac{\widehat{\rho}_s(f)}{\rho_s(f)} \leq 1 - \eta$; iii) $M_{lm}(s,f) \leq \widehat{d}$; iv) $\frac{\partial^2 f_{1,k}}{\partial s_l \partial s_m}(s) \leq \widehat{d}$; v) $v_{s,k}(f) \leq 1/\eta$; vi) $u_{s,k}(f) \geq \eta$; vii) $v_{s,k}(f) \geq \eta$; viii) $\lambda'(b)u(f) \leq \lambda'(b)\mathbf{1} \leq \widehat{d}$.*

*Proof.* Lemma 1 for $s_0 = \mathbf{1}$ was proved by M.P. Quine in [13, Lemma 3].

Let $d(f^1, f^2) = \max_{k\in\{1,2,\ldots,d\}}(\sup_{i\in X}|p_k(i,f^1) - p_k(i,f^2)|)$. M.P. Quine proves that with this metric $\mathcal{K}$ is a compact, therefore every sequence $f^{(n)}$ has a convergent subsequence $f^{(n_k)}$. Let $f^(n) \to f^*$ in metrics $d(\cdot,\cdot)$. It is easy to show that $M_{ml}(s, f^{(n)}) \to M_{ml}(s, f^*)$ if $f^{(n)} \to f^*$ as $n \to \infty$. Using this fact and condition $(A)$, it is easy to show that for fixed $s \in C$ there exists a constant $a(s)$, such that $[M^U(s,f)]_{ml} \geq a(s), m, l \in \{1,2,\ldots,d\}, f \in \mathcal{K}$. Given that $M_{ml}(s,f)$ are increasing in $s$, $a(s)$ must be non-decreasing. Therefore, for all $s \geq s_0$, we have

$$a(s) \geq a(s_0) > 0. \tag{2}$$

The product of two compact spaces is compact; hence, $C_{s_0} \times \mathcal{K}$ is compact. Since eigenvalues are continuous functions of their matrices, $\rho_s(f)$ attains its lower bound $\eta = \rho_{s^*}(f^*)$

for some point $s^* \in C_{s_0}$ and $f^* \in \mathcal{K}$. Using (2) and the Perron-Frobenius theorem, we ensure its positivity and thus prove $(i)$. The rest of the estimates follow similarly, using Quine's technique. $\square$

Conditions $(i)$ and $(ii)$ of Lemma 1, together with condition $(C)$, ensure that the conditions of the theorem in cite [2] are satisfied for the matrices $\frac{M(s,f)}{\rho_s}$. As a result for all $f \in \mathcal{K}$ and all $s \in C_{s_0}$, there exists a sequence $\delta_n$ converging to zero such that

$$(1-\delta_n)u_s(f)v_s'(f) \le \frac{M^n(s,f)}{\rho_s^n} \le (1+\delta_n)u_s(f)v_s'(f). \tag{3}$$

Now we provide the Taylor expansions for $f(s)$.

The Taylor expansion (or a Taylor-type expansion at the point $\mathbf{1}$, see Joffe and Spitzer [3] and Quine [13]) of the vectors $f(s)$ at a point $s_0$, where $s, s_0 \in C$, $s_0 > s$, gives

$$f(s_0) - f(s) = M(s_0)(s_0 - s) - E(s)(s_0 - s), \tag{4}$$

where $0 \le E(s) \le M(s_0)$, $E(s) \to 0$, $s \to s_0$, $s \le t \implies E(t) \le E(s)$,

$$f(s_0)-f(s) = M(s_0)(s_0-s) - w[s_0, s_0-s] + e_s[s_0, s_0-s] = M(s_0)(s_0-s) - \widehat{w}[s_0, s_0-s], \tag{5}$$

with

$$w_k[s_0, s] = \frac{1}{2}\sum_{l,m=1}^{d} \frac{\partial^2 f_{1,k}}{\partial s_l \partial s_m}(s_0) s_l s_m, \quad 0 \le e_s[\cdot] \le w[\cdot], \quad e_s[s_0, \cdot] \to \mathbf{0} \ \text{ as } s \to s_0. \tag{6}$$

The Taylor expansion (or modification of the Joffe and Sitzer method at the point $\mathbf{1}$) for the gradients $M_k(s) = (M_{k1}(s), \dots, M_{kd}(s))'$ for $k \in \{1, \dots, d\}$ gives

$$M_k(s) - M_k(s_0) = w_k(s_0)(s - s_0) - E_k(s)(s - s_0),$$

where $w_k(s_0) = \|w_{klm}(s_0)\|_{l,m=1}^d$, $w_{klm}(s_0) = \frac{\partial^2 f_{1,k}}{\partial s_l \partial s_m}(s_0)$ and $0 \le E_k(s) \le w_k(s_0)$, $E_k(s) \to \mathbf{0}$, $s \to s_0$, or in matrix form

$$M(s) - M(s_0) = w(s_0, s) - \widehat{E}(s_0, s) = \widehat{w}(s_0, s), \tag{7}$$

where $w(s_0, s) = \|w_{km}(s_0, s)\|_{k,m=1}^d$, $w_{km}(s_0, s) = [w_k(s_0)(s - s_0)]_m$,

$$\widehat{E}(s_0, s) = \|\widehat{E}_{km}(s_0, s)\|_{k,m=1}^d \text{ and } \widehat{E}_{km}(s_0, s) = [E_k(s)(s - s_0)]_m.$$

