A. Santiago-Santos, N. T. Tapia-Bonilla

SOME RESULTS ON $\frac{1}{n}$-HOMOGENEITY

A. Santiago-Santos, N. T. Tapia-Bonilla. *Some results on $\frac{1}{n}$-homogeneity*, Mat. Stud. **63** (2025), 181–198.

Given a positive integer n, a non-empty topological space is said to be $\frac{1}{n}$-homogeneous provided there are exactly n orbits for the action of the group of homeomorphisms of the space onto itself. Now, for a non-empty topological space X, the cone of X, $\mathrm{Cone}(X)$, is the quotient space that is obtained by identifying all the points $(x,1)$ in $X\times[0,1]$ to a single point. The suspension of X, $\mathrm{Sus}(X)$, is the quotient space that is obtained by identifying all the points $(x,1)$ in $X\times[-1,1]$ to a single point, and all the points $(x,-1)$ to another point. The quotient space Z_X, is the space that is obtained by identifying all the points $(x,1)$ and all the points $(x,-1)$ in $X\times[-1,1]$ to one point. In this paper we determine general properties of the quotient spaces Z_X and we investigate $\frac{1}{n}$-homogeneity on the quotient spaces Z_X, $\mathrm{Sus}(Z_X)$ and $\mathrm{Cone}(Z_X)$, among certain classes of compact metric spaces. In particular, we obtain the degree of homogeneity of the harmonic sequence, the n-rose finite graph and the space X which is the union of a sequence $\{C_n\}_{n=1}^{\infty}$ of circles in the plane joined by a point and converging to a limit circle C_0.

1. Introduction. Let $\mathcal{H}(X)$ denote the group of homeomorphisms of a non-empty space X onto itself. An *orbit of* X is the action of $\mathcal{H}(X)$ at a point x of X, namely $\{h(x)\colon h\in\mathcal{H}(X)\}$. The symbol $\mathcal{O}_X(x)$ denotes the orbit of space X that contains x. Given a positive integer n, a non-empty topological space X is a said to be $\frac{1}{n}$-*homogeneous* provided that X has exactly n orbits, in which case we say that the *degree of homogeneity of* X, denoted by $\mathrm{d_H}(X)$, is n (this notation was introduced in [28]).

For a non-empty topological space X, the cone of X, $\mathrm{Cone}(X)$, is the quotient space that is obtained by identifying all the points $(x,1)$ in $X\times I$ to a single point ([22, p. 41, 3.15]). The suspension of X, $\mathrm{Sus}(X)$, is the quotient space that is obtained by identifying all the points $(x,1)$ in $X\times J$ to a single point, and all the points $(x,-1)$ to another point ([22, p. 42, 3.16]). Moreover, we denote the vertex of $\mathrm{Cone}(X)$ by v_X and the vertices of $\mathrm{Sus}(X)$ by v_X^1 and v_X^{-1}. In this paper we consider the following general problem:

Problem. *Determine the degree of homogeneity of some classes of spaces.*

For recent and related results on this problem we refer to [1, 6, 10, 14, 15, 20, 21, 26–30]. In this paper we continue the work on this problem. Throughout this manuscript we study the quotient space $Z_X=\mathrm{Sus}(X)/_{\{v_X^1,v_X^{-1}\}}$, we investigate $\frac{1}{n}$-homogeneity of Z_X, $\mathrm{Sus}(Z_X)$ and $\mathrm{Cone}(Z_X)$. As a consequence of this results we obtain the degree of homogeneity the harmonic sequence, the cone and the suspension over the harmonic sequence. Moreover, the degree of homogeneity of the finite graph, the n-rose and the space X which is the union

2020 *Mathematics Subject Classification:* 54B15, 54F15, 54F35.
Keywords: continuum; $\frac{1}{n}$-homogeneous; degree of homogeneity; suspension; locally connected.
doi:10.30970/ms.63.2.181-198

of a sequence $\{C_n\}_{n=1}^{\infty}$ of circles in the plane joined by a point and converging to a limit circle C_0.

This paper is organized as follows. In Sections 1 we present some notation, terminology and general results that we will use in this paper. In Sections 2 we show some examples about the space quotient Z_X and we present examples of $\frac{1}{n}$-homogeneous spaces. Moreover, we devote Sections 2 to study $\frac{1}{n}$-homogeneity of Z_X for among certain classes of compact metric spaces. Finally, in Section 3, Section 4 and Section 5 we present the main results of this article.

1.1. Notation and terminology. In this section we present general notation and we recall the concept of cone and suspension of a nonempty space. We also define terminology that we will use frequently. For notation and terminology not given here or in Section 1, see [22].

The symbol $\mathbb{N}$ denotes the set of positive integers; $\mathbb{R}$ denotes the set of real numbers; $A \times B$ denotes the Cartesian product of A and B; $\overline{A}$ denotes the closure of A; iM and ∂M denote the interior and boundary manifolds, respectively, of a manifold M. Throughout the paper, I denotes the closed interval $[0,1]$ and J denotes the closed interval $[-1,1]$. Moreover, all the spaces considered in the present work are non-empty.

On the other hand, a *continuum* is a compact and connected metric space. The term *nondegenerate* refers to a space that contains more than one point. An *arc* is a space homeomorphic to the closed interval $[0,1]$. An arc A in X is called a *free arc* if $\mathrm{int}(A)$ is open in X. A *simple closed curve* is a space homeomorphic to the unit circle S^1. A *dendrite* is a locally connected continuum that contains no simple closed curve (see [22, p. 165, 10.1]). A *local dendrite* is a continuum for which each of its points has a (closed) neighborhood that is a dendrite (see [13, p. 303]). *Dimension* means topological dimension for separable metric spaces [8].

Let X be a topological space and let κ be a cardinal number. A point $x \in X$ is said to be of *order less than or equal to* κ provided that x has a basis of open neighborhoods in X whose boundaries have at most κ elements; in this case we write $\mathrm{ord}_x(X) \leq \kappa$. If κ is the smallest cardinal number for which x has such neighborhoods, we will say that $\mathrm{ord}_x(X) = \kappa$. A point p of a topological space X is a *ramification point* of X if $\mathrm{ord}_p(X) \geq 3$, it is an *ordinary point* of X if $\mathrm{ord}_p(X) = 2$ and it is an *end point* of X if $\mathrm{ord}_p(X) = 1$. The sets of ramification points, ordinary points and end points of X will be denoted by $R(X)$, $OR(X)$ and $E(X)$, respectively. If X is a local dendrite we denote $E_D(X) = E(X) \cap \overline{R(X)}^X$ and $E_{ND}(X) = E(X) \setminus E_D(X)$. We say that X is *contractible* in $Y \supset X$ provided there exist a map $H\colon X \times [0,1] \to Y$ and a point $p \in Y$ such that $H(x,0) = x$ and $H(x,1) = p$ for each $x \in X$. In this case we say that X *is contracted* to p in Y. Further, if $Y = X$ we simply say that X is *contractible.* A point p of X is a *local cut point* of X if there exists a connected neighborhood U of p such that $U \setminus \{p\}$ is not connected ([31, p. 61]).

A metric space X is *locally contractible at a point* p provided that each neighborhood U of p in X contains a neighborhood V of p such that V is contractible in U to a point ([3, p. 28]). A metric space X is called *locally contractible* if X is locally contractible at each of its points.

In the rest of this section, we recall basic facts on quotient spaces and results that will be used throughout the paper. For a topological space X, the cone of X, $\mathrm{Cone}(X)$, is the quotient space that is obtained by identifying all the points $(x,1)$ in $X \times I$ to a single point ([22, p. 41, 3.15]). The suspension of X, $\mathrm{Sus}(X)$, is the quotient space that is obtained by identifying all the points $(x,1)$ in $X\times J$ to a single point, and all the points $(x,-1)$ to another point ([22, p. 42, 3.16]). Moreover, we denote the vertex of $\mathrm{Cone}(X)$ by v_X and the vertices of $\mathrm{Sus}(X)$ by v_X^1 and v_X^{-1}. We often assume without saying so that $X\times(-1,1)$ is a subspace of $\mathrm{Sus}(X)$. With this in mind, we write points in $\mathrm{Sus}(X)$ that are not the vertices as ordered pairs (x,t). Also, we consider $\mathrm{Cone}(X)$ as a subspace of $\mathrm{Sus}(X)$. When $A\subseteq X$, we consider $\mathrm{Sus}(A)$ as a subspace of $\mathrm{Sus}(X)$ with the same vertices, v_X^1 and v_X^{-1}, as in $\mathrm{Sus}(X)$.

Let $n\in\mathbb{N}$ with $n\geq 3$ and let X be an n-point (discrete) space. We define:

- the *n-theta* as the suspension over X; in the case that $n=3$, we will say that X is the *Greek letter theta*. We will always denote an m-theta by θ_m.
- the *simple n-od* as the cone over X; in the case that $n=3$, we will say that X is a *simple triod*, and the vertex of the cone is called the *core* of the simple triod. We will always denote a simple n-od by T_n.

Let $n\in\mathbb{N}$ with $n\geq 2$. A *n-rose* (also known as a bouquet of n circles) is the quotient space C/S, where C is a disjoint union of circles and S is a set consisting of one point from each circle. The circles of the rose are called *petals*. The rose with two petals is known as the *figure eight*.

Continuing on, we give some preliminary results which will be used in the proof of the main results.

Let us begin this subsection with the following result which is proved in [29, p. 133, 3.2].

