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FURTHER RESULTS ON LEFT AND RIGHT GENERALIZED DRAZIN INVERTIBLE OPERATORS

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In this paper we present some new characteristics and expressions of left and right generalized Drazin invertible bounded operators on a Banach space X. An explicit formula relating the left and the right generalized Drazin inverses to spectral idempotents is provided. In addition, we give a characterization of operators in $\mathcal{B}_l(X)$ (resp. $\mathcal{B}_r(X)$) with equal spectral idempotents, where $\mathcal{B}_l(X)$ (resp. $\mathcal{B}_r(X)$) denotes the set of all left (resp. right) generalized Drazin invertible bounded operators on X. Next, we give some sufficient conditions which ensure that the product of elements of $\mathcal{B}_l(X)$ (resp. $\mathcal{B}_r(X)$) remains in $\mathcal{B}_l(X)$ (resp. $\mathcal{B}_r(X)$). Finally, we extend Jacobson's lemma for left and right generalized Drazin invertibility. The provided results extend certain earlier works given in the literature.

1. Introduction. Let X (resp. H) be an infinite-dimensional complex Banach (resp. Hilbert) space throughout this paper. $\mathcal{B}(X)$ (resp. $\mathcal{B}(H)$) denotes the set of all bounded linear operators on X (resp. H). For $A \in \mathcal{B}(X)$ write $\mathcal{N}(A)$, $\mathcal{R}(A)$ and $\sigma(A)$ as the null space, the range and the spectrum of A, respectively. A^* is the adjoint operator of A and I denotes the identity operator on X. If M is an A-invariant subspace of X, then A_M denotes the restriction of A to M. For a subset Ω of $\mathbb{C}$ we write $\operatorname{iso}(\Omega)$ for its isolated points and $\operatorname{acc}(\Omega)$ for its accumulation points. Further, $\sigma_{\mathrm{ap}}(A)$ is the approximate point spectrum of A and $\sigma_{\mathrm{su}}(A)$ is its surjectivity spectrum

$$\sigma_{\mathrm{ap}}(A) = \{\lambda \in \mathbb{C}\colon \lambda I - A \text{ is not bounded below}\},$$
$$\sigma_{\mathrm{su}}(A) = \{\lambda \in \mathbb{C}\colon \lambda I - A \text{ is not surjective}\}.$$

$A \in \mathcal{B}(X)$ is said to be bounded below if there exists $M > 0$ such that $\|Ax\| \geq M\|x\|$ for all $x \in X$, which is also equivalent to A injective and $\mathcal{R}(A)$ is closed in X. In the case of Hilbert spaces, boundedness from below (resp. surjectivity) is equivalent with the left (resp. right) invertibility. $\sigma(A) = \sigma_{\mathrm{ap}}(A) \cup \sigma_{\mathrm{su}}(A)$ and by the closed range theorem we easily get that $\sigma_{\mathrm{ap}}(A)$ and $\sigma_{\mathrm{su}}(A)$ satisfy the duality relations $\sigma_{\mathrm{ap}}(A) = \sigma_{\mathrm{su}}(A^*)$ and $\sigma_{\mathrm{su}}(A) = \sigma_{\mathrm{ap}}(A^*)$.

The quasinilpotent part $\mathcal{H}_0(A)$ of an operator $A \in \mathcal{B}(X)$ is the set

$$\mathcal{H}_0(A) = \left\{x \in X\colon \lim_{n\to\infty} \|A^n x\|^{1/n} = 0\right\}.$$

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The analytical kernel $\mathcal{K}(A)$ of A is the following set

$$\mathcal{K}(A)=\left\{\begin{array}{c} x\in X\colon \exists\{x_n\colon n\in\mathbb{N}\}\subset X,\ \exists\delta_x>0 \text{ such that } Ax_1=x,\\ Ax_{n+1}=x_n \text{ and } \|x_n\|\le\delta_x^n\|x\| \text{ for all } n\in\mathbb{N}\end{array}\right\}.$$

It easily follows from the definitions that $\mathcal{H}_0(A)$ and $\mathcal{K}(A)$ are generally not closed subspaces of X, $A(\mathcal{H}_0(A))\subset\mathcal{H}_0(A)$, $A(\mathcal{K}(A))=\mathcal{K}(A)$ and that A is quasinilpotent (i.e. $\sigma(A)=\{0\}$) if and only if $\mathcal{H}_0(A)=X$ or $\mathcal{K}(A)=\{0\}$. Moreover, if A is bounded from below then $\mathcal{H}_0(A)=\{0\}$. $\mathcal{K}(A)=X$ as soon as A is surjective. Furthermore, it is well known that the quasinilpotent part and analytical kernel play an important role in local spectral theory and Fredholm theory see e.g. [1]. If M and N are two closed A-invariant subspaces of X such that $X=M\oplus N$, we say that A is completely reduced by the pair (M,N) and it is denoted by $(M,N)\in\operatorname{Red}(A)$, where $\oplus$ is the algebraic direct sum. In this case we write $A=A_M\oplus A_N$ and we have $\mathcal{N}(A)=\mathcal{N}(A_M)\oplus\mathcal{N}(A_N)$, $\mathcal{R}(A)=\mathcal{R}(A_M)\oplus\mathcal{R}(A_N)$, $A^n=A_M^n\oplus A_N^n$ for all $n\in\mathbb{N}$, and A is bounded below (resp. surjective) if and only if A_M and A_N are bounded below (resp. surjective).