Finally, we provide the representation for the immigration p.g.f., given by M.P. Quine [13]

$$1 - g(s) = \lambda'(\mathbf{1} - s) - d'[s](\mathbf{1} - s), \tag{8}$$

where $0 \le d_k[s] \le \frac{d_2}{2}\sum_{l=1}^d (1 - s_l)$ $(\forall k \in \{1, 2, \dots, d\})$.

**Lemma 2.** *Let $0 < s_0 \in C$. Then there exist a positive constant $c_m$, such that for any $x, y \in C_{s_0}$ and any $f \in \mathcal{K}$ the following holds*

$$|v_x'(f)M(y, f)u_x(f) - \rho_y(f)| \le c_m|x - y|^2. \tag{9}$$

*Proof.* In view of normalisation $v_s'u_s$, we can rewrite $v_{s_1}'M(s_2)u_{s_1} - \rho_{s_1}$ as

$$v_x'M(y)u_x - \rho_y = v_x'M(y)(u_x - u_y) + (v_x - v_y)'M(y)u_y = (v_x - v_y)'M(y)(u_x - u_y) +$$
$$+\rho_y(v_y'(u_x - u_y) + (v_x - v_y)'u_y) = (v_x - v_y)'M(y)(u_x - u_y) + \rho_y(v_y'u_x + v_x'u_y - 2). \tag{10}$$

The inequalities $(i)$ and $(ii)$ of Lemma 1 guarantee (a consequence of the Implicit function theorem) that the vectors $v_s$ are differentiable functions of $s$ on the compact $C_{s_0} \times \mathcal{K}$. Therefore, we can express their Taylor expansions as

$$v_x = v_y + (V(y) + E_v(x))(x - y), \tag{11}$$

where $V(y) = \left\|V_{kl}(y)\right\|_{k,l\in\{1,\dots,d\}}$, $V_{kl}(y) = \frac{\partial v_k(y)}{\partial y_l}$, $[E_v(x)]_{kl} \to 0, x \to y$. Note that $V_{kl}(x)$ must be bounded on the compact $C_{s_0} \times \mathcal{K}$. Multiplying the transposed equation (11) by $u_y$, we obtain $v_x' u_y - 1 = ((V(y) + E_v(x))(x-y))' \, u_y$. Similarly, we can show that $v_y' u_x - 1 = = -((V(y) + E_v(x))(x-y))' \, u_x$. From these two equations, we get

$$v_y' u_x + v_x' u_y - 2 = ((V(y) + E_v(x))(x-y))' \, (u_y - u_x). \tag{12}$$

The vectors $u_x$ are also differentiable on the compact $C_{s_0} \times \mathcal{K}$, so their derivatives are uniformly bounded, and hence, (12) implies

$$|v_y' u_x + v_x' u_y - 2| \le c_m^1 |x-y|^2. \tag{13}$$

Lemma 1 $(iii)$ also shows that

$$|(v_x - v_y)' M(y)(u_x - u_y)| \le c_m^2 |x-y|^2. \tag{14}$$

The relations (10), (13) and (14) yield (9). □

**Lemma 3.** *Let* $\theta_n = n \ln \rho_\mu$. *If* $\sigma(n) \to 0$, *then*

$$\theta_n = -n|1-\rho|(1+\alpha(n)), \tag{15}$$

*If* $\rho > 1$, *then*

$$\mathbf{1} - \mu = \frac{(\rho - 1)}{q} u(\mathbf{1} + \alpha), \tag{16}$$

$$(1 - \rho_\mu) = (\rho - 1)(1 + \alpha). \tag{17}$$

*Proof.* From [13], we know that

$$\mathbf{1} - f_n(s) = v'(\mathbf{1} - f_n(s))(u + \alpha(n,s)) = \psi_n(\mathbf{1} - s)(u + \alpha(n,s)), s \in C, \tag{18}$$

where $\psi_n(s) = \frac{\rho^n v' s}{1 + \pi_n q v' s}$, $\pi_0 = 0$, $\pi_n = \sum_{j=1}^n \rho^{j-2}$. Since $\mu = f(\mu) = \dots = f_n(\mu)$, we get

$$\mathbf{1} - \mu = \mathbf{1} - f_n(\mu) = \rho^n \frac{v'(\mathbf{1} - \mu)}{1 + \pi_n q v'(\mathbf{1} - \mu)} (u + \alpha(n)). \tag{19}$$