Lemma 1. *Let X be a disconnected topological space. Then v_X is a cut point of* $\mathrm{Cone}(X)$.

Lemma 2. [14, p. 484, 3.0.1] *Let X be a metric space and let $x\in X$. Assume that x does not belong to any arc in X. Let A be a pathwise connected subcontinuum of* $\mathrm{Sus}(X)\setminus\{v_X^1,v_X^{-1}\}$ *such that $x\in\pi^*(A)$. Note that $\pi^*(A)$ is a pathwise connected subcontinuum of X that contains x. Thus, $\pi^*(A)=\{x\}$. Therefore $A\subseteq$* $\mathrm{Sus}(\{x\})$.

Lemma 3. [14, p. 484, 3.0.2] *Let X be a metric space without arcs. Then for each $x\in X$ and each $t\in(-1,1)$, the point (x,t) is not the core of a simple triod in* $\mathrm{Sus}(X)$.

Lemma 4. *Suppose that $A_1,\dots,A_n$ are closed subsets of a topological space X such that $X=\bigcup_{i=1}^n A_i$. Moreover, suppose that for each $i\in\{1,\dots,n\}$, B_i is a closed subset of a topological space Y and $f_i\colon A_i\to B_i$ are homeomorphisms such that:*
(i) for each $i,j\in\{1,\dots,n\}$ and $x\in A_i\cap A_j$, $f_i(x)=f_j(x)$,
(ii) for each $i,j\in\{1,\dots,n\}$, $f_i(A_i\cap A_j)=B_i\cap B_j=f_j(A_i\cap A_j)$.
Then, the function $f\colon X\to\bigcup_{i=1}^n B_i$ given by $f(x)=f_i(x)$, if $x\in A_i$, is a homeomorphism.

The following results are about absolute neighborhood retract (ANR) spaces.

Lemma 5. [3, p. 122, 10.4] *Let X be a metric, compact, finite dimensional space. Then X is an ANR if and only if X is locally contractible.*

Lemma 6. [15, p. 215, 2.2] *Let X be a one-dimensional continuum. Then X is an ANR if and only if X is a local dendrite.*

Lemma 7. [19, p. 226, 5.4.1] *Let X be an ANR and let U be an open subspace of X. Then U is an ANR.*

We will now present essential sets that will be employed repeatedly throughout the paper.

Notation 1. *Let X be a metric space. Denote*

$$M = \{w \in \mathrm{Sus}(X) \colon w \text{ is locally connected in } \mathrm{Sus}(X)\} \text{ and } N = \mathrm{Sus}(X) \setminus M.$$

Note that the sets M and N are invariant under homeomorphisms of $\mathrm{Sus}(X)$ *onto* $\mathrm{Sus}(X)$.

2. General properties of Z_X and $\frac{1}{n}$-homogeneity. Given a topological space X, we consider the quotient space $Z_X = \mathrm{Sus}(X)/_{\{v_X^1, v_X^{-1}\}}$, topologized with the quotient topology.

Proposition 1. *If $X = \{0\} \cup \{\frac{1}{n} \colon n \in \mathbb{N}\}$, then Z_X is the union of a sequence $\{C_n\}_{n=1}^{\infty}$ of circles in the plane joined by a point and converging to a limit circle C_0* [9, p. 24, Figure 361].

Notation 2. *Given a topological space X, $q_{Z_X} \colon \mathrm{Sus}(X) \to Z_X$ denotes the quotient map. Let α_X denote the point $q_{Z_X}\left(\{v_X^1, v_X^{-1}\}\right)$.*

Remark 1. Given a topological space X, note that $Z_X \setminus \{\alpha_X\}$ is a space homeomorphic to $\mathrm{Sus}(X) \setminus \{v_X^1, v_X^{-1}\}$ using the appropriate restriction of q_{Z_X} $\left(q_{Z_X}|_{\mathrm{Sus}(X)\setminus\{v_X, v_X^{-1}\}}\right)$.

Recall that an onto map $f \colon X \to Y$ between continua is said to be simple provided that $f^{-1}(y)$ has at most two points for all $y \in Y$.

Remark 2. The function $q_{Z_X} \colon \mathrm{Sus}(X) \to Z_X$ is a simple map.

Remark 3. Let X be a non-empty metric space such that $R(X) \neq \varnothing$. Let $\mathcal{V}$ be a basic neighborhood from α_X in Z_X. Observe that $q_{Z_X}^{-1}(\mathcal{V})$ is a open subset of $\mathrm{Sus}(X)$ such that $\{v_X^1, v_X^{-1}\} \subseteq q_{Z_X}^{-1}(\mathcal{V})$. There exists $t \in (-1, 1)$ such that

$$(X \times (-1, -t)) \cup (X \times (t, 1)) \cup (\{v_X^1, v_X^{-1}\}) \subseteq q_{Z_X}^{-1}(\mathcal{V}).$$

Let $t_0 = \frac{1+t}{2}$ and let $U = (X \times [-1, -t_0) \cup (X \times [t_0, 1)) \cup (\{v_X^1, v_X^{-1}\})$. Note that $q_{Z_X}(U) \subseteq \mathcal{V}$ and

$$\begin{aligned} q_{Z_X}(U) &= q_{Z_X}(X \times (-1, -t_0]) \cup q_{Z_X}(X \times [t_0, 1)) \cup q_{Z_X}(\{v_X^1, v_X^{-1}\}) = \\ &= q_{Z_X}((X \times (-1, -t_0]) \cup (X \times [t_0, 1)) \cup \{\alpha_X\}. \end{aligned}$$

This implies that $q_{Z_X}(U)$ is a connected neighborhood and $q_{Z_X}(U)$ is a non planar neighborhood of α_X.

Proposition 2. *Let $n \in \mathbb{N}$. Let X be a discrete space with n points.*

1) if $n = 1$, then Z_X is a space homeomorphic to a simple closed curve.

2) if $n \geq 2$, then Z_X is a space homeomorphic to n-rose.

As a consequence of [29, p. 133, 3.4] we obtain the following result.

Proposition 3. *Let X be a metric space without isolated points. If $\chi \in Z_X \setminus \{\alpha_X\}$, then χ is not a local cut point.*

Proposition 4. *Let $n \in \mathbb{N}$ with $n \geq 2$. If X be a discrete space with n points, then α_X is the only point that is the core of a simple triod in Z_X.*

The next result is used in the proof of Lemma 18.

Lemma 8. *Let X be a metric space. Then X is locally connected if and only if Z_X is locally connected.*

Proof. Suppose that X is locally connected. By [15, p. 222, 4.3] we obtain that $\mathrm{Sus}(X)$ is locally connected. Thus, by [13, p. 257, 5] and since $q_{Z_X}(\mathrm{Sus}(X)) = Z_X$, we obtain that Z_X is locally connected.

Reciprocally, suppose that Z_X is locally connected. By [13, p. 230, 3] we have that $Z_X \setminus \{\alpha_X\}$ is locally connected. Thus, by Remark 1 we obtain that $\mathrm{Sus}(X) \setminus \left\{v_X^1, v_X^{-1}\right\}$ is locally connected. Therefore, X is locally connected. □

Lemma 9. *If X is a metric space, then α_X is a local cut point of Z_X.*

Proof. Let $\mathcal{V}$ be a basic neighborhood from α_X in Z_X. Let U be as in Remark 3 such that $q_{Z_X}(U) \subseteq \mathcal{V}$. By Remark 3 we know that $q_{Z_X}(U)$ is a connected neighborhood of α_X. Moreover, $q_{Z_X}(U)\setminus\{\alpha_X\} = q_{Z_X}(X\times(-1,-t_0])\cup q_{Z_X}(X\times[t_0,1))$ is not connected. Therefore, α_X is a local cut point Z_X. □

Lemma 10. *Let X be a metric space. Then*

1) Z_X is locally contractible at the point α_X.

2) Z_X is locally connected at the point α_X.

Proof. Let $\mathcal{V}$ be a basic neighborhood from α_X in Z_X. We prove that Z_X is locally contractible at the point α_X. Let U be as in Remark 3 such that $q_{Z_X}(U) \subseteq \mathcal{V}$. Observe that $q_{Z_X}(U)$ is contractible to α_X. Therefore, Z_X is locally contractible at the point α_X and Z_X is locally connected at the point α_X.

Similarly, 2) is validated. □

A topological space X is *connected im kleinen at a point* p provided that each neighborhood of p contains a connected neighborhood of p [22, p. 75, 5.10]. The following corollary is an easy consequence of Lemma 10.

Corollary 1. *If X is a metric space, then Z_X is connected im Kleinen at the point α_X.*

Recall the following definition which can be found in [25].

Definition 1. A continuum Y is *n-homogeneous at a point $p \in Y$* (n a positive integer) provided that for any two n-element subsets A and B of Y such that $p \in A \cap B$, there is a homeomorphism h of Y onto Y such that $h(A) = B$ and $h(p) = p$.