A operator $A\in\mathcal{B}(X)$ is said to be *Drazin invertible* if $A=A_1\oplus A_2$ where A_1 is invertible and A_2 is nilpotent, for details see [1]. In 1996, Koliha introduced the notion of generalized Drazin invertible operators ([6]), $A\in\mathcal{B}(X)$ is generalized Drazin invertible if and only if $0\notin\operatorname{acc}\sigma(A)$, which is also equivalent to the fact that $(\mathcal{K}(A),\mathcal{H}_0(A))\in\operatorname{Red}(A)$, where $A=A_{\mathcal{K}(A)}\oplus A_{\mathcal{H}_0(A)}$, $A_{\mathcal{K}(A)}$ is invertible and $A_{\mathcal{H}_0(A)}$ is quasinilpotent. He showed that if A is generalized Drazin invertible, its generalized Drazin inverse (denoted A^{gD}) is unique in $\mathcal{B}(X)$ and $A^{gD}=(A+P(A))^{-1}(I-P(A))$ where $P(A)$ is the spectral projection of A associated with 0.

Two decades later, in 2015, Miloud, Benharrat and Messirdi introduced and studied in [7] the concept of left and right generalized Drazin invertible operators. Their definitions are linked to complementation of the spectral subspaces, the quasinilpotent part and the analytical kernel.

Definition 1. Let $A\in\mathcal{B}(X)$. A is said to be *left* (resp. *right*) *generalized Drazin invertible* if $\mathcal{H}_0(A)$ (resp. $\mathcal{K}(A)$) is closed and complemented with a subspace M (resp. N) of X such that $A(M)\subset M$ and $A(M)$ is closed (resp. $A(N)\subset N\subseteq\mathcal{H}_0(A)$).

Left and right generalized Drazin invertible operators have been characterized respectively via approximate point spectrum and surjectivity spectrum ([7], Theorem 3.8, Theorem 3.10). It follows from the definition that if A is left (resp. right) generalized Drazin invertible, it generally admits several left (resp. right) generalized Drazin inverses. $\{A^{\lg D}\}$ (resp. $\{A^{\mathrm{rg}\,D}\}$) denotes the set of all left (resp. right) generalized Drazin inverses of A. Moreover, $A\in B(X)$ is generalized Drazin invertible if and only if it is both left and right generalized Drazin invertible, in this case $\{A^{\lg D}\}=\{A^{\mathrm{rg}\,D}\}=A^{\mathrm{g}\,D}$. $\mathcal{B}_l(X)$ (resp. $\mathcal{B}_r(X)$) is the set of all left (resp. right) generalized Drazin invertible operators of $\mathcal{B}(X)$.

The previous representation of $A^{\mathrm{g}\,D}$ via the spectral projection of A, leds us to seek an analogous relationship for left and right generalized Drazin invertible operators. The first objective of this work is to associate to any operator of $\mathcal{B}_l(X)$ or $\mathcal{B}_r(X)$ adequate bounded projections on X, which give an explicit expression of the corresponding inverses. Any projection satisfying these conditions will be called a *spectral idempotent*. We also investigate necessary and sufficient conditions for left or right generalized Drazin invertible operators to have the same spectral idempotents. Explicit representations of the spectral idempotents and that of the corresponding left and right generalized Drazin inverses are obtained. In addition, we give some sufficient conditions which ensure that the product of elements of $\mathcal{B}_l(X)$ (resp. $\mathcal{B}_r(X)$) remains in $\mathcal{B}_l(X)$ (resp. $\mathcal{B}_r(X)$). Note that, recent results of [7] relating to the left and right generalized Drazin invertibility are improved from our theorems.

Jacobson's lemma states that if $A, B \in \mathcal{B}(X)$, then $I - AB$ is invertible if and only if $I - BA$ is invertible. Jacobson's lemma was extended to Drazin invertible operators in [4] and to generalized Drazin invertible operators in [10]. Our second goal of this paper is to prove that Jacobson's lemma also holds for left and right generalized Drazin invertible operators. We show that the left (resp. right) generalized Drazin spectrum of AB coincides with the left (resp. right) generlized Drazin spectrum of BA, that is $\lambda I - AB$ is left (resp. right) generalized Drazin invertible if and only if $\lambda I - BA$ is left (resp. right) generalized Drazin invertible. The case $\lambda = 0$ is also investigated.

The paper is organized as follows. In section 2, we present algebraic characterizations and explicit expressions of the spectral idempotents and the corresponding left and right generalized Drazin invertible operators. We apply our results to study the operators of $\mathcal{B}_l(X)$ and $\mathcal{B}_r(X)$ with equal spectral idempotents. In section 3, we investigate when the product of two left or right generalized Drazin invertible operators is again left or right generalized Drazin invertible. In section 4, we extend Jacobson's lemma for the left and the right generalized Drazin inverses.

2. Characterization of left and right generalized Drazin invertible operators. In this section, we start with a characterization of left (resp. right) generalized Drazin invertible operators initially shown in [7]. These generalized inverses were also studied in [8] and [2] respectively, via Fredholm theory and local spectral theory. Many results appearing in the rest of this work are new.