Premultiplying the left and right sides of (19) by $v'$ and taking into consideration the normalisation condition $v'u = 1$, we get the equation $v'(\mathbf{1}-\mu) = \rho^n \frac{v'(\mathbf{1}-\mu)}{1+\pi_n q v'(\mathbf{1}-\mu)}(1+\alpha(n))$. Solving this equation for $v'(\mathbf{1}-\mu)$ gives $v'(\mathbf{1}-\mu) = -(1 - \rho^n(1+\alpha(n)))/q\pi_n$. Since $\rho^n \le \widehat{c}$ as $\sigma(n,s) \to 0$, then $1 - \rho^n(1+\alpha(n)) = (1-\rho^n)(1+\alpha(n))$, we have $v'(\mathbf{1} - \mu) = (\rho - 1)q^{-1}(1 + \alpha(n))$, which gives (16) by the first equality in (18) ($\alpha(n) = \alpha$, since $\mathbf{1} - \mu$ doesn't directly depend on $n$).

Since $\mathbf{1} - \mu = \alpha$ by (16), and $\rho_s$ are differentiable functions of $s$ on the compact, it must be the case that $\rho = \rho_\mu + \alpha$. Therefore, to prove (17), it is sufficient to show that $\rho_\mu < 1$.

Suppose the opposite $\rho_\mu > 1$. Let $u_\mu$ be the right eigenvector of $\rho_\mu$, which we know to be positive. Take a small $\beta > 0$, $k \in \{1, 2, \dots, d\}$, then

$$f_{1,k}(\mu - \beta u_\mu) = f_{1,k}(\mu) - \beta \sum_{l=1}^d M_{kl}(\mu) u_{\mu,l} + o(\beta) =$$
$$= \mu_k - \beta \rho_\mu u_{\mu,k} + o(\beta) = \mu_k - \beta u_{\mu,l} - (\rho_\mu - 1)\beta u_{\mu,l}(1 + o(1)) < \mu_k - \beta u_{\mu,l}. \tag{20}$$

The relation (20) implies that the map $f = f(s)$ maps the set $\mathbf{0} \le s \le \mu - \alpha u_\mu$ to itself. Therefore, by Brauer's theorem, there exists a fixed point $s^* = f(s^*)$ on $\mathbf{0} \le s \le \mu - \beta u_\mu$. But $\mu$ itself is the smallest fixed point on $\mathbf{0} \le s \le \mathbf{1}$, which contradicts our assumption that $\rho_\mu > 1$ and proves (17).

The relation (15) follows directly from (17) and the Taylor expansion for the logarithm. □

It is known [15, p. 7] that $t_0(s) = s$ and $t_n(s) = sf(t_{n-1}(s))$. Additionally, according to Pakes [11], there exists a unique solution $h^* = h^*(s)$ to the equation in the one-dimensional case $h(s) = sf(h(s)), s < \mathbf{1}$. This solution is the limit of a strictly decreasing sequence $t_n(s)$. The same result can easily be extended to the multi-dimensional case (Wang proves similar equality in [15]).

**Lemma 4.** *Let $\sigma(n,s) \to 0$. Then for $s \in C_{s_0}$,*

$$v'_\mu \mu(\mathbf{1} - s) = v'_\mu(\mu - h^*)\alpha(s), \quad v'_\mu(\mu - h^*) = \alpha(s), \tag{21}$$

$$v'_{h^*}(t_n(s) - h^*) = \alpha(n,s). \tag{22}$$

*Proof.* Expansion (5) gives

$$h^* = sf(h^*) = s\left(\mu - M(\mu)(\mu - h^*) + \widehat{w}[\mu, \mu - h^*]\right). \tag{23}$$

Let $s_m = \min_{k\in\{1,2,\dots,d\}}(s_k)$. Then, after subtracting $sh^*$ from the previous equation and premultiplying it by $v'_\mu$, we get

$$\begin{aligned} &v'_\mu\left(s(\mu - M(\mu)(\mu - h^*)) - sh^*\right) = v'_\mu s((I - M(\mu))(\mu - h^*)) \geq \\ &\geq s_m v'_\mu((I - M(\mu))(\mu - h^*)) = s_m(1 - \rho_\mu)v'_\mu(\mu - h^*) \geq 0. \end{aligned} \tag{24}$$