Lemma 11. *If X is a homogeneous continuum, then Z_X is 2-homogeneous at α_X.*

Proof. Let A and B be sets of two elements in Z_X such that $\alpha_X \in A \cap B$. Note that $A = \{\alpha_X, \chi_1\}$ and $A = \{\alpha_X, \chi_2\}$, where $\chi_1, \chi_2 \in Z_X \setminus \{\alpha_X\}$. By Remark 1 we have that $\mathrm{Sus}(X) \setminus \{v_X, v_X^{-1}\}$ is homeomorphic to $Z_X \setminus \{\alpha_X\}$. Thus, by [14, p. 485, 4.1.2] we conclude that $Z_X \setminus \{\alpha_X\}$ is contained in a single orbit of Z_X. Since $\chi_1, \chi_2 \in Z_X \setminus \{\alpha_X\}$, there exist $G\colon Z_X \to Z_X$ a homeomorphism such that $G(\chi_1) = \chi_2$. Let $H\colon Z_X \to Z_X$ be given by

$$H(w) = \begin{cases} \alpha_X, & \text{if } w = \alpha_X; \\ G(w), & \text{if } w \neq \alpha_X. \end{cases}$$

Then H is a homeomorphism. Moreover, $H(A) = H(\{\alpha_X, \chi_1\}) = \{\alpha, \chi_2\} = B$. This concludes the proof that Z_X is 2-homogeneous at α_X. □

On the other hand, recall that a topological space X is $\frac{1}{n}$-homogeneous provided that there are exactly n orbits for the action of the group of homeomorphisms of X onto itself.

Lemma 12. [26, p. 52, 3.0.6] *Let X be a topological space. If A is contained in an orbit of X, then*
(i) $A \times (0,1)$ is contained in an orbit of $\mathrm{Cone}(X)$.
(ii) $A \times \{0\}$ is contained in an orbit of $\mathrm{Cone}(X)$.

Lemma 13. [27, p. 67, 7.2] *For a topological space X the following statements hold*
(i) The vertices v_X^1 and v_X^{-1} belong to the same orbit of $\mathrm{Sus}(X)$.
(ii) Let $A \subseteq X$. If A is contained in some orbit of X, then $A \times (-1,1)$ is contained in some orbit of $\mathrm{Sus}(X)$.

The harmonic sequence has degree of homogeneity two. The Proposition 5 provides an proof of this fact.

Proposition 5. *Let X be the harmonic sequence $\{0\} \cup \{\frac{1}{n} \colon n \in \mathbb{N}\}$. Then $\mathrm{d_H}(X) = 2$. Moreover, the orbits of X are $\{0\}$ and $\{\frac{1}{n} \colon n \in \mathbb{N}\}$.*

Proof. Define the following subsets of X.

$$M' = \{x \in X \colon x \text{ accumulation point of } X\} \text{ and } N' = X \setminus M'.$$

Note that $M' = \{0\}$ and $N' = X \setminus \{0\}$. Since M' and N' are invariant under homeomorphisms of X we have that $M' = \{0\}$ is a orbit of X. In the rest of the proof we will show that N' is the other orbit de X. For this, let $p, q \in N'$. Since N' is homogeneous there exist a homeomorphism $f \colon N' \to N'$ such that $f(p) = q$. Let $h_{pq} \colon X \to X$ be given by

$$h_{pq}(x) = \begin{cases} 0, & \text{if } x = 0; \\ f(x), & \text{if } x \in N'. \end{cases} \tag{1}$$

Observe that $\{0\}$ and N' are closed subsets in X. Moreover, note that $X = \{0\} \cup N'$ and also $\mathrm{id}_{\overline{N'}}$ and f are homeomorphisms. Since $\overline{N'} \cap \{0\} = \{0\}$, by Lemma 4 we conclude

$$h_{pq} \text{ is a homeomorphism.} \tag{2}$$

Moreover, $h_{pq}(p) = f(p) = q$. Therefore, $\mathcal{O}_X(p) = \mathcal{O}_X(q)$. Thus, N' is contained in a single orbit of X. However, since $M' = \{0\}$ and M' is invariant under homeomorphisms of X onto X, we obtain that N' is an orbit of X. □

The suspension over the harmonic sequence has degree of homogeneity three. The Proposition 6 provides an proof of this fact, even more, we determine orbits specifically.

Proposition 6. *Let X be the harmonic sequence $\{0\} \cup \{\frac{1}{n} \colon n \in \mathbb{N}\}$. Then $\mathrm{d_H}(\mathrm{Sus}(X)) = 3$. Moreover, the orbits of $\mathrm{Sus}(X)$ are $\{v_X^1, v_X^{-1}\}$, $\{0\} \times (-1,1)$ and $(X \setminus \{0\}) \times (-1,1)$.*

Proof. Observe that v_X^1 and v_X^{-1} are only local cut points the core of the simple triod in $\mathrm{Sus}(X)$ (see Lemma 3). Thus, $\{v_X^1, v_X^{-1}\}$ is a orbit of $\mathrm{Sus}(X)$. By Proposition 5 we know that the orbits of X are $O_X^1 = \{0\}$ y $O_X^2 = X \setminus \{0\}$. On the other hand, by (ii) of Lemma 13 we have that there exist $\mathcal{O}^1_{\mathrm{Sus}(X)}$, and $\mathcal{O}^2_{\mathrm{Sus}(X)}$, orbits of $\mathrm{Sus}(X)$ sucht that $O_X^1 \times (-1,1) \subseteq \mathcal{O}^1_{\mathrm{Sus}(X)}$ and $O_X^2 \times (-1,1) \subseteq \mathcal{O}^2_{\mathrm{Sus}(X)}$. Let M and N be as defined in Notation 1. Note that $M = \{0\} \times (-1,1)$ and $N = \mathrm{Sus}(X \setminus \{0\})$. Since M and N are invariant under homeomorphisms of $\mathrm{Sus}(X)$ onto $\mathrm{Sus}(X)$, we obtain that $\mathcal{O}^1_{\mathrm{Sus}(X)} \neq O^2_{\mathrm{Sus}(X)}$.

Therefore, $\mathrm{Sus}(X)$ is $\frac{1}{3}$-homogeneous. □

The cone over the harmonic sequence has degree of homogeneity five. The Proposition 7 provides an proof of this fact, even more, we determine orbits specifically.

Proposition 7. *Let X be the harmonic sequence $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Then* $\mathrm{d_H}(\mathrm{Cone}(X)) = 5$. *Moreover, the orbits of* $\mathrm{Cone}(X)$ *are* $\{v_X\}$, $\{0\} \times (0,1)$, $\{(0,0)\}$, $(X \setminus \{0\}) \times \{0\}$ *and* $(X \setminus \{0\}) \times (0,1)$.

Proof. Observe that v_X is only point that is the core of the simple triod in $\mathrm{Cone}(X)$ (see Lemma 3). Then

$$\{v_X\} \text{ is an orbit of } \mathrm{Cone}(X). \tag{3}$$

In the rest of the proof we will show that $\{(0,0)\}$, $\{0\} \times (0,1)$, $(X \setminus \{0\}) \times \{0\}$ and $(X \setminus \{0\}) \times (0,1)$ are the others orbits of $\mathrm{Cone}(X)$. By Proposition 5 we know that the orbits of X are $O_X^1 = \{0\}$ y $O_X^2 = X \setminus \{0\}$. In consequence, by Lemma 12 we have that there exist $\mathcal{O}^1_{\mathrm{Cone}(X)}$, $\mathcal{O}^2_{\mathrm{Cone}(X)}$, $\mathcal{O}^3_{\mathrm{Cone}(X)}$, and $\mathcal{O}^4_{\mathrm{Cone}(X)}$, orbits of $\mathrm{Cone}(X)$ sucht that

$$O_X^1 \times \{0\} \subseteq \mathcal{O}^1_{\mathrm{Cone}(X)}, \quad O_X^2 \times \{0\} \subseteq \mathcal{O}^2_{\mathrm{Cone}(X)},$$
$$O_X^1 \times (0,1) \subseteq \mathcal{O}^3_{\mathrm{Cone}(X)}, \quad O_X^2 \times (0,1) \subseteq \mathcal{O}^4_{\mathrm{Cone}(X)}.$$

Denote

$$M = \{w \in \mathrm{Cone}(X) \colon w \text{ is locally connected in } \mathrm{Cone}(X)\} \text{ and } N = \mathrm{Cone}(X) \setminus M.$$

Note that the sets M and N are invariant under homeomorphisms of $\mathrm{Cone}(X)$ onto $\mathrm{Cone}(X)$. Note that $M = \{0\} \times [0,1)$ and $N = (X \setminus \{0\} \times [0,1))$. Since M and N are invariant under homeomorphisms of $\mathrm{Cone}(X)$ onto $\mathrm{Cone}(X)$, we obtain that $\mathcal{O}^1_{\mathrm{Cone}(X)} \neq O^2_{\mathrm{Cone}(X)}$, $\mathcal{O}^1_{\mathrm{Cone}(X)} \neq O^4_{\mathrm{Cone}(X)}$ and $\mathcal{O}^3_{\mathrm{Cone}(X)} \neq O^4_{\mathrm{Cone}(X)}$. In the rest of the proof we will show that $O^1_{\mathrm{Cone}(X)} \neq O^3_{\mathrm{Cone}(X)}$ and $O^2_{\mathrm{Cone}(X)} \neq O^4_{\mathrm{Cone}(X)}$.