Theorem 1 ([7]). *Let $A \in \mathcal{B}(X)$, then the following assertions are equivalent:*

1) $A \in \mathcal{B}_l(X)$ *(resp.* $A \in \mathcal{B}_r(X)$*),*
2) $0 \in \mathrm{iso}\,(\sigma_{\mathrm{ap}}(A))$ *(resp.* $0 \in \mathrm{iso}\,(\sigma_{\mathrm{su}}(A))$*),*
3) $A = A_M \oplus A_{\mathcal{H}_0(A)}$ *(resp.* $A = A_{\mathcal{K}(A)} \oplus A_N$*) where* A_M *is bounded below (resp.* $A_{\mathcal{K}(A)}$ *is surjective) and* $A_{\mathcal{H}_0(A)}$ *(resp.* A_N*) is quasinilpotent. In this case we use the notation* $(M, \mathcal{H}_0(A)) \in \mathrm{lRed}(A)$ *(resp.* $(\mathcal{K}(A), N) \in \mathrm{rRed}(A)$*),*
4) *There exists a bounded projection* $P(A)$ *on* X *such that* $AP(A) = P(A)A$, $A + P(A)$ *is bounded below (resp. surjective),* $AP(A)$ *is quasinilpotent and* $\mathcal{R}(P(A)) = \mathcal{H}_0(A)$ *(resp.* $\mathcal{N}(P(A)) = \mathcal{K}(A)$*).*

Since the properties to be bounded below or to be surjective are dual each other, then A is left generalized Drazin invertible if and only if A^* is right generalized Drazin invertible.

Remark 1. 1) Note that if A is generalized Drazin invertible, there exists only one projection $P(A) \in \mathcal{B}(X)$ that commutes with A such that $A + P(A)$ is invertible and $AP(A)$ is quasinilpotent. Indeed, $P(A) = I - AA^{\mathrm{g}D} = I - A^{\mathrm{g}D}A$, so the uniqueness of the generalized inverse $A^{\mathrm{g}D}$ of A proves the uniqueness of the spectral idempotent $P(A)$ of A. Nevertheless, when $A \in \mathcal{B}_l(X)$ or $A \in \mathcal{B}_r(X)$ the bounded projection $P(A)$ on X associated to A, satisfying assertion (4) of Theorem 1, is not necessarily unique.
2) Given two closed subspaces M and M' of X such that $(M, \mathcal{H}_0(A)) \in \mathrm{lRed}(A)$ and $(M', \mathcal{H}_0(A)) \in \mathrm{lRed}(A)$, then M and M' have $\mathcal{H}_0(A)$ as a common complement in X.

If $P(A)$ and $P'(A)$ are bounded spectral idempotents on X associated to A, satisfying assertion (4) of Theorem 1, then $\mathcal{R}(P(A)) = \mathcal{R}(P'(A)) = \mathcal{H}_0(A)$, $P(A)P'(A) = P'(A)$ and $P'(A)P(A) = P(A)$. Furthermore, M and M' are isomorphic [[5], Corollary 2.10]. In particular, if $P(A) = P'(A)$ then $M = M'$. The same conclusion is valid for right generalized Drazin invertible operators.

Proposition 1. *Let $A \in \mathcal{B}_l(X)$ (resp. $A \in \mathcal{B}_r(X)$). Then, the bounded spectral idempotent $P(A)$ on X associated to A, satisfying assertion (4) of Theorem 1 is unique if and only if A is generalized Drazin invertible.*

Proof. It suffices to show for example that if A is left generalized Drazin invertible with a unique bounded spectral idempotent $P(A)$ on X commuting with A such that $A+P(A)$ is bounded below, $AP(A)$ is quasinilpotent and $\mathcal{R}(P(A)) = \mathcal{H}_0(A)$, then it is necessary that $\mathcal{N}(P(A)) = \mathcal{K}(A)$. Indded, as $P(A)$ is unique it is clear that $A+P(A)$ is injective with closed range and a fortiori invertible. So, A is generalized Drazin invertible, $P(A)$ is the spectral projection of A corresponding to 0, $\mathcal{R}(P(A)) = \mathcal{H}_0(A)$ and $\mathcal{N}(P(A)) = \mathcal{K}(A)$. □

The following algebraic characterizations of left and right generalized Drazin invertiblity are given in [8] in the Hilbert spaces setting.

Lemma 1. *$A \in \mathcal{B}_l(H)$ (resp. $A \in \mathcal{B}_r(H)$) if and only if there exist two operators $L_l, Q_l \in \mathcal{B}(H)$ (resp. $L_r, Q_r \in \mathcal{B}(H)$) such that Q_l (resp. Q_r) is quasinilpotent, $AL_lA = L_lA^2 = A - Q_l$ (resp. $AL_rA = A^2L_r = A - Q_r$) and $L_lAL_l = L_l^2A = L_l$ (resp. $L_rAL_r = AL_r^2 = L_r$). L_l (resp. L_r) is a left (resp. right) generalized Drazin inverse of A (not necessarily unique).*

From Lemma 1, we may deduce the following result which gives explicit expressions of the spectral idempotents involved in Theorem 1.

Proposition 2. *Let $A \in \mathcal{B}_l(H)$ (resp. $A \in \mathcal{B}_r(H)$), then $P_l(A) = I - L_lA$ (resp. $P_r(A) = I - AL_r$) is a bounded spectral idempotent on H such that:*

1) *$AP_l(A) = P_l(A)A$ (resp. $AP_r(A) = P_r(A)A$),*
2) *$A + P_l(A)$ (resp. $A + P_r(A)$) is left invertible (resp. right invertible),*
3) *$AP_l(A)$ (resp. $AP_r(A)$) is quasinilpotent,*
4) *$\mathcal{R}(P_l(A)) = \mathcal{H}_0(A)$ (resp. $\mathcal{N}(P_r(A)) = \mathcal{K}(A)$).*

As a consequence, we obtain the following useful relations between the spectral idempotents and the corresponding left or right generalized Drazin inverses.