Since $h^*(s) \leq h^*(\mathbf{1}) = \mu$, relations (23) and (24) imply $v'_\mu s\widehat{w}[\mu, \mu - h^*] \leq v'_\mu h^*(\mathbf{1} - s) \leq v'_\mu \mu(\mathbf{1} - s)$. Using (23), the previous inequality, and (17), we derive $v'_\mu(\mu - h^*) = v'_\mu\mu(\mathbf{1} - s) + v'_\mu s\left(M(\mu)(\mu - h^*) - \widehat{w}[\mu, \mu - h^*]\right) = v'_\mu\mu(\mathbf{1} - s) + v'_\mu(\mu - h^*)(1 + \alpha(s))$.
From those equations follows the first part of (21). Using (5), we have

$$\mu - h^* = \mu - sf(h^*) = \mu(\mathbf{1} - s) + s\left(M(\mu) - E_{h^*}\right)(\mu - h^*),$$

from which follows $v'_\mu(\mu - h^*) \leq v'_\mu(\mathbf{1} - s) + v'_\mu M(\mu)(\mu - h^*) = v'_\mu(\mathbf{1} - s) + \rho_\mu v'_\mu(\mu - h^*)$. Thus, $v'_\mu(\mu - h^*) \leq v'_\mu(\mathbf{1} - s)/(1 - \rho_\mu)$. Since the components of $\mathbf{1} - s$ have rate of at least $\frac{1-\rho}{n}$ as $\sigma(n,s) \to 0$, last inequality and Lemma 1 (*vii*) prove the second part of (21).

Since $h^* \leq t_n(s), \mu \leq \mathbf{1}$, $\mu - h^* = \alpha(s)$ by (21), and $\mathbf{1} - \mu = \alpha$ by (16), we obtain (22). □

**Lemma 5.** *Let $\sigma(n,s) \to 0$, then*

$$\mu - h^* = v'_\mu(\mu - h^*)u_\mu(\mathbf{1} + \alpha(s)). \tag{25}$$

$$t_n(s) - h^* = v'_{h^*}(t_n(s) - h^*)u_{h^*}(\mathbf{1} + \alpha(n,s)); \tag{26}$$

*Proof.* Let us proceed to prove (25). Choose sequence $f^n \in \mathcal{K}$, for which $\rho(f^n) \to 1$. Denote $\mu(f^n) = \mu_n, h^*(f^n) = h^*_n$. Since $\mu = f(\mu)$ and $h^* = sf(h^*)$, using (4) repeatedly, we derive

$$\begin{aligned} &\mu_n - h^*_n = \mu_n(\mathbf{1} - s) + s(M(\mu_n) - E(h^*_n))(\mu_n - h^*_n) = \dots \\ &\dots = \sum_{k=0}^{n-1}\left(s(M(\mu_n) - E(h^*_n))\right)^k \mu_n(\mathbf{1} - s) + \left(s(M(\mu_n) - E(h^*_n))\right)^n(\mu_n - h^*_n). \end{aligned}$$

From the remark at the beginning of the chapter, $1 - s_j = O(\min\{\frac{1}{n^2}, \frac{1-\rho}{n}\})$, we see that $s_j^k = 1 + \alpha(n,s), k \in \{0, 1, .., n\}$. Then the previous inequality gives

$$\mu_n - h^*_n = \Big(\sum_{k=0}^{n-1}(M(\mu_n) - E(h^*_n))\Big)^k \mu_n(\mathbf{1} - s) + (M(\mu_n) - E(h^*_n))^n(\mu_n - h^*_n)(\mathbf{1} + \alpha(n,s)). \tag{27}$$

Let $P(n) = \rho^{-1}_{\mu_n}M(\mu_n)$, $A(n,s) = \rho^{-1}_{\mu_n}E(h^*_n), B(n,s) = P(n) - A(n,s)$. From (3) it follows that $P^m(n) \to R(n) = u'_{\mu_n}v_{\mu_n}$, $m \to \infty$, uniformly for $n$. Having this, a simple but tedious application of Quine' and Spitzer's methods to (27), gives (25).

The proof of (26) follows similarly. □

We define $a_* = a_*(s) = \rho_{h^*} v'_{h^*} s u_{h^*}$, $q_{h^*} = v'_{h^*} s w[h^*, u_{h^*}]$,
$R_1 = R_1(s,n) = q_{h^*}(1 - a_*^n)$, $\quad R_2 = R_2(s) = (1 - a_*)/v'_{h^*}(s - h^*)$, $\quad R = R_1/R_2$.
To prove Lemma 7 and the main theorem, we need the next lemma from [7].

**Lemma 6** (Karpenko, Nagaev [7])**.** *Let $\pi_k(\rho, s)$ be a sequence of non-negative functions (or vector functions) such that* $\overline{\lim}_{\sigma(n,s)\to 0} \sup\{\pi_k(\rho,s)\colon f \in \mathcal{K}_\rho\} < \infty$ *for all $k \in \mathbb{N}$ and*

$$\lim_{\sigma(n,s)\to 0} \inf \left\{ \sum_{k=1}^n \pi_k(\rho,s)\colon f \in \mathcal{K}_\rho \right\} = \infty.$$

*Then* $\sum_{k=1}^n \pi_k(\rho,s)\alpha(k,s) = \sum_{k=1}^n \pi_k(\rho,s)\alpha(n,s)$.