Consider the following subsets of $\mathrm{Cone}(X)$:

$$P = \{w \in \mathrm{Cone}(X) \colon w \text{ is a local cut point in } \mathrm{Cone}(X)\} \text{ and } Q = \mathrm{Cone}(X) \setminus P.$$

Note that $P = X \times \{0\}$ y $Q = (X \times (0,1)) \cup \{v_X\}$. Since the sets P and Q are invariant under homeomorphisms of $\mathrm{Cone}(X)$ onto $\mathrm{Cone}(X)$. We conclude that $O^1_{\mathrm{Cone}(X)} \neq O^3_{\mathrm{Cone}(X)}$ and $O^2_{\mathrm{Cone}(X)} \neq O^4_{\mathrm{Cone}(X)}$. Therefore, $\mathrm{Cone}(X)$ is $\frac{1}{5}$-homogeneous. □

We conclude this section with the theorem that is stated below; hence, it is essential to remember the concept that comes after. A *hairy point* F_ω, is the union of countably infinitely many arcs $A_1, A_2, \ldots$ such that $\lim\limits_{i\to\infty} \mathrm{diam}(A_i) = 0$; when the arcs A_i emanate from a single point p, and are otherwise disjoint one from another ([9, p. 46]). Let $\mathcal{A}_0$ be the class of local dendrites that are either arcs, simple closed curves or a hairy point F_ω; we also consider the class $\mathcal{F}_0 = \mathcal{A}_0 \cup \{\theta_m \colon m \geq 3\} \cup \{T_n \colon n \geq 3\}$.

Theorem 3. *Let X be a local dendrite.*
(i) If X is a simple closed curve or $X \in \{F_\omega\} \cup \{T_n \colon n \in \mathbb{N} \setminus \{1,2\}\}$, then $\mathrm{d_H}(\mathrm{Cone}(X)) = \mathrm{d_H}(X) + 1$.
(ii) If $X \in \{\theta_m \colon m \in \mathbb{N} \setminus \{1,2\}\}$, then $\mathrm{d_H}(\mathrm{Cone}(X)) = 4 = \mathrm{d_H}(X) + 2$.
(iii) If $X \notin \mathcal{F}_0$, if $\mathrm{d_H}(X)$ *is finite and if k is the number of orbits of X contained in $E_{ND}(X)$, then* $\mathrm{d_H}(\mathrm{Cone}(X)) = 1 + 2\,\mathrm{d_H}(X) - 2k$.
(iv) If $d_H(X)$ is infinite or X is an arc, then $\mathrm{d_H}(\mathrm{Cone}(X)) = \mathrm{d_H}(X)$.

3. $\frac{1}{n}$-homogeneity of Z_X. In this section we present a result on the degree of homogeneity of Z_X. We start this subsection with the following notations.

Notation 4. *Let X be a metric space. Consider the following subsets of Z_X:*

$$M' = \{w \in Z_X \colon w \text{ i is locally connected in } Z_X\} \text{ and } N' = Z_X \setminus M'.$$

Note that the sets M' and N' are invariant under homeomorphisms of Z_X onto Z_X.

Notation 5. *Let X be a metric space. Consider the following subsets of Z_X:*
$$K = \{w \in Z_X \colon w \text{ is locally contractible in } Z_X\} \text{ and } L = Z_X \setminus K.$$
Note that the sets K and L are invariant under homeomorphisms of Z_X onto Z_X.

Notation 6. *Let X be a metric space. Consider the following subsets of Z_X:*
$$P = \{w \in Z_X \colon w \text{ is a local cut point in } Z_X\} \text{ and } Q = Z_X \setminus K.$$
Note that the sets P and Q are invariant under homeomorphisms of Z_X onto Z_X.

The following result which is a consequence of Proposition 2.

Proposition 8. *If X is a discrete space with one point, then $\mathrm{d_H}(Z_X) = 1$.*

Proposition 9. *Let $X = \{0\} \cup \{\frac{1}{n} \colon n \in \mathbb{N}\}$. Then $\mathrm{d_H}(Z_X) = 3$. Moreover, the orbits of Z_X are $\{\alpha_X\}$, $\{0\} \times (-1, 1)$ and $(X \setminus \{0\}) \times (-1, 1)$.*

Proof. By Proposition 6 we know that $\mathrm{d_H}(\mathrm{Sus}(X)) = 3$. Moreover, their orbits are the sets

$$\mathcal{O}^1_{\mathrm{Sus}(X)} = \{v^1_X, v^{-1}_X\}, \mathcal{O}^2_{\mathrm{Sus}(X)} = \{0\} \times (-1, 1) \text{ and } \mathcal{O}^3_{\mathrm{Sus}(X)} = (X \setminus \{0\}) \times (-1, 1).$$

Let P and Q be as defined in Notation 6. This implies that $P = \{\alpha_X\}$ and $Q = \{0\} \times (-1, 1)$. Since P is invariant under homeomorphisms of Z_X onto Z_X it follows that
$$\mathcal{O}^1_{Z_X} = P. \tag{4}$$
Hence, in the rest of the proof we will prove that the other two orbits of Z_X are in Q. By Remark 1 we know that $Z_X \setminus \{\alpha_X\}$ is homeomorphic to $\mathrm{Sus}(X) \setminus \{v^1_X, v^{-1}_X\}$. Denote

$$M = \{w \in \mathrm{Sus}(X) \setminus \{v^1_X, v^{-1}_X\} \colon w \text{ is locally connected in } \mathrm{Sus}(X) \setminus \{v^1_X, v^{-1}_X\}\}$$

and $N = \big(\mathrm{Sus}(X) \setminus \{v^1_X, v^{-1}_X\}\big) \setminus M$.

It follows that
$$M = \mathcal{O}^3_{\mathrm{Sus}(X)} \text{ and } N = \mathcal{O}^2_{\mathrm{Sus}(X)}. \tag{5}$$
Note that the sets M and N are invariant under homeomorphisms of $\mathrm{Sus}(X) \setminus \{v^1_X, v^{-1}_X\}$ onto $\mathrm{Sus}(X) \setminus \{v^1_X, v^{-1}_X\}$. Finally, from (4) and (5) we have that the orbits of Z_X are $\{\alpha_Z\}$, $\{0\} \times (-1, 1)$ and $(X \setminus \{0\}) \times (-1, 1)$. □

The next result is used in the proof of Lemma 15.

Theorem 7. *Let X be a homogeneous metric space. If X is nonlocally contractible, then $\mathrm{d_H}(Z_X) = 2$. Moreover, the orbits of Z_X are $Z_X \setminus \{\alpha_X\}$ and $\{\alpha_X\}$.*

Proof. Suppose that X is nonlocally contractible. Let K and L be as defined in Notation 5. By Lemma 10 we conclude that $K = \{\alpha_X\}$ and $L = Z_X \setminus \{\alpha_X\}$. Since K and L are invariant under homeomorphisms of Z_X onto Z_X it follows that K is an orbit of Z_X. In the rest of the proof we will show that L is the other orbits of Z_X. By Remark 1 we know that $Z_X \setminus \{\alpha_X\}$ is a space homeomorphic to $\mathrm{Sus}(X) \setminus \{v^1_X, v^{-1}_X\}$. Let $p, q \in L$ and let $h_{pq} \colon Z_X \to Z_X$ be given by
$$h_{pq}(x) = \begin{cases} x, & \text{if } x \in \{\alpha_X\}; \\ q_{Z_X}(x), & \text{if } x \in Z_X \setminus \{\alpha_X\}. \end{cases} \tag{6}$$
Observe that h_{pq} is a homeomorphism. Moreover, $h_{pq}(p) = q_{Z_X}(p) = q$. Therefore, $\mathcal{O}_{Z_X}(p) = \mathcal{O}_{Z_X}(q)$. Thus, L is contained in a single orbit of Z_X. □

Our next result follows from Theorem 7.

Corollary 2. *If X is the Menger Curve, then* $\mathrm{d_H}(Z_X) = 2$.

Since local contractibility at every point implies connectedness im kleinen (and thus, local connectedness), then we have the following consequence of Theorem 7.

Theorem 8. *Let X be a homogeneous metric space. If X is nonlocally connected, then* $\mathrm{d_H}(Z_X) = 2$. *Moreover, the orbits of Z_X are $Z_X \setminus \{\alpha_X\}$ and $\{\alpha_X\}$.*

Bing and Jones produced the circle of pseudo-arcs, a circle-like continuum with a continuous decomposition into pseudo-arcs such that the decomposition space is a simple closed curve [17, p. 172]. Thus, it's important to recall that a solenoid, the pseudoarc, a circle of pseudoarcs and the Cantor set are homogeneous of finite dimension (see [22, p. 25, 2.16], [16, p. 167, 12.31], [17, p. 172] and [7, p. 100, 2.37], respectively). The followig corollary is a consequence from Theorem 8.

Corollary 3. *If X is the pseudoarc (Cantor set, a circle of pseudoarcs or a solenoid), then* $\mathrm{d_H}(Z_X) = 2$.

The next result is used in the proof of Lemma 16.