Theorem 2. *Let $A \in \mathcal{B}_l(H)$ (resp. $A \in \mathcal{B}_r(H)$) with $(M, \mathcal{H}_0(A)) \in \mathrm{lRed}(A)$ (resp. $(\mathcal{K}(A), N) \in \mathrm{rRed}(A)$). Then, there exist two operators $L_M \in \mathcal{B}(M)$ and $S_l \in \mathcal{B}(H)$ (resp. $L_{\mathcal{K}(A)} \in \mathcal{B}(\mathcal{K}(A))$ and $S_r \in \mathcal{B}(H)$) such that the set $\left\{A^{\mathrm{lg}\,D}\right\}$ (resp. $\left\{A^{\mathrm{rg}\,D}\right\}$) is given by*

$$\left\{A^{\mathrm{lg}\,D}\right\} = \left\{(L_l + S_l)\,(I - P_l(A))\right\} = \left\{L_M \oplus 0_{|\mathcal{H}_0(A)}\right\},$$
$$(\text{resp. } \left\{A^{\mathrm{rg}\,D}\right\} = \left\{(L_r + S_r)\,(I - P_r(A))\right\} = \left\{L_{\mathcal{K}(A)} \oplus 0_{|N}\right\}).$$

Proof. We have $P_l(A) = I - L_lA$ (resp. $P_r(A) = I - AL_r$), $A = A_M \oplus A_{\mathcal{H}_0(A)}$ and $I = I_M \oplus I_{\mathcal{H}_0(A)}$ (resp. $A = A_{\mathcal{K}(A)} \oplus A_N$ and $I = I_{\mathcal{K}(A)} \oplus I_N$). Let L_M (resp. $L_{\mathcal{K}(A)}$) be a left (resp. right) inverse of A_M (resp. $A_{\mathcal{K}(A)}$). $L_M \in \mathcal{B}(M)$ (resp. $L_{\mathcal{K}(A)} \in \mathcal{B}(\mathcal{K}(A))$) exists since A_M

(resp. $A_{\mathcal{K}(A)}$) is by assumption left (resp. right) invertible. Using Lemma 1 and Proposition 2, we get

$$L_l = L_M \oplus 0_{\mathcal{H}_0(A)} \text{ (resp. } L_r = L_{\mathcal{K}(A)} \oplus 0_N), \ P_l(A) = 0_M \oplus I_{\mathcal{H}_0(A)} \text{ (resp. } P_r(A) = 0_{\mathcal{K}(A)} \oplus I_N).$$

Thus,

$$L_l\left(A + P_l(A)\right) = I_M \oplus 0_{\mathcal{H}_0(A)} \text{ and } L_l\left(A + P_l(A)\right) + \left(0_M \oplus I_{\mathcal{H}_0(A)}\right) = I$$
$$\text{(resp. } \left(A + P_r(A)\right) L_r = I_{\mathcal{K}(A)} \oplus 0_N \text{ and } \left(A + P_r(A)\right) L_r + \left(0_{\mathcal{K}(A)} \oplus I_N\right) = I).$$

Since $A_{\mathcal{H}_0(A)}$ (resp. A_N) is quasinilpotent, $I_{\mathcal{H}_0(A)} + A_{\mathcal{H}_0(A)}$ (resp. $I_N + A_N$) is boundedly invertible, one can define S_l (resp. S_r) by

$$S_l = 0_M \oplus \left(I_{\mathcal{H}_0(A)} + A_{\mathcal{H}_0(A)}\right)^{-1} \text{ (resp. } S_r = 0_{\mathcal{K}(A)} \oplus \left(I_N + A_N\right)^{-1}).$$

Then,

$$S_l\left(A + P_l(A)\right) = 0_M \oplus I_{\mathcal{H}_0(A)} \text{ (resp. } \left(A + P_r(A)\right) S_r = 0_{\mathcal{K}(A)} \oplus I_N)),$$
$$\left(L_l + S_l\right)\left(A + P_l(A)\right) = I \text{ (resp. } \left(A + P_r(A)\right)\left(L_r + S_r\right) = I)$$

which means that $A + P_l(A)$ (resp. $A + P_r(A)$) is left (resp. right) invertible and $L_l + S_l$ (resp. $L_r + S_r$) is a left (resp. right) inverse of $A + P_l(A)$ (resp. $A + P_r(A)$). Moreover,

$$S_l L_l = L_l S_l = 0, \ \left(L_l + S_l\right)\left(I - P_l(A)\right) = \left(L_l + S_l\right) L_l A = L_l^2 A = L_l = L_M \oplus 0_{|\mathcal{H}_0(A)}$$
$$S_r A L_r = 0, \ \left(L_r + S_r\right)\left(I - P_r(A)\right) = \left(L_r + S_r\right) A L_r = L_r A L_r = A L_r^2 = L_r = L_{\mathcal{K}(A)} \oplus 0_{|N}.$$

□

Remark 2. As $L_M A_M = I_M \neq A_M L_M$ and $A_{\mathcal{K}(A)} L_{\mathcal{K}(A)} = I_{\mathcal{K}(A)} \neq L_{\mathcal{K}(A)} A_{\mathcal{K}(A)}$, we have $\left\{A^{\lg D}\right\} A \neq A \left\{A^{\lg D}\right\}$ and $\left\{A^{\operatorname{rg} D}\right\} A \neq A \left\{A^{\operatorname{rg} D}\right\}$.