Define

$$q(s_0) = q(s_0, f) = \frac{1}{2} \sum_{k,l,m=1}^{d} \frac{\partial^2 f_{1,k}}{\partial s_l \partial s_m}(s_0)\, u_{s_0,l} u_{s_0,m} v_{s_0,k},$$

$$W = W(s, \rho_\mu) = \frac{4q(\mu) v'_\mu(\mathbf{1} - s)}{(1 - \rho_\mu)^2}, \quad V = V(s, \rho_\mu) = \sqrt{1 + W}.$$

**Lemma 7.** *Let $\sigma(n,s) \to 0$, then*

$$t_n(s) - h^* = \frac{(1 - a_*) a_*^n (1 + \alpha(n,s))}{R_2(1 - R(1 + \alpha(n,s)))}(\mathbf{1} + \alpha(n,s)); \tag{28}$$

$$\mu - h^* = \frac{(1 - \rho_\mu)(V - 1)}{2q(\mu)} u_\mu(\mathbf{1} + \alpha(s)). \tag{29}$$

*Proof.* Relations (5), (6), and (26) give

$$t_{n+1}(s) - h^* = sM(h^*)(t_n(s) - h^*) + (v'_{h^*}(t_n(s) - h^*))^2 s w[h^*, u_{h^*}](1 + \alpha(n,s)). \tag{30}$$

After multiplying (30) from the left-hand side by $v'_{h^*}$, we obtain

$$v'_{h^*}(t_{n+1}(s) - h^*) = \rho_{h^*} v'_{h^*}(h^* - h_n(s)) \frac{v'_{h^*} s M(h^*)(t_n(s) - h^*)}{\rho_{h^*} v'_{h^*}(t_n(s) - h^*)} + \\ + (v'_{h^*}(t_n(s) - h^*))^2 q_{h^*}(1 + \alpha(n,s)). \tag{31}$$

Taking into account the normalisation $v'_{h^*} u_{h^*} = 1$ and using (26), we get

$$\rho_{h^*} \frac{v'_{h^*} s M(h^*)(t_n(s) - h^*)}{\rho_{h^*} v'_{h^*}(t_n(s) - h^*)} = a_* \Big(1 - \Big(1 - \frac{(v_{h^*} s)' u_{h^*}(\mathbf{1} + \alpha(n,s))}{(v_{h^*} s)' u_{h^*} v'_{h^*} u_{h^*}(\mathbf{1} + \alpha(n,s))}\Big)\Big) = \\ = a_* \Big(1 - \frac{((v_{h^*}(\mathbf{1} - s))' - (v_{h^*}(\mathbf{1} - s))' u_{h^*} v'_{h^*}) u_{h^*} \alpha(n,s)}{(v_{h^*} s)' u_{h^*} v'_{h^*} u_{h^*}(\mathbf{1} + \alpha(n,s))}\Big).$$

Let $x_n = v'_{h^*}(t_n(s) - h^*)$, $b_n(s) = 1 - \frac{((v_{h^*}(\mathbf{1}-s))' - (v_{h^*}(\mathbf{1}-s))' u_{h^*} v'_{h^*}) u_{h^*} \alpha(n,s)}{(v_{h^*} s)' u_{h^*} v'_{h^*} u_{h^*}(\mathbf{1} + \alpha(n,s))}$. Then, using the previous equation and (31), we obtain the equation

$$x_{n+1} = a_* b_n(s) x_n + q_{h^*} x_n^2 (1 + \alpha(n,s)). \tag{32}$$

Now proceed as in [7, p. 442]. Let $y_n = 1/x_n$. Then (32) give

$$y_n = a_* b_n(s) y_{n+1} + q_{h^*} \frac{x_n}{x_{n+1}}(1 + \alpha(n,s)). \tag{33}$$

The relation (22) shows that $\frac{x_n^2}{x_{n+1}} = \alpha(n)$. Note that $a_* = 1+\alpha(s), q_{h^*} = q(h^*)+\alpha(s)$. Assumption made on the rate of convergence of $\mathbf{1}-s$ and parts $(v)$-$(vii)$ of Lemma 1 shows that $\prod_{l=1}^{k} b_l(s) = 1+\alpha, k\in\{1,\dots,n\}$. This and (32) gives $\frac{x_n}{x_{n+1}} = \frac{1+\alpha(n,s)}{a_* b_n(s)} = 1+\alpha(n,s)$, so we can rewrite (33) as $y_{n+1} = a_*^{-1}b_n^{-1}(s)y_n - q_{h^*}(1+\alpha(n,s))$. Iteration of this equation yields

$$y_{n+1} = -q_{h^*}a_*^{-n}\sum_{k=0}^{n} a_*^k \prod_{l=0}^{k} b_{n-l}^{-1}(1+\alpha(k,s)) + a_*^{-n-1}\prod_{l=0}^{n} b_{n-l}^{-1}(s)y_0.$$