Theorem 9. *Let X be a homogeneous metric space such that $|X| \geq 2$. If X does not contain arcs, then* $\mathrm{d_H}(Z_X) = 2$. *Moreover, their orbits are $Z_X \setminus \{\alpha_X\}$ and $\{\alpha_X\}$.*

Proof. Suppose that X does not contain arcs. Consider the following subsets of Cone(X):

$$\hat{A} = \{w \in Z_X \colon \text{ is the core of a simple triod in } Z_X\}, \quad \hat{B} = Z_X \setminus \hat{A}.$$

By Lemma 3 we obtain that $\hat{A} = \{\alpha_X\}$ and $\hat{B} = Z_X \setminus \{\alpha_X\}$. It follows that $\mathcal{O}^1_{Z_X} = \{\alpha_X\}$. In the rest of the proof we will show that $\hat{B}$ is the other orbit of Z_X. By Remark 1 we know that $Z_X \setminus \{\alpha_X\}$ is a space homeomorphic to $\mathrm{Sus}(X) \setminus \{v^1_X, v^{-1}_X\}$. Let $p, q \in \hat{B}$ and let $h_{pq} \colon Z_X \to Z_X$ be given by

$$h_{pq}(x) = \begin{cases} x, & \text{if } x \in \{\alpha_X\}; \\ q_{Z_X}(x), & \text{if } x \in Z_X \setminus \{\alpha_X\}. \end{cases} \tag{7}$$

Observe that h_{pq} is a homeomorphism. Moreover, $h_{pq}(p) = f(p) = q$. Therefore, $\mathcal{O}_{Z_X}(p) = \mathcal{O}_{Z_X}(q)$. Thus, $\hat{B}$ is contained in a single orbit $\mathcal{O}^2_{Z_X}$ of Z_X. □

Our next result follow from Theorem 9.

Corollary 4. *Let $n \in \mathbb{N}$, $n \geq 2$. If X is a discrete space with n points, then* $\mathrm{d_H}(Z_X) = 2$. *Moreover, their orbits are $Z_X \setminus \{\alpha_X\}$ and $\{\alpha_X\}$.*

Our next result follow from Proposition 2 and Corollary 4.

Corollary 5. *Let $n \in \mathbb{N}$, $n \geq 2$. If X is the n-rose, then* $\mathrm{d_H}(X) = 2$.

Our next result follow from Proposition 8 and Corollary 4.

Corollary 6. *Let $n \in \mathbb{N}$ and let X be a discrete space with n points. Then* $\mathrm{d_H}(Z_X) = 1$ *if and only if $n = 1$.*

The following theorem will be used in the proof of Lemma 17.

Theorem 10. *Let X be a homogeneous metric space without isolated points. If X is disconnected, then* $\mathrm{d_H}(Z_X) = 2$.

Proof. Suppose that X is disconnected. Let P and Q be as defined in Notation 6. By Proposition 3 and Lemma 9 we obtain that $P = \{\alpha_X\}$ and $Q = Z_X \setminus \{\alpha_X\}$. Since P is invariant under homeomorphisms of Z_X onto Z_X it follows that $\mathcal{O}^1_{Z_X} = \{\alpha_X\}$. Let $p, q \in N'$ and let $h_{pq}\colon Z_X \to Z_X$ be given by

$$h_{pq}(x) = \begin{cases} x, & \text{if } x \in \{\alpha_X\}; \\ q_{Z_X}(x), & \text{if } x \in Z_X \setminus \{\alpha_X\}. \end{cases} \tag{8}$$

Observe that h_{pq} is a homeomorphism. Moreover, $h_{pq}(p) = f(p) = q$. Therefore, $\mathcal{O}_{Z_X}(p) = \mathcal{O}_{Z_X}(q)$. Thus, Q is contained in a single orbit $\mathcal{O}^2_{Z_X}$ of Z_X. In consequence $\mathrm{d_H}(Z_X) = 3$. □

Remark 4. Let X be a metric space and $p \notin X$ an isolated point. Let $Y = X \cup \{p\}$. Then:
(a) each $t \in (-1, 1)$, the point (p, t) is a local cut point of $\mathrm{Sus}(X)$.
(b) v^1_X and v^{-1}_X are local cut points of $\mathrm{Sus}(X)$.

As a consequence of Remark 4 we obtain the following remark.

Remark 5. Let X be a metric space and $p \notin X$. Let $Y = X \cup \{p\}$. If p is an isolated point of Y. Then:
(a) each each $t \in (-1, 1)$, the point $q_{Z_X}(\{p\} \times (-1, 1))$ is a local cut point of Z_Y,
(b) α_X is a local cut point of Z_Y.

Theorem 11. *Let X be a homogeneous, metric space and p a point such that $p \notin X$. Let $Y = X \cup \{p\}$. If p is an isolated point of Y, then* $\mathrm{d_H}(Z_Y) = 3$.

Proof. Suppose that p is an isolated point of Y. Let P and Q be as defined in Notation 6. By Lemma 9 and Remark 5 we obtain that $P = \{\alpha_Y\} \cup \{q_{Z_Y}(\{p\} \times (-1, 1))\}$ and $Q = q_{Z_Y}(X \times (-1, 1))$. Note that the sets P and Q are invariant under homeomorphisms of Z_Y onto Z_Y. Thus, it follows that $\mathcal{O}^1_{Z_Y} = Q$. In the rest of the proof we will prove that the other two orbits of Z_Y are in P. For this, consider the following subsets of P:

$$P_1 = \{w \in P \colon w \in \overline{Q}\} \text{ and } P_2 = P \setminus P_1.$$

Note that the sets P_1 and P_2 are invariant under homeomorphisms of P onto P. Moreover, $P_1 = \{\alpha_Y\}$ and $P_2 = q_{Z_Y}(\{p\} \times (-1, 1))$. It follows that $P_1 = \mathcal{O}^2_{Z_Y}$ and $P_2 = \mathcal{O}^3_{Z_Y}$. Therefore, $\mathrm{d_H}(Z_Y) = 3$. □

Our next results follow from Theorem 11.

Corollary 7. *Let X be a simple closed curve p a point such that $p \notin X$. If $Y = X \cup \{p\}$, then* $\mathrm{d_H}(Z_Y) = 3$.

In [2, p. 322, III] it is shown that the Menger curve is homogeneous. Thus

Corollary 8. *Let X be the Menger curve and p a point such that $p \notin X$. If $Y = X \cup \{p\}$, then* $\mathrm{d_H}(Z_Y) = 3$.

Since a solenoid, the pseudoarc, a circle of pseudoarcs and the Cantor set are homogeneous of finite dimension (see [22, p. 25, 2.16], [16, p. 167, 12.31], [17, p. 172] and [7, p. 100, 2.37], respectively) then as a particular case of Theorem 11 we have the following results.

Corollary 9. *Let X be a continuum and p a point such that $p \notin X$. If $Y = X \cup \{p\}$ and X is the solenoid (the pseudoarc, a circle of pseudoarcsc or the Cantor set), then* $\mathrm{d_H}(Z_Y) = 3$.

Recall that the m-dimensional torus, often called the m-torus, is the product space of m circles. That is: $T^m = \underbrace{S^1 \times \cdots \times S^1}_{m}$.

Corollary 10. *Let X be a m-torus (or the Hilbert cube) and p a point such that $p \notin X$. If $Y = X \cup \{p\}$, then* $\mathrm{d_H}(Z_Y) = 3$.

Theorem 12. *Let X be a locally connected, homogeneous, metric space and $p \notin X$. Let $Y = X \cup \{p\}$. If p is an accumulation point of Y and X does not contain arcs, then* $\mathrm{d_H}(Z_Y) = 3$.

Proof. Suppose that p is an accumulation point of Y and X does not contain arcs. Let M' and N' be as defined in Notation 4. Note that the sets M' and N' are invariant under homeomorphisms of Z_Y onto Z_Y. By Lemma 10 we obtain that $K = \{\alpha_Y\} \cup \{q_{Z_Y}(X \times (-1,1))\}$ and $L = q_{Z_Y}(\{p\} \times (-1,1))$. Since the sets K and L are invariant under homeomorphisms of Z_Y onto Z_Y. Thus, it follows that $\mathcal{O}^1_{Z_Y} = L$. In the rest of the proof we will prove that the other two orbits of Z_Y are in K. For this, consider the following subsets of K:

$$K_1 = \{w \in K \colon w \text{ is the core of the simple triod in } K\} \text{ and } K_2 = K \setminus K_1.$$

Note that the sets K_1 and K_2 are invariant under homeomorphisms of K onto K. Moreover, $K_1 = \{\alpha_Y\}$ and $K_2 = q_{Z_Y}(X \times (-1,1))$. It follows that $K_1 = \mathcal{O}^2_{Z_Y}$ and $K_2 = \mathcal{O}^3_{Z_Y}$. Therefore, $\mathrm{d_H}(Z_Y) = 3$. □

As a consequence of the theorem proved (Theorem 12) and Proposition 1, we have the following:

Corollary 11. *If $X = \{0\} \cup \{\frac{1}{n} \colon n \in \mathbb{N}\}$, then the homogeneity degree of the union of a sequence $\{C_n\}_{n=1}^{\infty}$ of circles in the plane joined by a point and converging to a limit circle C_0 is three.*

Lemma 14. *Let X be a homogeneous separable metric space. If $\mathrm{d_H}(Z_X) \neq 2$, then X is locally contractible.*

Proof. Suppose that $\mathrm{d_H}(Z_X) \neq 2$ and X is not locally contractible. Since X is homogeneous, by Theorem 7 we obtain that $\mathrm{d_H}(Z_X) = 2$, a contradiction. Therefore, X is locally contractible. □

Lemma 15. *Let X be a homogeneous separable metric space. If $\mathrm{d_H}(Z_X) \neq 2$, then X is locally connected.*

Proof. Suppose that $\mathrm{d_H}(Z_X) \neq 2$ and X is not locally connected. Since X is homogeneous, by Theorem 8 we obtain that $\mathrm{d_H}(Z_X) = 2$, a contradiction. Therefore, X is locally connected. □

Our next result follows from Lemma 5 and Lemma 14.

Corollary 12. *Let X be a homogeneous, compact, finite dimensional metric space. If $\mathrm{d_H}(Z_X) \neq 2$, then X is ANR.*

Our next result follows from Lemma 6 and Corollary 12.