Example 1. Consider respectively the right and left shift on l^2 given by

$$S\left(x_1, x_2, x_3 ...\right) = \left(0, x_1, x_2, x_3, ...\right) \text{ and } T\left(x_1, x_2, x_3, ...\right) = \left(x_2, x_3, ...\right).$$

Let $Q\left(x_1, x_2, x_3, ...\right) = \left(\frac{x_2}{2}, \frac{x_3}{3}, ...\right)$, since $\|Q^n\| = \frac{1}{(n+1)!}$, $n \in \mathbb{N}$, hence Q is a quasi-nilpotent operator on l^2. We have, $TS = I_{l^2}$, identity on l^2, then S is left invertible with the left inverse T, but has no right inverse, and T is right invertible with right inverse S, but has no left inverse. Define $A = S \oplus Q$ and $B = T \oplus Q$ on $X = l^2 \oplus l^2$. Then A is left generalized Drazin invertible and B is right generalized Drazin invertible on X, with

$$L_l(A) = T \oplus 0_{l^2}, \ L_r(B) = S \oplus 0_{l^2} \text{ and } P_l(A) = 0_{l^2} \oplus I_{l^2} = P_r(B).$$
$$T \oplus 0_{|l^2(\mathbb{N}^*)} \in \left\{A^{lGD}\right\} \text{ and } S \oplus 0_{|l^2(\mathbb{N}^*)} \in \left\{B^{rGD}\right\}.$$

Definition 2. Let $A, B \in \mathcal{B}_*(X)$, $* \in \{l, r\}$. We say that A and B *have equal spectral idempotents* if there exists some projection $P \in \mathcal{B}(X)$ such that pairs (A, P) and (B, P) satisfy assertion (4) of Theorem 1.

We investigate necessary and sufficient conditions for left or right generalized Drazin invertible operators to have the same spectral idempotents. For this purpose we characterize the set $(P_*(A))^{-1} = \{B \in \mathcal{B}_*(H) \colon P_*(A) = P_*(B)\}$ where A is a given operator in $\mathcal{B}_*(H)$, $* \in \{l, r\}$. Let us first note that if $A \in \mathcal{B}(H)$ is generalized Drazin invertible then $P(A)A^{\mathrm{g}D} = A^{\mathrm{g}D}P(A) = 0$, therefore we observe that $(P(A))^{-1} = \left\{B \in \mathcal{B}(H) \colon BB^{\mathrm{g}D} = AA^{\mathrm{g}D}\right\}$. In our second main result we provide similar characterizations on left and right generalized Drazin invertible operators. We use the same notations introduced previously.

Theorem 3. *Let $A \in \mathcal{B}_*(H)$, $* \in \{l, r\}$. Then, $P_*(A)\left\{A^{*\mathrm{g}D}\right\} = \left\{A^{*\mathrm{g}D}\right\}P_*(A) = \{0\}$ and*

$$(P_*(A))^{-1} = \begin{cases} B \in \mathcal{B}_*(H) \colon \{B^{*\mathrm{g}D}\}B = \{A^{*\mathrm{g}D}\}A\} & \text{if } \ * = l \\ \{B \in \mathcal{B}_*(H) \colon B\{B^{*\mathrm{g}D}\} = A\{A^{*\mathrm{g}D}\}\} & \text{if } \ * = r \end{cases}$$

where $\Phi\left\{\Phi^{*GD}\right\} = \{\Phi\left(L_*(\Phi) + S_*(\Phi)\right)(I - P_*(\Phi))\}$, with $L_*(\Phi)$ and $S_*(\Phi)$ are respectively the operators L_* and S_* associated with $\Phi \in \{A, B\}$, $* \in \{l, r\}$.

Proof. As $S_lL_l = L_lS_l = 0$, $L_lAL_l = L_l^2A = L_l$, $S_rAL_r = 0$ and $L_rAL_r = AL_r^2 = L_r$, we have

$$P_l(A)\left\{A^{\mathrm{lg}D}\right\} = \{(I - L_lA)(L_l + S_l)(I - P_l(A))\} =$$
$$= \{(I - L_lA)(L_l + S_l)L_lA\} = \left\{L_l^2A + S_lL_lA - L_lAL_l^2A - L_lAS_lL_lA\right\} = \{0\},$$
$$P_r(A)\left\{A^{\mathrm{rg}D}\right\} = \{(I - AL_r)(L_r + S_r)(I - P_r(A))\} =$$
$$= \{(I - AL_r)(L_r + S_r)AL_r\} = \left\{L_rAL_r + S_rAL_r - AL_r^2AL_r - AS_rAL_r\right\} = \{0\}.$$

Let $B \in \mathcal{B}_*(H)$, $* \in \{l, r\}$. We then observe

$$P_l(A) = P_l(B) \Longleftrightarrow L_l(A)A = L_l(B)B \Longleftrightarrow$$
$$\Longleftrightarrow \{L_l(A)\}A = \{L_l(B)\}B \Longleftrightarrow \left\{A^{lGD}\right\}A = \left\{B^{lGD}\right\}B,$$
$$P_r(A) = P_r(B) \Longleftrightarrow AL_r(A) = BL_r(B) \Longleftrightarrow A\{L_r(A)\} = B\{L_r(B)\} \Longleftrightarrow$$
$$\Longleftrightarrow A\left\{A^{rGD}\right\} = B\left\{B^{rGD}\right\}.$$

□

Remark 3. If $A, B \in \mathcal{B}(H)$ are generalized Drazin invertible, then $\left\{\Phi^{lGD}\right\} = \left\{\Phi^{rGD}\right\} = \Phi^{GD}$, $\Phi \in \{A, B\}$. We deduce from Theorem 3 that

$$P_l(A) = P_r(A) = P(A) = P(B) = P_l(B) = P_r(B) \Longleftrightarrow AA^{\mathrm{g}D} = BB^{\mathrm{g}D}$$

where P is the spectral projection associated with 0. So, our result is a generalization of Theorem 6.1 from [6].