So, using Lemma 6, we get $y_{n+1} = -q_{h^*}a_*^{-n}\sum_{k=0}^{n} a_*^k(1+\alpha(n,s)) + a_*^{-n-1}y_0(1+\alpha)$, and consequently

$$x_n = \Big(a_*^{-n}x_0^{-1} - q_{h^*}\frac{(a_*^{-n}-1)}{(a_*-1)}\Big)^{-1}(1+\alpha(n,s)). \tag{34}$$

Using the fact that $x_0 = s-h^*$ and rearranging (34) we get that

$$v'_{h^*}(t_n(s)-h^*) = \frac{(1-a_*)a_*^n(1+\alpha(n,s))}{R_2(1-R(1+\alpha(n,s)))}. \tag{35}$$

This relation, along with (26), prove (28).

Representations (5) for $h^*$ and (21) give $h^* = s(\mu - M(\mu)(\mu-h^*)+w[\mu,\mu-h^*](1+\alpha(s)))$. Let $x = v'_\mu(\mu-h^*)$. Applying similar steps to (31)-(32), we get the equation

$$q(\mu)x^2(1+\alpha(s)) + (1-\rho_\mu)x - v'_\mu(\mathbf{1}-s)(1+\alpha(s)) = 0.$$

This equation has one positive solution

$$x = (1-\rho_\mu)\frac{\sqrt{1+W(1+\alpha(s))^2}-1}{2q(\mu)(1+\alpha(s))}.$$

The relation (29) follows directly from this representation. □

**Lemma 8.** *If* $-\theta_n(V-1)\le\widehat{c}$, $-\theta_n V\ge\widehat{c}$ *and* $\sigma(n,s)\to 0$, *then*

$$1-a_* = (1-\rho_\mu)V(1+\alpha(s)); \tag{36}$$

$$a_*^n = e^{\theta_n V}(1+\alpha(n,s)), \quad 1-a_*^n = (1-e^{\theta_n V})(1+\alpha(n,s)). \tag{37}$$

*Proof.* Using (21), we can rewrite the representation (7) as

$$M(h^*) = M(\mu) + w(h^*,\mu)(1+\alpha(s)).$$

Multiplying this equation by $v'_\mu$ from the left-hand side and by $u_\mu$ from the right-hand side, and using (29), we get

$$v'_\mu M(h^*)u_\mu = \rho_\mu - (1-\rho_\mu)(V-1)(1+\alpha(s)). \tag{38}$$

Next we multiply the equation (38) by $(v_{h^*}s)'u_{h^*}$ and rearrange to get

$$(v_{h^*}s)'u_{h^*}v'_\mu M(h^*)u_\mu = \rho_\mu - (1-\rho_\mu)(V-1)\Big(1+\rho_\mu\frac{(1-(v_{h^*}s)'u_{h^*})}{(1-\rho_\mu)(V-1)}+\alpha(s)\Big).$$

Simple calculations show that $\rho_\mu\frac{(1-(v_{h^*}s)'u_{h^*})}{(1-\rho_\mu)(V-1)} = \alpha(s)$. Therefore, we have

$$(v_{h^*}s)'u_{h^*}v'_\mu M(h^*)u_\mu = \rho_\mu - (1-\rho_\mu)(V-1)(1+\alpha(s)). \tag{39}$$

The relations (9) and (29) show that

$$v'_\mu M(h^*)u_\mu - \rho_{h^*} = O\left((1-\rho_\mu)^2(V-1)^2\right) \tag{40}$$

and thus,

$$1-a_* = 1-(v_{h^*}s)'u_{h^*}v'_\mu M(h^*)u_\mu + (v_{h^*}s)'u_{h^*}v'_\mu M(h^*)u_\mu - a_* = (1-\rho_\mu)V(1+\alpha(s)), \tag{41}$$

proving (36). The relations (40) and (41) imply that we can replace the left-hand side of (39) by $a_*$. We can express $\rho_\mu - 1$ as $\ln \rho_\mu(1+\alpha)$ and get $a_* = \rho_\mu(1+(V-1)\ln \rho_\mu(1+\alpha(s)))$. Given the first condition of the lemma, $-(V-1)\ln \rho_\mu = \alpha(n,s)$, so we can use the approximation, $\ln(1+(V-1)\ln \rho_\mu(1+\alpha(s))) = (V-1)\ln \rho_\mu(1+\alpha(n,s))$. Thus, $a_*^n = \rho_\mu^n e^{\theta_n(V-1)}(1+\alpha(n,s)) = e^{\theta_n V}(1+\alpha(n,s))$, which proves (37). The second part of (37) follows from the previous equation and the second condition of the lemma. □