Corollary 13. *Let X be a homogeneous, one dimensional continuum. If $\mathrm{d_H}(Z_X) \neq 2$, then X is a local dendrite.*

Lemma 16. *Let X be a homogeneous separable metric space. If $\mathrm{d_H}(Z_X) \neq 2$, then X contains arcs.*

Proof. Suppose that $\mathrm{d_H}(Z_X) \neq 2$ and X does not contain arcs. Since X is homogeneous, by Theorem 9 that $\mathrm{d_H}(Z_X) = 2$, a contradiction. Therefore, X contains arcs. □

Lemma 17. *Let X be a homogeneous separable metric space without isolated points. If $\mathrm{d_H}(Z_X) \neq 2$, then X is connected.*

Proof. Suppose that $\mathrm{d_H}(Z_X) \neq 2$ and X does not connected. Since X is homogeneous, by Theorem 10 that $\mathrm{d_H}(Z_X) = 2$, a contradiction. Therefore, X is connected. □

Lemma 18. *Let X be a separable metric space. If $\mathrm{d_H}(Z_X) = 1$, then Z_X is locally connected.*

Proof. Suppose that $\mathrm{d_H}(Z_X) = 1$ and that Z_X is not locally connected. Since $\mathrm{d_H}(Z_X) = 1$ we have that Z_X is not locally connected in any point, a contradiction to Lemma 10. Therefore, Z_X is locally connected. □

4. $\frac{1}{n}$-homogeneity of $\mathrm{Sus}(Z_X)$. In this section we present a result on the degree of homogeneity of $\mathrm{Sus}(Z_X)$. We start this section with the following result which is an easy consequence of Proposition 2.

Proposition 10. *If X is a discrete space with one point, then $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 1$.*

Lemma 19. *Let X be a homogeneous metric space with no arcs such that $|X| \geq 2$. Then $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$.*

Proof. Suppose that X does not contain arcs. Let $\mathcal{Z} = \mathrm{Sus}(Z_X)$. By Theorem 9 we know that $\mathrm{d_H}(Z_X) = 2$. Moreover, its orbits are $Z_X \setminus \{\alpha_X\}$ and $\{\alpha_X\}$. By Lemma 13, we have that there exist orbits $\mathcal{O}^1_{\mathcal{Z}}$ and $\mathcal{O}^2_{\mathcal{Z}}$, of $\mathcal{Z}$, such that:

$$\{\alpha_X\} \times (-1,1) \subseteq \mathcal{O}^1_{\mathcal{Z}} \quad \text{and} \quad (Z_X \setminus \{\alpha_X\}) \times (-1,1) \subseteq \mathcal{O}^2_{\mathcal{Z}}. \tag{9}$$

Consider the following subsets of $\mathcal{Z}$:

$$\mathcal{A} = \{w \in \mathcal{Z} \colon w \text{ has a planar neighborhood in } \mathcal{Z}\}; \quad \mathcal{B} = \mathcal{Z} \setminus \mathcal{A}.$$

Given that $|X| \geq 2$, we have that v^1_X is the core of the simple triod in $\mathrm{Sus}(X)$. This implies that $\alpha_X \in \mathcal{B}$. In consequence:

$$\mathcal{A} = \mathcal{Z} \setminus \mathrm{Sus}\left(\{\alpha_X\}\right); \quad \mathcal{B} = \mathrm{Sus}\left(\{\alpha_X\}\right). \tag{10}$$

Since the sets $\mathcal{A}$ and $\mathcal{B}$ are invariant under homeomorphisms of $\mathcal{Z}$ in $\mathcal{Z}$, from (9), (10) we obtain that:

$$\mathcal{O}^1_{\mathcal{Z}} \neq O^2_{\mathcal{Z}}. \tag{11}$$

Now, if $v^1_{Z_X} \in \mathcal{O}^1_{\mathcal{Z}}$, this implies that $v^{-1}_{Z_X} \in \mathcal{O}^1_{\mathcal{Z}}$(see Lemma 13-(i)). In consequence $\mathcal{O}^1_{\mathcal{Z}} = \mathcal{B}$. This would imply that the arc $\mathcal{B}$ is homogeneous; a contradiction. Therefore, $v^1_{Z_X} \notin \mathcal{O}^1_{\mathcal{Z}}$. Thus, $\{v^1_{Z_X}, v^{-1}_{Z_X}\}$ is the other orbit of $\mathcal{Z}$. Finally, $\mathrm{d_H}(\mathcal{Z}) = 3$. □

As a consequence of Lemma 19 we get the following corollary.

Corollary 14. *If X is a discrete space containing more than one point, the* $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$.

As a consequence of Proposition 2 and Corollary 14 we get the following result.

Corollary 15. *Let $n \in \mathbb{N}$, $n \geq 2$. If X is the n-rose, then* $\mathrm{d_H}(\mathrm{Sus}(X)) = 3$.

Note that Corollary 15 gives us another alternative way to calculate the degree of homogeneity of the suspension of the finite graph, the n-rose.

Theorem 13. *Let X be a homogeneous metric space. If X is nonlocally contractible, then* $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$.

Proof. Suppose that X is nonlocally contractible. Let $\mathcal{Z} = \mathrm{Sus}(Z_X)$. By Theorem 7 we have that $\mathrm{d_H}(Z_X) = 2$. Moreover, the orbits Z_X are $\{\alpha_X\}$ and $Z_X \setminus \{\alpha_X\}$. By Lemma 13 we have that there exist $\mathcal{O}^1_{\mathcal{Z}}$ and $\mathcal{O}^2_{\mathcal{Z}}$, orbits of $\mathcal{Z}$, such that:

$$\{\alpha_X\} \times (-1,1) \subseteq \mathcal{O}^1_{\mathcal{Z}}, \quad (Z_X \setminus \{\alpha_X\}) \times (-1,1) \subseteq \mathcal{O}^2_{\mathcal{Z}}. \tag{12}$$

Consider the following subsets of $\mathcal{Z}$:

$$\mathcal{M} = \{w \in \mathcal{Z} \colon \mathcal{Z} \text{ is locally contractible in } w\} \qquad \text{and } \mathcal{N} = \mathcal{Z} \setminus \mathcal{M}.$$

By Lemma 10 we obtain that $\mathcal{Z}$ is locally contractible in the points $\{\alpha_X\} \times (-1,1)$. Since $\mathcal{Z}$ is locally contractible in $v^1_{\mathcal{Z}}$ and $v^{-1}_{\mathcal{Z}}$ we have that:

$$\mathcal{M} = \mathrm{Sus}(\{\alpha_X\}), \quad \mathcal{N} = (Z_X \setminus \{\alpha_X\}) \times (-1,1). \tag{13}$$

Since the sets $\mathcal{M}$ and $\mathcal{N}$ are invariant under homeomorphisms of $\mathcal{Z}$ in $\mathcal{Z}$ from (12), (13) it follows that

$$(Z_X \setminus \{\alpha_X\}) \times (-1,1) = \mathcal{O}^2_{\mathcal{Z}}. \tag{14}$$

If $v^1_{Z_X} \in \mathcal{O}^1_{\mathcal{Z}}$, this implies that $v^{-1}_{Z_X} \in \mathcal{O}^{\mathcal{Z}}_1$(see Lemma 13-(i)). Thus, $\mathcal{O}^1_{\mathcal{Z}} = \mathcal{M}$ but $\mathcal{M}$ is an arc, a contradiction. Therefore, $\{v^1_{Z_X}, v^{-1}_{Z_X}\}$ is the other orbit of $\mathcal{Z}$. Finally, $\mathrm{d_H}(\mathcal{Z}) = 3$. □

Our next result follow from Theorem 13.

Corollary 16. *If X is the Menger curve, then* $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$.

Since local contractibility at every point implies connectedness im kleinen (see [15, p. 3, 2.3]) and thus, local connectedness, then we have the following results.

Corollary 17. *Let X be a homogeneous metric space. If X is nonlocally connnected, then* $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$.

Now Corollary 17 yields the following results.

Corollary 18. *If X is the pseudoarc, then* $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$.

Corollary 19. *If X is the Cantor set, then* $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$.

Corollary 20. *If X is a circle of pseudoarcs, then* $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$.

Corollary 21. *If X is a solenoid, then* $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$.

Theorem 14. *Let X be a homogeneous metric space. Let $Y = X \cup \{p\}$, where p is an isolated point of Y. If X is nonlocally connected, then* $\mathrm{d_H}(\mathrm{Sus}(Z_Y)) = 4$.