3. Product of left and right generalized Drazin invertible operators. In this section we give sufficient conditions for a product of two left (resp. right) generalized Drazin invertible operators to be left (resp. right) generalized Drazin invertible in a Banach space. Some results on the product are established in [9], our proofs are more detailed and accessible. We will first prove the following preparatory results where we describe some basic properties of subspaces $\mathcal{H}_0(AB)$ and $\mathcal{K}(AB)$ if $A, B \in \mathcal{B}(X)$.

Lemma 2. *Let $A, B \in \mathcal{B}(X)$. 1) $\mathcal{H}_0(A) \subseteq \mathcal{H}_0(AB)$.*
2) If $AB = BA$, $\mathcal{H}_0(A) \cap \mathcal{H}_0(B) \subseteq \mathcal{H}_0(AB)$ and $\mathcal{H}_0(A) \subseteq \mathcal{H}_0(A^n)$, $n \in \mathbb{N}^*$.
3) If $AB = BA$ and B is invertible, $\mathcal{H}_0(AB) = \mathcal{H}_0(A)$.

Proof. 1) We have for $x \in X$: $\left\| (AB)^n x \right\|^{1/n} = \left\| B^n A^n x \right\|^{1/n} \leq \left\| B \right\| \left\| A^n x \right\|^{1/n}$, $n \in \mathbb{N}^*$.

Thus, $\lim_{n\to\infty} \left\| (AB)^n x \right\|^{1/n} = 0$ as soon as $x \in \mathcal{H}_0(A)$.

2) $\left\| (AB)^n x \right\|^{1/n} \leq \min\left(\left\| B \right\| \left\| A^n x \right\|^{1/n}, \left\| A \right\| \left\| B^n x \right\|^{1/n} \right)$, $x \in X$. So if $x \in \mathcal{H}_0(A) \cap \mathcal{H}_0(B)$, we have

$$\lim_{n\to\infty} \left\| A^n x \right\|^{1/n} = \lim_{n\to\infty} \left\| B^n x \right\|^{1/n} = 0 \Longrightarrow \lim_{n\to\infty} \left\| (AB)^n x \right\|^{1/n} = 0$$

which means that $x \in \mathcal{H}_0(AB)$.

Taking $A = B$, it is easy to verify the second inclusion.

3) It is clear from (2) that $\mathcal{H}_0(A) \subseteq \mathcal{H}_0(AB)$. If we apply the same argument to operators AB and B^{-1}, we obtain the result. □

Lemma 3. *Let $A, B \in \mathcal{B}(X)$ be such that $AB = BA$, then:* 1) $\mathcal{K}(AB) \subseteq \mathcal{K}(A) \cap \mathcal{K}(B)$; 2) $\mathcal{K}(A^p) = \mathcal{K}(A)$, *for all* $p \in \mathbb{N}^*$; 3) $\mathcal{K}(AB) = \mathcal{K}(A)$ *if* B *is invertible.*

Proof. 1) Let $x \in \mathcal{K}(AB)$. So, there exist $\delta_x > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$x_0 = x, ABx_{n+1} = x_n \text{ and } \left\| x_n \right\| \leq \delta_x^n \left\| x \right\| \text{ for all } n \in \mathbb{N}.$$

Take $y_0 = x$. We have $x = ABx_1$, so $y_0 = Ay_1$ where $y_1 = Bx_1$. Thus, $y_1 = B\left(ABx_2\right) = AB^2x_2$, so $y_1 = Ay_2$ with $y_2 = B^2x_2$. By induction we construct a new sequence $(y_n)_{n\in\mathbb{N}}$ in X such that $x = y_0$, $y_n = Ay_{n+1}$, and $y_n = B^n x_n$ for all $n \in \mathbb{N}$. Furthermore,

$$\left\| y_n \right\| = \left\| B^n x_n \right\| \leq \left\| B \right\|^n \left\| x_n \right\| \leq \left(\left\| B \right\| \delta_x \right)^n \left\| x \right\| \text{ for all } n \in \mathbb{N}.$$

Therefore, $x \in \mathcal{K}(A)$. By virtue of the commutativity, we also have $x \in \mathcal{K}(A)$. 2) From (1) we have $\mathcal{K}(A^p) \subseteq \mathcal{K}(A)$ for all $p \in \mathbb{N}^*$. So, we need only to prove the reverse inclusion. Let $x \in \mathcal{K}(A)$, then there exist $\delta_x > 0$ and a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_0 = x$, $Ax_{n+1} = x_n$ and $\left\| x_n \right\| \leq \delta_x^n \left\| x \right\|$ for all $n \in \mathbb{N}$. Let $(y_n)_{n\in\mathbb{N}}$ be the sequence in X defined by $y_n = x_{np}$. So we have

$$y_0 = x,\ y_n = Ax_{np+1} = A^2x_{np+2} = ... = A^p x_{n(p+1)} = A^p y_{n+1}, \quad \left\| y_n \right\| = \left\| x_{np} \right\| \leq \delta_x^{np} \left\| x \right\|$$

for all $n \in \mathbb{N}$. Then $x \in \mathcal{K}(A^p)$.