**Lemma 9.** *If $\rho \gtrless 1$ and $\sigma(n,s) \to 0$, then in the conditions of Lemma 8,*

$$t_n(s) - h^* = \frac{V(V \pm 1)e^{V\theta_k}(1-\rho_\mu)}{q(V \mp 1 + (V \pm 1)e^{V\theta_k})} u(\mathbf{1} + \alpha(n,s)), \tag{42}$$

$$s - h^* = \frac{(1-\rho_\mu)(V \pm 1)}{2q} u(\mathbf{1} + \alpha(s)). \tag{43}$$

*Proof.* The relations $\mathbf{1} - \mu = \alpha$, $\mu - h^* = \alpha(s)$ and the continuity of $w(s)$ and $u_s$ give

$$q(\mu) = q \cdot (1+\alpha), \quad u_\mu = u(\mathbf{1} + \alpha); \tag{44}$$

$$q(h^*) = q \cdot (1+\alpha(s)), \quad u_{h^*} = u(\mathbf{1} + \alpha(s)). \tag{45}$$

Now, the representation (42) follows directly from (28), using (36), (37) as well as (45).

To prove (43) we represent $s - h^*$ as

$$s - h^* = s(\mathbf{1} - f(h^*)) = s(\mathbf{1} - \mu + \mu - h^* + h^* - f(h^*)) = s(\mathbf{1} - \mu + \mu - h^* - f(h^*)(\mathbf{1} - s)).$$

The relation (21) allows us to rewrite this equation as

$$s - h^* = s(\mathbf{1} - \mu + (\mu - h^*)(\mathbf{1} + \alpha(s))). \tag{46}$$

Splitting (46) into cases with $\rho \gtrless 1$, and using (16), (17), (29), (44), yields (43). □

**4. Proof of the theorem.** By $\phi_n(s)$ we denote p.g.f. of the total progeny in $n$-th generation for processes with immigration. It satisfies (see for example [12]) next functional equation

$$\phi_n(s) = \prod_{k=0}^{n-1} g(t_k(s)). \tag{47}$$

*Proof.* From (47), we have $E\big(\exp\{-\sum_{k=1}^d \tau_k Y_{n,k}/m_{n,k}\}\big) = \phi_n(s_n)$. Note that since $\tau/m_n = \alpha(n)$ as $\sigma(n) \to 0$, we have

$$1 - s_n = \frac{\tau_j}{m_{n,j}}(1 + t_j \alpha(n)). \tag{48}$$

Then, similarly to the one-dimensional case (Pakes [12, p. 287]),

$$\ln \phi_n(s_n) = -\sum_{k=0}^{n-1}(1 - g(t_k(s_n))) - \sum_{k=0}^{n-1} r_{n,k}(s_n),$$

where

$$0 \le r_{n,k}(s_n) \le \frac{(1 - g(\tau_k(s_n)))^2}{g(\tau_k(s_n))} \le \frac{(1 - g(\tau_k(s_n)))\,(1 - g(h^*(s_n)))}{g(h^*(s_n))}, \ \sigma(n) \to 0.$$

Using the expansion (8), we get

$$\ln \phi_n(s_n) = -\sum_{k=0}^{n-1} \lambda'(\mathbf{1} - t_k(s_n)) + \sum_{k=0}^{n-1} d'[t_k(s_n)](\mathbf{1} - t_k(s_n)) - \sum_{k=0}^{n-1} r_{n,k}(s_n), \tag{49}$$

where the second and the third sums tend to zero if the first sum is bounded.

Note that by (26), (43) and (45), we have $s_n - t_k(s_n) = v'(s_n - t_k(s_n))u(\mathbf{1} + \alpha(k, s_n))$. From this equation and Lemma 1 (*viii*), we also get that $n\lambda'(\mathbf{1} - s_n) \to 0$ as $\sigma(n) \to 0$. Then equation (49) can be rewritten as

$$\ln \phi_n(s_n) = -\lambda' u \sum_{k=0}^{n-1} v'(s_n - t_k(s_n))(1 + \alpha(k, s_n)) + \sum_{k=0}^{n-1} d'[t_k(s_n)](\mathbf{1} - t_k(s_n)) - \sum_{k=0}^{n-1} r_{n,k}(s_n). \tag{50}$$

First, consider the case where $r < 1$. Combined representation for $\ell_n(r)$ gives

$$m_n = \frac{\rho_{\mu,n}^{\mp n} - 1 \mp n(1 - \rho_{\mu,n})}{(1 - \rho_{\mu,n})^2} v \quad \text{for} \quad \rho \gtrless 1.$$