Proof. Suppose that X is nonlocally connected. Let $\mathcal{Z} = \mathrm{Sus}(Z_Y)$. By Theorem 11 we have that $\mathrm{d_H}(Z_Y) = 3$. Moreover, the orbits Z_Y are $\{\alpha_Y\}$, $q_{Z_Y}(\{p\}\times(-1,1))$ and $q_{Z_Y}(X\times(-1,1))$. By Lemma 13 we have that there exist $\mathcal{O}^1_{\mathcal{Z}}$, $\mathcal{O}^2_{\mathcal{Z}}$ and $\mathcal{O}^3_{\mathcal{Z}}$, orbits of $\mathcal{Z}$, such that:

$$\{\alpha_Y\} \times (-1,1) \subseteq \mathcal{O}^1_{\mathcal{Z}}, \quad q_{Z_Y}(\{p\} \times (-1,1)) \subseteq \mathcal{O}^2_{\mathcal{Z}}, \tag{15}$$

$$q_{Z_Y}(X \times (-1,1)) \subseteq \mathcal{O}^3_{\mathcal{Z}}. \tag{16}$$

Consider the following subsets of $\mathcal{Z}$:

$$\hat{M} = \{w \in \mathcal{Z}\colon \mathcal{Z} \text{ is locally connected in } w\} \text{ and } \hat{N} = \mathcal{Z} \setminus \mathcal{M}.$$

By Lemma 10 we obtain that $\mathcal{Z}$ is locally connected in the points $\{\alpha_Y\}\times(-1,1)$. Moreover, $\mathcal{Z}$ is locally connected in the points $q_{Z_Y}(\{p\}\times(-1,1))$ and since $\mathcal{Z}$ is locally connected in $v^1_{Z_Y}$ and $v^{-1}_{Z_Y}$ we have that:

$$\hat{M} = \mathrm{Sus}(\{\alpha_Y\} \cup q_{Z_Y}(\{p\} \times (-1,1))), \tag{17}$$

$$\hat{N} = q_{Z_Y}(X \times (-1,1)) \times (-1,1). \tag{18}$$

Since the sets $\hat{M}$ and $\hat{N}$ are invariant under homeomorphisms of $\mathcal{Z}$ in $\mathcal{Z}$ from (15)–(18) it follows that

$$q_{Z_Y}(X \times (-1,1)) \times (-1,1) = \mathcal{O}^3_{\mathcal{Z}}. \tag{19}$$

In the rest of the proof we will prove that the other orbits of $\mathcal{Z}$ are in $\hat{M}$. For this, first we see that $\mathcal{O}(v^1_{Z_Y}) = \{v^1_{Z_Y}, v^{-1}_{Z_Y}\}$. Let $g\colon \mathcal{Z} \to \mathcal{Z}$ be a homeomorphism. Note that $v^1_{Z_Y} \in \overline{\mathcal{O}^1_{Z_Y}}$. Then $g(\overline{\mathcal{O}^1_{Z_Y}}) = \overline{g(\mathcal{O}^1_{Z_Y})}$. By (19) we have that $\overline{g(\mathcal{O}^1_{Z_Y})} = \overline{K'}$, where $\overline{K'} = (q_{Z_Y}(X \times (-1,1)) \times (-1,1)) \cup \{v^1_{Z_Y}, v^{-1}_{Z_Y}\}$. Now, by (17) we know that $g(v^1_{Z_X}) \in \hat{M}$ in consequence $g(v^1_{Z_Y}) \in \{v^1_{Z_Y}, v^{-1}_{Z_Y}\}$. Thus by Lemma 13-(i) we obtain that,

$$\mathcal{O}(v^1_{Z_Y}) = \{v^1_{Z_Y}, v^{-1}_{Z_Y}\}. \tag{20}$$

Now, we consider the following subsets of $\hat{M}$

$$\hat{M}_1 = \{w \in \hat{M}\colon w \text{ has a planar neighborhood in } \mathcal{M}\} \text{ and } \hat{M}_2 = \hat{M} \setminus \hat{M}_1.$$

Since $q_{Z_Y}(\{p\}\times(-1,1))$ is an free arc we obtain that $\hat{M}_1 = q_{Z_X}(\{p\}\times(-1,1))\times(-1,1)$ and $\hat{M}_2 = \mathrm{Sus}(\alpha_X)$. By (20) and since the sets $\hat{M}_1$ and $\hat{M}_2$ are invariant under homeomorphisms of $\mathcal{Z}$ in $\mathcal{Z}$, this implies that

$$\{\alpha_Y\} \times (-1,1) = \mathcal{O}^1_{\mathcal{Z}} \quad \text{and} \quad q_{Z_Y}(\{p\} \times (-1,1)) = \mathcal{O}^2_{\mathcal{Z}}. \tag{21}$$

Therefore from (19), (20) and (21) we have that $\mathrm{d_H}(\mathcal{Z}) = 4$. □

As a consequence of Theorem 14 we obtain the following corollaries.

Corollary 22. *Let $Y = X \cup \{p\}$, where X is the pseudoarc, then* $\mathrm{d_H}(\mathrm{Sus}(Z_Y)) = 4$.

Corollary 23. *Let $Y = X \cup \{p\}$, where X is the Cantor set, then* $\mathrm{d_H}(\mathrm{Sus}(Z_Y)) = 4$.

Corollary 24. *Let $Y = X \cup \{p\}$, where X is a circle of pseudoarcs, then* $\mathrm{d_H}(\mathrm{Sus}(Z_Y)) = 4$.

Corollary 25. *Let $Y = X \cup \{p\}$, where X is a solenoid, then* $\mathrm{d_H}(\mathrm{Sus}(Z_Y)) = 4$.

Similarly, the following result is proved.

Theorem 15. *Let X be a a locally connected, homogeneous, metric space. Let $Y = X \cup \{p\}$, where p is an accumulation point of Y. If X does not cointains arcs, then* $\mathrm{d_H}(\mathrm{Sus}(Z_Y)) = 4$.

As a consequence of Theorem 15 we obtain the following result.

Proposition 11. *Let $X = \{0\} \cup \{\frac{1}{n} \colon n \in \mathbb{N}\}$. Then* $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 4$.

Lemma 20. *Let X be a homogeneous metric space. If $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 2$, then X is locally contractible.*

Proof. Suppose that $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 2$ and X is not locally contractible. Then, by Theorem 13 we obtain that $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$, a contradiction. Therefore, X is locally contractible. □

Lemma 21. *Let X be a homogeneous metric space. If $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 2$, then X is locally connected.*

Proof. Suppose that $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 2$ and X is not locally connected. Then, by Corollary 17 we obtain that $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$, a contradiction. Therefore, X is locally connected. □

As a consequence of Lemma 21 and Lemma 8 we obtain the following result.

Corollary 26. *Let X be a homogeneous metric space. If $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 2$, then Z_X is locally connected.*

Lemma 22. *Let X be a homogeneous metric space such that $|X| \geq 2$. If $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 2$, then X contains arcs.*

Proof. Suppose that $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 2$ and X does not contains arcs. Then, by Theorem 19 we obtain that $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 3$, a contradiction. Therefore, X contains arcs. □

Our next result follows from Lemma 5 and Lemma 20.

Corollary 27. *Let X be a homogeneous, compact, finite dimensional metric space.*
If $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 2$, then X is ANR.

Our next result follows from Lemma 6 and Corollary 27.

Corollary 28. *Let X be a homogeneous, one dimensional continuum. If $\mathrm{d_H}(\mathrm{Sus}(Z_X)) = 2$, then X is a local dendrite.*

5. $\frac{1}{n}$-homogeneity of $\mathrm{Cone}(Z_X)$. In this section we present results on the homogeneity of $\mathrm{Cone}(Z_X)$. For this, we start with the following results.

Proposition 12. *If X is a discrete space with one point, then* $\mathrm{d_H}(\mathrm{Cone}(Z_X)) = 2$.

Lemma 23. *Let $n \in \mathbb{N}$, $n \geq 2$. If X is a discrete space with n points, then*
$$\mathrm{d_H}(\mathrm{Cone}(Z_X)) = 5.$$

Proof. Suppose that X is a discrete space with n points. Note that $\mathrm{d_H}(X) = 1$. By Proposition 2 we obtain that Z_X is a n-rose. Thus by Corollary 5 we know that $\mathrm{d_H}(Z_X) = 2$. Note that Z_X is a finite graph. In consequence, by Theorem 3 we get that $\mathrm{d_H}(\mathrm{Cone}(Z_X)) = 5$. □

As a consequence of Proposition 2 and Lemma 23 we have the following result.

Corollary 29. *Let $n \in \mathbb{N}$, $n \geq 2$. If X is the n-rose, then* $\mathrm{d_H}(\mathrm{Cone}(X)) = 5$.

Now, recall that a point p of a space X is *homotopically labile* (see [4]; in [11] and [12] such points are called *unstable*) if for every neighborhood U of p there exists a map $h\colon X \times [0,1] \to X$ such that:

(*i*) $h(x,0) = x$, for each $x \in X$, (*ii*) $h(x,t) = x$, whenever $x \notin U$ and $t \in [0,1]$,

(*iii*) $h(x,t) \in U$, whenever $x \in U$ and $t \in [0,1]$, (*iv*) $h(x,1) \neq p$, for each $x \in X$.

A point that is not homotopically labile is *homotopically stable* (or simply *stable*). Some results about homotopically labile points are the following.

Lemma 24. [12, p. 134, 2] *Let z be a homotopically labile point of a space Z and let Y be a Tychonoff space. Then for any $y_0 \in Y$ the pair (z, y_0) is a homotopically labile point of $Z \times Y$.*

Lemma 25. *If X is a nonempty compact metric space, then each point $(w,0)$ is homotopically labile in $Z_X \times [0,1)$, for each $w \in \mathrm{Sus}(X)$.*

Proof. Since Z_X is a continuum (see [22, p. 43, 3.16]) and the point 0 is homotopically labile in $[0,1)$ from Lemma 24 we have the conclusion. □

The following result will be used to prove Lemma 26, whose proof the reader can find in [11] or [5].