3) If we apply the inclusion (1) by taking B invertible, with A replaced by AB and B replaced by its inverse B^{-1}, we obtain $\mathcal{K}(AB) = \mathcal{K}(A)$. □

Theorem 4. *Let $A \in \mathcal{B}_l(X)$ and $(M, \mathcal{H}_0(A)) \in \mathrm{lRed}(A)$ (resp. $A \in \mathcal{B}_r(X)$ and $(\mathcal{K}(A), N) \in \mathrm{rRed}(A)$). If $B \in \mathcal{B}(X)$ is an invertible operator commuting with A such that $B\left(M\right) \subset M$ (resp. $B\left(N\right) \subset N$) then $AB \in \mathcal{B}_l(X)$ (resp. $AB \in \mathcal{B}_r(X)$).*

Proof. By Lemma 2 (resp. Lemma 3), we have $\mathcal{H}_0(AB) = \mathcal{H}_0(A)$ (resp. $\mathcal{K}(AB) = \mathcal{K}(A)$). So, it follows that $\mathcal{H}_0(AB)$ (resp. $\mathcal{K}(AB)$) is closed in X and complemented with M (resp. N). It is clear that $AB\left(M\right) \subset M$ (resp. $AB\left(N\right) \subset N \subseteq \mathcal{H}_0(AB)$) and $AB\left(M\right) = BA\left(M\right)$ (resp. $AB\left(N\right) = BA\left(N\right)$) is closed since $A\left(M\right)$ (resp. $A\left(N\right)$) is closed and B is invertible. According to Definition 1, it follows that $AB \in \mathcal{B}_l(X)$ (resp. $AB \in \mathcal{B}_r(X)$). □

Corollary 1. *Let $A \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$. Then $A \in \mathcal{B}_*(X)$ if and only if $\lambda A \in \mathcal{B}_*(X)$, $* \in \left\{ l, r \right\}$.*

Proof. $\lambda = 0$ is trivial. The result follows from Theorem 4 if we put $B = \lambda I$, $\lambda \neq 0$. □

Theorem 5. *Let A and B be commuting operators in $\mathcal{B}_l(X)$ (resp. $\mathcal{B}_r(X)$) with equal spectral idempotents. Then $AB \in \mathcal{B}_l(X)$ (resp. $AB \in \mathcal{B}_r(X)$) and has equal spectral idempotent with A and B.*

Proof. According to Theorem 1 and the assumption, there exists a projection $P \in \mathcal{B}(X)$ such that $AP = PA$, $BP = PB$, $A+P$ and $B+P$ are bounded below (resp. surjective), AP and BP are quasinilpotent and $\mathcal{R}(P) = \mathcal{H}_0(A) = \mathcal{H}_0(B)$ (resp. $\mathcal{N}(P) = \mathcal{K}(A) = \mathcal{K}(B)$). Then, $ABP = APB = PAB$. Since BP is quasinilpotent, then ABP is also quasinilpotent. Further, we have $AB + P = (A+P)(B+P) - (A+B)P$. Since $(A+P)(B+P)$ is bounded below (resp. surjective) and commutes with the quasinilpotent operator $(A+B)P$, then $AB+P$ is also bounded below (resp. surjective). It results immediately from Theorem 1 that AB is left (resp. right) generalized Drazin invertible operator. □

Corollary 2. *If $A \in \mathcal{B}_l(X)$ (resp. $A \in \mathcal{B}_r(X)$) then $A^n \in \mathcal{B}_l(X)$ (resp. $A^n \in \mathcal{B}_r(X)$) for all $n \in \mathbb{N}$.*

Remark 4. If $A, B \in \mathcal{B}_l(H)$ (resp. $A, B \in \mathcal{B}_r(H)$) with equal spectral idempotents and $AB = BA$, then it follows from Theorem 3 and Theorem 5 that $\{(AB)^{\lg D}\}AB = \{A^{\lg D}\}A = \{B^{\lg D}\}B$ (resp. $AB\{(AB)^{\mathrm{rg}\, D}\} = A\{A^{\mathrm{rg}\, D}\} = B\{B^{\mathrm{rg}\, D}\}$).

4. Jacobson's lemma for left and right generalized Drazin invertible operators. We present in this section some interesting spectral results related to left and right generalized Drazin inverses of the product of two bounded operators. Let us note that the products AB and BA of two bounded operators A and B on X share some spectral properties. First, it is well known that $\sigma(AB)\setminus\{0\} = \sigma(BA)\setminus\{0\}$. Under some conditions this result has obvious analogues in which the spectrum is replaced by a number of distinguished parts of the spectrum such as the approximate point spectrum and surjectivity spectrum. In another main result we affirm that Jacobson's lemma holds for left and right generalized Drazin invertibility. Let $A \in \mathcal{B}(X)$, denote by $\sigma_{\lg D}(A) = \{\lambda \in \mathbb{C} \colon (\lambda I - A) \notin \mathcal{B}_l(X)\}$the left generalized Drazin spectrum of A and by $\sigma_{\mathrm{rg}\, D}(A) = \{\lambda \in \mathbb{C} \colon (\lambda I - A) \notin \mathcal{B}_r(X)\}$ the right generalized Drazin spectrum of A;

$$\sigma_{\mathrm{g}\, D}(A) = \sigma_{\lg D}(A) \cup \sigma_{\mathrm{rg}\, D}(A) = \{\lambda \in \mathbb{C} \colon \lambda I - A \text{ is not generalized Drazin invertible}\}$$

is the generalized Drazin spectrum of A.