From this, (44) and (48), it follows that $W = 4q\tau(1 + \tau\alpha(n))/\big(\rho_{\mu,n}^{\mp n} - 1 \mp n(1 - \rho_{\mu,n})\big)$, and thus

$$V = \sqrt{1 + 4q\tau \frac{1 + \tau\alpha(n)}{\rho_{\mu,n}^{\mp n} - 1 \mp n(1 - \rho_{\mu,n})}} = z_\pm(\tau(1 + \tau\alpha(n)), e^{\theta_n}). \tag{51}$$

We express $v'(s_n - t_k(s_n))$ as $v'(s_n - h^*) + v'(h^* - t_k(s_n))$. From (43) we get

$$v'(s_n - h^*) = \frac{(1 - \rho_{\mu_n})(V \pm 1)}{2q}(1 + \alpha(n)),$$

and therefore

$$\sum_{k=0}^{n-1} v'(s_n - h^*) = n\frac{(1 - \rho_{\mu_n})(V \pm 1)}{2q}(1 + \alpha(n)) \to -\frac{(z_\pm(\tau, r, f^*) \pm 1)\ln r}{2q}, \sigma(n) \to 0, \tag{52}$$

since $e^{\theta_n} \to r$ as $\sigma(n) \to 0$ by (15).

We are again under the conditions of Lemma 8. Combining (35), (36), (37), and (43), we obtain

$$v'(t_k(s_n) - h^*(s_n)) = \frac{V(V \pm 1)e^{V\theta_k}(1 - \rho_{\mu_n})}{q(V \mp 1 + (V \pm 1)e^{V\theta_k})}(1 + \alpha(k, s_n)). \tag{53}$$

Note that $n(1 - \rho_{\mu_n}) = -\ln r(1 + \alpha(n))$. Using this, (42), (51), and Lemma 6 we get

$$\sum_{k=0}^{n-1} v'(t_k(s_n) - h^*) =$$

$$= -\frac{\ln r}{qn} \sum_{k=0}^{n-1} \frac{z_\pm(\tau(1 + \tau\alpha(n)), e^\theta)(z_\pm(\tau(1 + \tau\alpha(n)), e^\theta) \pm 1)r^{kz_\pm(\tau(1+\tau\alpha(n)),e^\theta)/n}(1 + \alpha(k, s_n))}{z_\pm(\tau(1 + \tau\alpha(n)), e^\theta) \mp 1 + r^{z_\pm(\tau(1+\tau\alpha(n)),e^\theta)k/n}(z_\pm(\tau(1 + \tau\alpha(n)), e^\theta) \pm 1)}$$

$$\underset{\sigma(n)\to 0}{\longrightarrow} -\frac{\ln r}{q(f^*)} \int_0^1 \frac{z_\pm(\tau, r, f^*)(z_\pm(\tau, r, f^*) \pm 1)r^{xz_\pm(\tau,r,f^*)}}{z_\pm(\tau, r, f^*) \mp 1 + r^{xz_\pm(\tau,r,f^*)}(z_\pm(\tau, r, f^*) \pm 1)} dx =$$

$$= -\frac{1}{q(f^*)} \ln \frac{w_\pm(\tau, r, f^*) + r^{z_\pm(\tau,r,f^*)}}{1 + w_\pm(\tau, r, f^*)}, \tag{54}$$

where $\theta \equiv \theta_n$. Combining (50), (52), (54) and Lemma 1 (*viii*) yields the result.

Consider the case $r = 1$. Representation for $\ell_n(r)$ in this case, together with the relations (15), (44) and (48), show that $W = \frac{8q\tau(1+\tau\alpha(n))}{\theta_n^2} \to \infty$ and $V = \sqrt{8q\tau(1 + \tau\alpha(n))}/\theta_n$. Then, by (43)

$$\sum_{k=0}^{n-1} v'(s_n - h^*) = \sqrt{\frac{2\tau}{q}}(1 + \tau\alpha(n)) \underset{\sigma(n)\to 0}{\longrightarrow} \sqrt{\frac{2\tau}{q(f^*)}}. \tag{55}$$

The relations (43), (53) and Lemma 6 yield

$$\sum_{k=0}^{n-1} v'(t_k(s_n) - h^*) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\sqrt{8q\tau} e^{-\sqrt{8q\tau}k/n}}{q\left(1 + e^{-\sqrt{8q\tau}k/n}\right)} (1 + \alpha(n)(s_n))$$

$$\underset{\sigma(n)\to 0}{\longrightarrow} \frac{1}{q(f^*)} \int_0^1 \frac{2\sqrt{2q(f^*)\tau} e^{-2x\sqrt{2q(f^*)\tau}}}{1 + e^{-2x\sqrt{2q(f^*)\tau}}} dx = -\frac{1}{q(f^*)} \ln \frac{1 + e^{-2\sqrt{2q(f^*)\tau}}}{2}. \qquad (56)$$

Combining (50), (55) and (56) yields the result. □

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