Theorem 16. *If X is a metric space, nonempty, locally compact and finite dimensional, then X has a homotopically stable point.*

Lemma 26. *If X is a nonempty finite dimensional compact metric space, then $Z_X \times [0,1)$ is not homogeneous.*

Proof. By Lemma 25 we know that each point $(w,0)$ is homotopically labile in $Z_X \times [0,1)$. Now, since Z_X is a continuum (see [22, p. 43, 3.16]) in particularly is a compact space. Thus, Z_X is locally compact.

On the other hand, since X is finite dimensional it follows that Z_X is finite dimensional. Thus, $Z_X \times [0,1)$ is a metric space, nonempty, locally compact and finite dimensional, then by Theorem 16, $Z_X \times [0,1)$ has a homotopically stable point. In consequence $Z_X \times [0,1)$ is not homogeneous. □

As a consequence of Lemma 26 we get the following result.

Theorem 17. *Let X be a finite dimensional compact metric space. If $\mathrm{d_H}(\mathrm{Cone}(Z_X)) = 2$, then $\mathcal{O}(v_{\mathrm{Cone}(Z_X)})$ is nondegenerate.*

Proof. Assume that $\mathcal{O}(v_{\mathrm{Cone}(Z_X)}) = \{v_{\mathrm{Cone}(Z_X)}\}$. Then $Z_X \times [0,1)$ is the other orbit. Therefore, $Z_X \times [0,1)$ is homogeneous, a contradiction to Lemma 26. □

Lemma 27. *Let X be a homogeneous metric space. If $\mathrm{d_H}(\mathrm{Cone}(Z_X)) = 2$, then X is locally contractible.*

Proof. Suppose that $\mathrm{d_H}(\mathrm{Cone}(Z_X)) = 2$ and X is nonlocally contractible. Thus, by Theorem 7 we have that $\mathrm{d_H}(Z_X) = 2$. Moreover, the orbits of Z_X are $Z_X \setminus \{\alpha_X\}$ and $\{\alpha_X\}$. Consider the following subsets of $\mathrm{Cone}(Z_X)$:

$$A = \{w \in \mathrm{Cone}(Z_X)\colon \mathrm{Cone}(Z_X) \text{ is locally contractible in } w\}, \quad B = \mathrm{Cone}(Z_X) \setminus A.$$

By Lemma 10 we obtain that $A = \mathrm{Sus}(\alpha_X)$ and $B = (Z_X \setminus \{\alpha_X\}) \times [0, 1)$. Since $\mathrm{d_H}(\mathrm{Cone}(Z_X)) = 2$ this implies that A and B are orbits of $\mathrm{Cone}(Z_X)$, a contradiction because A is an arc. Therefore, X is locally contractible. □

The following lemma can be proved analogously to Lemma 27.

Lemma 28. *Let X be a homogeneous metric space. If $\mathrm{d_H}(\mathrm{Cone}(Z_X)) = 2$, then X is locally connected.*

Using the Lemma 28 we obtain:

Corollary 30. *Let X be a homogeneous metric space. If $\mathrm{d_H}(\mathrm{Cone}(Z_X)) = 2$, then X is locally connected.*

Our next result follows from Lemma 5 and Lemma 27.

Corollary 31. *Let X be a homogeneous, compact, finite dimensional metric space. If $\mathrm{d_H}(\mathrm{Cone}(Z_X)) = 2$, then X is ANR.*

Our next result follows from Lemma 6 and Corollary 31.

Corollary 32. *Let X be a homogeneous, one dimensional continuum. If $\mathrm{d_H}(\mathrm{Cone}(Z_X)) = 2$, then X is a local dendrite.*

REFERENCES

1. G. Acosta, L.C. Hoehn, Y. Pacheco Juárez, *Homogeneity degree of fans*, Topology Appl., **231** (2017), 320–328. https://doi.org/10.1016/j.topol.2017.09.002
2. R.D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math., **67** (1958), 313–324. https://doi.org/10.2307/1970007
3. K. Borsuk, Theory of Retracts, Monografie Mat., V.44, Polish Scientific Publishers, Warszawa, Poland, 1967.
4. K. Borsuk, J.W. Jaworowski, *On labile and stable points*, Fund. Math., **39** (1952), 159–175.
5. T. Ganea, *Homotopically stable points*, Rev. Math. Pures Appl., **5** (1960), 315–317. (in Russian)
6. Shijie Gu, *Degree of homogeneity on suspensions of manifolds*, Topology Appl., **238** (2018), 20–23. https://doi.org/10.1016/j.topol.2018.02.002
7. J.G. Hocking, G.S. Young, Topology, Dover Publications, New York, 1988.
8. W. Hurewicz, H. Wallman, Dimension theory, Princeton University Press, Princeton, N. J., 1969.
9. A. Illanes, S.B. Nadler, Jr., *Hyperspaces: fundamentals and recent advances*, Monographs and Textbooks in Pure and Applied Mathematics Series, Vol. 216, Marcel Dekker, Inc., New York and Basel, 1999.
10. A. Illanes, V. Martínez-de-la-Vega, *Models and homogeneity degree of hyperspaces of a simple closed curve*, Rocky Mountain J. Math., **54** (2024), №3, 765–785. https://doi.org/10.1216/rmj.2024.54.765
11. W. Kuperberg, *On stable points in finite dimensional locally compact spaces*, Thesis in the archive of the Warsaw University, 1963. (in Polish)

12. W. Kuperberg, *Homotopically labile points of locally compact metric spaces*, Fund. Math., **73** (1971), 133–136.
13. K. Kuratowski, Topology, V.II, Academic Press, New York, 1968.
14. M. de J. López Toriz, P. Pellicer-Covarrubias, A. Santiago-Santos, $\frac{1}{2}$*-homogeneous suspensions*, Topology Appl., **157** (2010), 482–493. doi:10.1016/j.topol.2009.10.007
15. M. de J. López, P. Pellicer-Covarrubias, A. Santiago-Santos, *Suspensiones* $\frac{1}{2}$*-homogéneas*, Chapter 9 in Topologia y sus Aplicaciones, (2013) 211–227. (in Spanish)
16. F. Leon Jones, Historia y desarrollo de la teoría de los continuos indecomponibles, Aportaciones Matemáticas: Investigación, 27, Sociedad Matemática Mexicana, 2004. (in Spanish)
17. W. Lewis, *Continuous collections of hereditarily indecomposable continua*, Topology Appl., **74** (1996), 169–176. https://doi.org/10.1016/S0166-8641(96)00068-5
18. S. Macias, Topics on continua, 2nd edition, Springer, 2018.
19. J. van Mill, Infinite-dimensional topology, Prerequisites and introduction, North-Holland, Amsterdam, 1989.
20. D. Michalik, *Homogeneity degree of cones*, Topology Appl., **232** (2017), 183–188. doi: 10.1016/j.topol.2017.10.010
21. D. Michalik, *Suspensions of locally connected curves: homogeneity degree and uniqueness*, Topology Proceedings, **54** (2019), 69–79.
22. S.B. Nadler, Jr., Continuum Theory: An Introduction, Monographs and Textbooks in Pure and Applied Mathematics Series, V.158, Marcel Dekker, Inc., New York, Basel and Hong Kong, 1992.
23. S.B. Nadler, Jr., P. Pellicer-Covarrubias, *Hyperspaces with exactly two orbits*, Glasnik Mat., **41** (2006), №1, 141–157. doi: 10.3336/gm.41.1.12
24. S.B. Nadler, Jr., P. Pellicer-Covarrubias, *Cones that are* $\frac{1}{2}$*-Homogeneous*, Houston J. Math., **33** (2007), №1, 229–247.
25. S.B. Nadler, Jr., P. Pellicer-Covarrubias e I. Puga, $\frac{1}{2}$*-Homogeneous continua with cut points*, Topology Appl., **154** (2007), 2154–2166. https://doi.org/10.1016/j.topol.2006.01.018
26. P. Pellicer-Covarrubias, A. Santiago-Santos, *Degree of homogeneity on cones*, Topology Appl., **175** (2014), 49–64. DOI: 10.1016/j.topol.2014.07.002
27. P. Pellicer-Covarrubias, R. Pichardo-Mendoza, A. Santiago-Santos, $\frac{1}{2}$*-homogeneity of* n*th suspensions*, Topology Appl., **161** (2014), 58–72. https://doi.org/10.1016/j.topol.2013.09.007
28. A. Santiago-Santos, *Degree of homogeneity on suspensions*, Topology Appl., **158** (2011), №16, 2125–2139. https://doi.org/10.1016/j.topol.2011.06.056
29. A. Santiago-Santos, N. Tapia-Bonilla, $\frac{1}{n}$*-Homogeneity of the 2-nd cones*, Topology Appl., **234** (2018), 130–154. https://doi.org/10.1016/j.topol.2017.11.033
30. A. Santiago-Santos, N.T. Tapia-Bonilla, *Topological properties on* n*-fold pseudo-hyperspace suspension of a continuum*, Topology Appl., **270** (2020), 106956. https://doi.org/10.1016/j.topol.2019.106956
31. G. Whyburn, Analytic topology, Amer. Math. Soc. Colloq. Publ., V.28, Amer. Math. Soc., Providence, R. I., 1942.

Instituto de Física y Matemáticas
Universidad Tecnológica de la Mixteca
Localidad Acatlima, Huajuapan de León, Oaxaca, México
alicia@mixteco.utm.mx

Facultad de Sistemas Biológicos e Innovación Tecnológica
Universidad Autónoma Benito Juárez de Oaxaca
Oaxaca de Juárez, México
noetapia7@gmail.com

Received 23.11.2024
Revised 06.04.2025