It is well known, see [7], that $\sigma_{\lg D}(A)$, $\sigma_{\mathrm{rg}\, D}(A)$ and $\sigma_{\mathrm{g}\, D}(A)$ are compact sets of the complex plane, $\sigma_{\lg D}(A) \subset \sigma_{\mathrm{ap}}(A)$ and $\sigma_{\mathrm{rg}\, D}(A) \subset \sigma_{\mathrm{su}}(A)$. Precisely,

$$\lambda \in \sigma_{\lg D}(A) \Longleftrightarrow \lambda \in \operatorname{acc} \sigma_{\mathrm{ap}}(A),\ \ \lambda \in \sigma_{\mathrm{rg}\, D}(A) \Longleftrightarrow \lambda \in \operatorname{acc} \sigma_{\mathrm{su}}(A),\ \ \lambda \in \sigma_{\mathrm{g}\, D}(A) \Longleftrightarrow \lambda \in \operatorname{acc} \sigma(A).$$

Let us first recall Jacobson's lemma about invertibility.

Lemma 4 ([3], Theorem 1)**.** *If $A, B, C \in \mathcal{B}(X)$ are such that $BAB = BCB$, then*

$$\sigma_{\mathrm{ap}}(BA)\setminus\{0\} = \sigma_{\mathrm{ap}}(CB)\setminus\{0\} \text{ and } \sigma_{\mathrm{su}}(BA)\setminus\{0\} = \sigma_{\mathrm{su}}(CB)\setminus\{0\}.$$

As a straightforward consequence of Lemma 4, we obtain the following result.

Theorem 6. *Let $A, B, C \in \mathcal{B}(X)$ be such that $BAB = BCB$. Then*

1) $\sigma_*(BA) = \sigma_*(CB)$, $\sigma_* \in \{\sigma_{\lg D},\ \sigma_{\mathrm{rg}\, D},\ \sigma_{\mathrm{g}\, D}\}$.

2) In particular if $A = C$ we have $\sigma_*(BA) = \sigma_*(AB)$, $\sigma_* \in \{\ \sigma_{\lg D},\ \sigma_{\mathrm{rg}\, D},\ \sigma_{\mathrm{g}\, D}\}$.

Proof. 1) Lemma 4 asserts that $\sigma_{\mathrm{ap}}(BA)\setminus\{0\} = \sigma_{\mathrm{ap}}(CB)\setminus\{0\}$, so it sufficies to show that BA is left generalized Drazin invertible if and only if CB is. Assume that $0 \notin \sigma_{\lg D}(BA)$, then $0 \in \operatorname{iso} \sigma_{\mathrm{ap}}(BA)$. Therefore, $\lambda I - BA$ is bounded below for $\lambda \in \mathbb{C}\setminus\{0\}$ small enough. Hence, $\lambda I - CB$ is also bounded below for $\lambda \in \mathbb{C}\setminus\{0\}$ small enough. Thus, $0 \in \operatorname{iso} \sigma_{\mathrm{ap}}(CB)$ and CB is left generalized Drazin invertible.

By duality, we have $\sigma_{\mathrm{rg}\,D}(BA) = \sigma_{\mathrm{rg}\,D}(CB)$. So, $\sigma_{\mathrm{g}\,D}(BA) = \sigma_{\mathrm{lg}\,D}(BA) \cup \sigma_{\mathrm{rg}\,D}(BA) = \sigma_{\mathrm{lg}\,D}(CB) \cup \sigma_{\mathrm{rg}\,D}(CB) = \sigma_{\mathrm{g}\,D}(CB)$.
2) is a consequence of the first assertion when $A = C$. □

Corollary 3. *Let $A, B \in \mathcal{B}(X)$. Then, $AB \in \mathcal{B}_*(X) \Longleftrightarrow BA \in \mathcal{B}_*(X)$, $* \in \{l, r\}$, AB is generalized Drazin invertible $\Longleftrightarrow$ BA is generalized Drazin invertible.*

Remark 5. Our previous result generalizes the Theorem 2.3 of [10].

Example 2. Let $S, T \in \mathcal{B}(X)$ and A be the operator defined on $X \oplus X$ by $A = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$.
Then, $A^2 = \begin{pmatrix} ST & 0 \\ 0 & TS \end{pmatrix} = ST \oplus TS$. Thus, if $\sigma_* \in \{\sigma_{\mathrm{lg}\,D},\ \sigma_{\mathrm{rg}\,D},\ \sigma_{\mathrm{g}\,D}\}$, we get

$$\sigma_*(A^2) = \sigma_*(ST \oplus TS) = \sigma_*(ST) \cup \sigma_*(TS) = \sigma_*(ST).$$

It is clear that $\sigma_*(A^2) = (\sigma_*(A))^2 = \{\lambda^2 \colon \lambda \in \sigma_*(A)\}$, so, $\sigma_*(A) = (\sigma_*(ST))^{1/2} = \left\{\lambda^{1/2} \colon \lambda \in \sigma_*(ST)\right\}$.

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